Pullbacks of hyperplane sections for Lagrangian fibrations are primitive

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Dedicated to Professor Claire Voisin

Abstract. Let $p: M \to B$ be a Lagrangian fibration on a hyperkähler manifold of maximal holonomy (also known as IHS), and H be the generator of the Picard group of B. We prove that $p^*(H)$ is a primitive class on M.

1 Introduction

Here we consider a hyperkähler manifold of maximal holonomy admitting a holomorphic fibration $\pi: M \to B$. The fibration structure is quite restricted due to the work of Matsushita, [Mat], who first noticed that the general fiber is a Lagrangian abelian variety of half of the dimension of the total space. The base has the same rational cohomology as $\mathbb{C}P^n$ and the Picard group $\operatorname{Pic}(B)$ has rank one. We prove that the pullback of the fundamental class of a hyperplane section is primitive, i.e., indivisible as an integral class.

Theorem 1.1: Let M be a hyperkähler manifold admitting a Lagrangian fibration $\pi: M \to B$ and H be the generator of $\operatorname{Pic}(B)$. Then the class $\pi^* H \in H^2(M, \mathbb{Z})$ is primitive.

The proof is based on the observation that if $\pi^* H \in H^2(M, \mathbb{Z})$ is not primitive, i.e., $\pi^* H = mH'$, then H' has trivial cohomology by Demailly, Peternell and Schneider's theorem. Applying the Hirzebruch-Riemann-Roch formula for an irreducible hyperkähler manifold, one would obtain a contradiction.

As an application of the primitivity of $\pi^* H$ one can see that if P_1 and P_2 are non-ample nef line bundles associated with different Lagrangian fibrations $\pi_i: M \to \mathbb{C}P^n$, then $P_1 \otimes P_2$ is ample and base point free.

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2 Basic notions

Definition 2.1: A *hyperkähler manifold of maximal holonomy* (or *irreducible holomorphic symplectic*) manifold M is a compact complex simply connected Kähler manifold with $H^{2,0}(M) = \mathbb{C}\sigma$, where σ is everywhere non-degenerate.

For the rest of the paper we consider hyperkähler manifolds of maximal holonomy. Due to the work of Matsushita we know that the fibration structure of hyperkähler manifolds is quite restricted.

Theorem 2.2: (D. Matsushita, [Mat]) Let M be a hyperkähler manifold and $f\colon M \to B$ a proper surjective morphism with a smooth base B. Assume that f has connected fibers and $0 < \dim B < \dim M$. Then f is Lagrangian and $\dim_{\mathbb{C}} B = n$, where $\dim_{\mathbb{C}} M = 2n$.

Definition 2.3: Following Theorem 2.2, we call the morphism $f\colon M \to B$ a *Lagrangian fibration* on the hyperkähler manifold M.

Remark 2.4: In [Mat], D. Matsushita also proved that the base B of a Lagrangian fibration has the same (rational) cohomology as $\mathbb{C}P^n$. In [Hw], J.-M. Hwang proved that when B is smooth, then it is actually isomorphic to $\mathbb{C}P^n$.

Definition 2.5: Given a hyperkähler manifold M, there is a non-degenerate primitive form q on $H^2(M,\mathbb{Z})$, called the *Beauville-Bogomolov-Fujiki form* (or *"BBF form"* for short) of signature $(3, b_2 - 3)$, satisfying the *Fujiki relation*

$$\int_M \alpha^{2n} = c \cdot q(\alpha)^n \qquad \text{for } \alpha \in H^2(M,\mathbb{Z}),$$

with $c > 0$ a constant depending on the topological type of M. This form generalizes the intersection pairing on K3 surfaces. A detailed description of the form can be found in [Be], [Bog] and [F].

