Hyper-Kaehler Fibrations and Hilbert Schemes

by

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Bachelor of Science, Sofia University, June 2000
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Submitted to the Department of Mathematics
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

June 2006

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Abstract

In this thesis, I consider hyper-Kähler manifolds of complex dimension 4 which are fibrations. It is known that the fibers are abelian varieties and the base is $\mathbb{P}^2$. We assume that the general fiber is isomorphic to a product of two elliptic curves. We are able to relate this class of hyper-Kähler fibrations to already known examples. We prove that such a hyper-Kähler manifold is deformation equivalent to a Hilbert scheme of two points on a K3 surface.

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Acknowledgments

I am grateful to my advisor, Prof. Gang Tian, for introducing me to this very interesting subject and for all of his support throughout the years I have been at MIT.

I thank my husband, Jason Starr, for all the support and understanding he had for me while I was a graduate student.

I am grateful to all the committee members for their useful remarks on the contents of this thesis.

And last but not least, I am thankful to MIT for all of the possibilities that have been given to me while being at this wonderful place.

Contents

Chapter 1

Introduction

Compact hyperkähler manifolds, or irreducible symplectic manifolds, are higher-dimensional analogues of K3 surfaces. They indeed share many of the well-known properties of K3 surfaces.

Irreducible symplectic manifolds occupy a distinguished place in the list of higher dimensional Kähler manifolds. Together with Calabi-Yau manifolds they are the only irreducible simply connected Kähler manifolds with $c_1(X) = 0$ (cf. [2]).

We study holomorphic symplectic manifolds which are fibred by abelian varieties. This structure is a higher dimensional analogue of an elliptic fibration on a K3 surface. We investigate when a holomorphic symplectic manifold is fibred in this way, and we study the geometry of these fibrations. We consider hyper-Kähler manifolds of complex diminsion 4. We prove that if the abelian varieties are products of elliptic curves, then the hyper-Kähler fibration is deformation equivalent to a Hilbert scheme of points on a K3 surface.

According to Matsushita ([12]), if X is a 4-dimensional hyper-Kähler manifold which admits a fibration, then the generic fiber is an abelian surface and the base is $\mathbb{P}^2$.

Chapter two introduces the main definitions and examples which we are going to use in our work.

Chapter three describes the isolated singularities which a fibration with trivial canonical sheaf can have if the fibers are abelian varieties.

Chapter four answers a question by Gang Tian about deformations of Hilbert schemes. If we start with a Hilbert scheme of a K3 surface which is a fibration and we deform it so that we preserve the fiber structure, the question is whether it is still isomorphic to a Hilber scheme of a K3 surface. The answer is negative: in general this is not the case.

Chapter five describes a special class of hyper-Kähler manifolds. We prove that a hyper-Kähler fibration with general fiber a product of two elliptic curves is deformation equivalent to a Hilbert scheme of points on a K3 surface. We make the assumptions that the fibration admits a section and that the general singular fiber is semi-stable. This is a step towards a classification of hyper-Kähler manifolds of complex dimension 4.

Chapter 2

Preliminaries

2.1 Hilbert Schemes of Points on a Surface

First, we recall the definition of the Hilbert scheme in general. Let X be a projective scheme over $\mathbb{C}$ and $\mathcal{O}_X(1)$ an ample line bundle on X. We consider the contravariant finctor $\mathcal{H}ilb_X$ from the category of schemes to the category of sets:

$$\mathcal{H}ilb_X : [\text{Schemes}] \to [\text{Sets}]$$

It associates a scheme U with a set of families of closed subschemes in X parametrized by U. For each polynomial P, let $\mathcal{H}ilb_X^P$ be the subfunctor of $\mathcal{H}ilb_X$ which associates a scheme U with a set of families of closed subschemes in X parametrized by U which have P as their Hilbert polynomial. The following theorem is due to Grothendieck:

Theorem 2.1. *The functor $\mathcal{H}ilb_X^P$ is representable by a projective scheme $Hilb_X^P$.*

This means that there exists a universal family $\mathcal{Z}$ on $Hilb_X^P$, and that every family on U is induced by a unique morphism $\phi : U \to Hilb_X^P$.