Definition 2.6: Let $f : M \longrightarrow B$ be a Lagrangian fibration. As shown in [Mat], $H^*(B,\mathbb{Q}) = H^*(\mathbb{C}P^n,\mathbb{Q})$. Let H be a primitive integer generator of $H^2(M,\mathbb{Q})$, and $\mathcal{O}(1)$ be a holomorphic line bundle on B with first Chern class H. When $B = \mathbb{C}P^n$, the bundle $\mathcal{O}(1)$ coincides with the usual $\mathcal{O}(1)$. We call H *the fundamental class of a hyperplane section.*

Remark 2.7: A *semiample bundle* is a base point free line bundle which has positive Kodaira dimension. Let $f : M \to B$ be a Lagrangian fibration, and $L := f^*(\mathcal{O}(k))$, $k > 0$. Clearly, L is a semiample nef line bundle. By Matsushita's theorem any semiample nef line bundle is either ample or obtained this way. The *SYZ conjecture* (due to Tyurin, Bogomolov, Hassett, Tschinkel, Huybrechts and Sawon; see [V]) claims that converse is also true: any nef line bundle on a hyperkähler manifold is either ample or semiample. This conjecture is a special case of Kawamata's abundance conjecture.

Remark 2.8: The Hirzebruch-Riemann-Roch formula for an irreducible hyperkähler manifold M states that for a line bundle L on M, $\chi(L) = \sum a_i q(c_1(L))^i$, where the coefficients a_i are constants depending on the topology of M (see [Hu]). In particular, if $q(c_1(L)) = 0$, then $\chi(L) = a_0 = \chi(\mathcal{O}_M) = n + 1$, where $2n = \dim M$.

Here we restate a theorem by Demailly, Peternell and Schneider applied to compact Kähler manifolds with trivial canonical bundle, which is the set-up we need. The more general version of this result is Theorem 2.1.1 in [DPS].

Theorem 2.9: ([DPS, Theorem 2.1.1]) Let (M, I, ω) be a compact Kähler manifold, K_M its canonical bundle, $\dim_{\mathbb{C}} M = n$, and E a non-trivial nef line bundle on M. Assume that E admits a Hermitian metric with semipositive curvature form. The cohomology class ω is considered as an element in $H^1(\Omega^1 M)$. Consider the corresponding multiplication operator $\eta \longrightarrow \omega \wedge \eta$ maping $H^p(\Omega^q M \otimes E)$ to $H^{p+1}(\Omega^{q+1} M \otimes E)$. Then $\eta \longrightarrow \omega^i \wedge \eta$ induces a surjective map $H^0(\Omega^{n-i} M \otimes E) \twoheadrightarrow H^i(E \otimes K_M)$.

3 Main Results

Proposition 3.1: Let M be a hyperkähler manifold admitting a Lagrangian fibration $\pi : M \to B$, and E a line bundle on M which is trivial on the general fiber of π. Then $E = \pi^* E' \otimes V$, where E' is a line bundle on B, and V a torsion line bundle.

Proof: Since $\operatorname{Pic}(M)$ is discrete, it suffices to show that $c_1(E)_{\mathbb{Q}}$ is a pullback of a cohomology class on B.

Let Δ be the discriminant locus of the Lagrangian fibration, i.e., the locus parametrizing singular fibers, and let $X = \pi^{-1}(B \setminus \Delta)$. Since E is trivial on the general fiber of π, then $E|_X$ is trivial and the push-forward $\pi_* E$ is a trivial line bundle on $B \setminus \Delta$.

Let $E' := (\pi^* \pi_* E)^{**}$ be the reflexization of $\pi^* \pi_* E$. By [OSS, Lemma 1.1.15], E' is a line bundle (any rank 1 reflexive sheaf over a smooth manifold is a line bundle). Since $E'\Big|_{\pi^{-1}(B\setminus\Delta)}$ is trivial, E' is a trivial line bundle for $\operatorname{codim}\Delta \geqslant 2$, and in this case $E' = E$. Indeed, by [OSS, Lemma 1.1.12], any reflexive sheaf F is normal, that is, coincides with $j_* j^* F$, where j is an embedding of an open set with complement of codimension $\geqslant 2$.

In the case when $\operatorname{codim}\Delta = 1$, the same argument gives $E' = \mathcal{O}(D)$, where D is a divisor with support in $\pi^{-1}(\Delta)$. Since $H^*(B, \mathbb{Q}) = H^*(\mathbb{C}P^n, \mathbb{Q})$, all classes in $H^2(B, \mathbb{Q})$ are proportional. Then D is cohomologous to $\pi^{-1}(kH)$, where H is a hyperplane section in B, and $c_1(E)_{\mathbb{Q}} \in \pi^*(H^2(B, \mathbb{Q}))$. ■

Theorem 3.2: Let M be a hyperkähler manifold admitting a Lagrangian fibration $\pi: M \to B$, and H the generator of $\operatorname{Pic}(B)$ (this group has rank 1, as shown by D. Matsushita; see Remark 2.4). Then the class $\pi^* H \in H^2(M, \mathbb{Z})$ is primitive.

Proof: Suppose that $\pi^* H$ is not primitive, and $\pi^* H = kH'$ in $H^2(M, \mathbb{Z})$. Denote by E the line bundle with $c_1(E) = H'$. By Proposition 3.1, E is a non-trivial torsion bundle on smooth fibers of π. We apply Theorem 2.9 to the manifold M with the torsion nef line bundle E to obtain the surjective map $H^0(\Omega^{2n-i} M \otimes E) \twoheadrightarrow H^i(E)$. The bundle TM restricted to a regular fiber S of $\pi: M \longrightarrow B$ can be expressed as an extension

$$0 \longrightarrow TS \longrightarrow TM\big|_S \longrightarrow NS \longrightarrow 0,$$

where NS is the normal bundle, which is trivial because S is a fiber of the submersion $\pi: M \longrightarrow B$, in $s \in B$ which gives $NS = \pi^* T_s B$. However, TS is dual to NS, because S is Lagrangian, hence $\Omega^k M$ is an extension of trivial bundles. Then $\Omega^k(M) \otimes E$ has no sections for all k, and Theorem 2.9 implies that $H^i(E) = 0$ for all i.

Last, we apply the Hirzebruch-Riemann-Roch formula for the hyperkähler manifold M with the line bundle E. Since $q(E) = q(L) = 0$ and $H^i(E) = 0$, from Remark 2.8 we obtain $n + 1 = \chi(E) = \sum(-1)^i \dim H^i(E) = 0$, a contradiction. Therefore, $\pi^* H$ is primitive. ■

4 Applications

In this section we describe some applications of the primitivity result.

Proposition 4.1: Let M be a hyperkähler manifold admitting a Lagrangian fibration $f\colon M \to \mathbb{C}P^n$. Then the map $\pi_2(M) \longrightarrow \pi_2(\mathbb{C}P^n)$ is surjective.

Proof: From Theorem 3.2 we know that $L = f^*\mathcal{O}(1)$ is primitive, i.e., $c_1(L)$ is not divisible. By Poincaré duality there is $\alpha \in H_2(M, \mathbb{Z})$ such that the pairing $\langle c_1(L), \alpha\rangle = 1$ in M. This is the same as the pairing $\langle r_*\alpha, c_1(\mathcal{O}(1))\rangle$ in $\mathbb{C}P^n$, which means that $r_*\alpha$ is the class of a line, therefore $H_2(M, \mathbb{Z}) \longrightarrow H_2(\mathbb{C}P^n, \mathbb{Z})$ is surjective. Since M and $\mathbb{C}P^n$ are simply connected, this induces a surjection on the homotopy groups $\pi_2(M) \longrightarrow \pi_2(\mathbb{C}P^n)$. ■

Remark 4.2: We conjecture that if the fibration $f\colon M \to \mathbb{C}P^n$ has no multiple fibers, then for a general curve $C \subset \mathbb{C}P^n$ there is a continuous section $C \to M$. The evidence is that for every curve class $[C]$ there is a class on M surjecting to $[C]$ by Proposition 4.1.

Definition 4.3: A pullback of a very ample bundle is called *very semiample.*

Corollary 4.4: Let E be a semiample line bundle on a hyperkähler manifold, which is not ample. Assume that the corresponding Lagrangian fibration has base $\mathbb{C}P^n$. Then E is very semiample.

Proof: Indeed, by Theorem 3.2, $E = f^*\mathcal{O}(i)$, where $i > 0$ and $f: M \longrightarrow \mathbb{C}P^n$ is a Lagrangian fibration. ■

Claim 4.5: Let P_1 and P_2 be non-ample nef line bundles associated with different Lagrangian fibrations with base $\mathbb{C}P^n$. Then $P_1 \otimes P_2$ is ample and base point free.

Proof: By the corollary above, the line bundles P_i are very semiample, hence globally generated. ■

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