Definition 2.1. *Let P be the constant polynomial given by $P(m) = n$ for all $m \in \mathbb{Z}$. We denote by $Hilb^n(X) = Hilb_X^P$ the corresponding Hilbert scheme and call it the Hilber scheme of n point in X.*

Let $x_1, x_2, \ldots, x_n \in X$ be n distinct points and consider $Z = \{x_1, x_2, \ldots, x_n\} \subset X$ as a closed subscheme. Since the structure sheaf of Z is given by

$$\mathcal{O}_Z = \bigoplus_{i=1}^{n} \text{skyscraper sheaf at } x_i,$$

we have $\mathcal{O}_Z \otimes \mathcal{O}_X(m) = \mathcal{O}_Z$, for all $m \in \mathbb{Z}$, and hence $Z \in \mathit{Hilb}^n(X)$. This is the reason why $\mathit{Hilb}^n(X)$ is called Hilbert scheme of n points in X.

In the case when X is a complex surface, we have a nice desription of its Hilbert scheme.

Theorem 2.2. *(Fogarty [4]) Suppose X is non-singular and dim $X = 2$, then the following hold:*
(1) $\mathit{Hilb}^n(X)$ is non-singular of dimension $2n$;
(2) $\pi : \mathit{Hilb}^n(X) \to S^nX$ is a resolution of singularities, where S^nX is the n-th symmetric product of X.

2.2 Hyper-Kähler manifolds

2.2.1 Basic Facts

Definition 2.2. *A complex manifold X is called irreducible symplectic if it satisfies the following conditions:*

(1) X is compact and Kähler;

(2) X is simply connected;

(3) $H^0(X, \Omega_X^2)$ is spanned by an everywhere non-degenerate 2-form ω.

Any holomorphic two-form σ induces a homomorphism $\mathcal{T}_X \to \Omega_X$, which we also denote by σ. The two-form is everywhere non-degenerate if and only if $\sigma : \mathcal{T}_X \to \Omega_X$ is bijective. Note that (3) implies $h^{2,0}(X) = h^{0,2}(X) = 1$ and $K_X \cong \mathcal{O}_X$. In particular, $c_1(X) = 0$. Any irreducible symplectic manifold X has even complex dimension which we will fix to be $2n$.

Definition 2.3. *A compact connected* $4n$*-dimensional Riemannian manifold* (M, g) *is called hyperkähler (irreducible hyperkähler) if its holonomy is contained in (equals)* $\mathrm{Sp}(n)$.

If (M, g) is hyperkähler, then the quaternions $\mathbb{H}$ act as parallel endomorphisms on the tangent bundle of M. This is a consequence of the holonomy principle: Every tensor at a point in M that is invariant under the holonomy action can be extended to a parallel tensor over M. In particular, any $\lambda \in \mathbb{H}$ with $\lambda^2 = -1$ gives rise to an almost complex structure on M. These almost complex structures are all integrable [16]. After fixing a standard basis I, J, and $K := IJ$ of $\mathbb{H}$ any $\lambda \in \mathbb{H}$ with $\lambda^2 = -1$ can be written as $\lambda = aI + bJ + cK$ with $a^2 + b^2 + c^2 = 1$. The metric g is Kähler with respect to every such $\lambda \in S^2$. The corresponding Kähler form is denoted by $\omega_\lambda := g(\lambda\,.\,,\,.\,)$.

Thus, a hyperkähler metric g on a manifold M defines a family of complex Kähler manifolds $(M, \lambda, \omega_\lambda)$, where $\lambda \in S^2 \cong \mathbb{P}^1$.

If X is irreducible symplectic and $\alpha \in H^2(X, \mathbb{R})$ is a Kähler class on X, then there exists a unique Ricci-flat Kähler metric g with Kähler class α. This follows from Yau's solution of the Calabi-conjecture. Let ω be the holomorphic symplectic form. Bochner-Weitzenböck formula gives:

$$\Delta \mid \omega \mid^2 = \mid \nabla\omega \mid^2,$$

where Δ is the Laplacian and ∇ is the Levi-Civita connection. Integrating both sides over X, we have $\nabla\omega = 0$, which means that ω is parallel. This shows that the holonomy group is contained in $SU(2n) \cap Sp(n, \mathbb{C}) = Sp(n)$, where $n = \frac{1}{2}\dim_{\mathbb{C}} X$. Then g is an irreducible hyperkähler metric on the underlying real manifold M. Moreover, for one of the complex structures, say I, one has $X = (M, I)$.

Conversely, if (M, g) is hyper-Kähler and I, J, K are complex structures as above, then $\sigma := \omega_J + i\omega_K$ is a holomorphic everywhere non-degenerate two-form on $X = (M, I)$. If M is compact and g is irreducible hyperkähler, then M is simply connected and $H^0((M, I), \Omega^2_{(M,I)}) = \sigma \cdot \mathbb{C}$, i.e. X is irreducible symplectic.

Thus, irreducible symplectic manifolds with a fixed Kähler class and compact irreducible hyperkähler manifolds are the same object. We will use the two names accordingly.

Let X be an irreducible symplectic manifold and let $0 \neq \sigma \in H^0(X, \Omega^2_X)$ be fixed. By the holonomy principle one easily obtains (cf. [2]):

$$H^0(X, \Omega^p_X) \cong \begin{cases} 0 & p \equiv 1 (\mathrm{mod}\ 2) \\ \Lambda^{p/2}\sigma \cdot \mathbb{C} & p \equiv 0 (\mathrm{mod}\ 2). \end{cases}$$

Due to work of Beauville [2] there exists a natural quadratic form q_X on the second cohomology of an irreducible symplectic manifold generalizing the intersection pairing on a K3 surface. It is a primitive integral quadratic form on $H^2(X, \mathbb{Z})$ of index $(3, b_2(X) - 3)$. Also, $q_X(\sigma) = 0$ and $q_X(\sigma + \bar{\sigma}) > 0$.

Fujiki [5] shows the following relation: For any integral class $\alpha \in H^{2j}(X, \mathbb{Z})$ one has the form of degree $2n - j$ that sends $\beta \in H^2(X, \mathbb{Z})$ to $\int \alpha\beta^{2n-j} \in \mathbb{Z}$. Fujiki shows that for any $\alpha \in H^{4j}(X, \mathbb{Z})$ contained in the subalgebra generated by the Chern classes of X there exists a constant $c \in \mathbb{Q}$ such that

$$\int \alpha\beta^{2(n-j)} = c q_X(\beta)^{n-j} \quad \text{for any } \beta \in H^2(X, \mathbb{Q}). \tag{2.1}$$

As an application of (2.1) one has that the Hirzebruch-Riemann-Roch formula on an irreducible symplectic manifold takes the following form: If L is a line bundle on

X, then

$$\chi(L) = \sum \frac{a_i}{(2i)!} q_X(c_1(L))^i,$$

where the a_i's are constants only depending on X.

A *deformation* of a compact manifold X is a smooth proper holomorphic map $\mathcal{X} \to S$, where S is an analytic space and the fibre over a distinguished point $0 \in S$ is isomorphic to X. We will say that a certain property holds for the *generic* fibre, if for an open (in the analytic topology) dense set $U \subset S$ and all $t \in U$ the fibre $\mathcal{X}_t$ has this property. The property holds for the *general* fibre if such a set U exists that is the complement of the union of countably many nowhere dense closed (in the analytic topology) subsets.

One knows that for any compact Kähler manifold X there exists a *semi-universal* deformation $\mathcal{X} \to Def(X)$, where $Def(X)$ is a germ of an analytic space and the fibre $\mathcal{X}_0$ over $0 \in Def(X)$ is isomorphic to X. The *Zariski tangent space* of $Def(X)$ is naturally isomorphic to $H^1(X, \mathcal{T}_X)$. If $H^0(X, \mathcal{T}_X) = 0$, i.e. if X does not allow infinitesimal automorphisms, then $\mathcal{X} \to Def(X)$ is universal, i.e. for any deformation $\mathcal{X}_S \to S$ of X there exists a uniquely determined holomorphic map $S \to Def(X)$ such that $\mathcal{X}_S \cong \mathcal{X} \times_{Def(X)} S$. By a result of Tian [18] the base space $Def(X)$ is smooth if $K_X \cong \mathcal{O}_X$. In this case one says that X deforms *unobstructed*.

Let Γ be a lattice of index $(3, b-3)$. By q_Γ we denote its quadratic form. A *marked* irreducible symplectic manifold is a tuple (X, φ) consisting of an irreducible symplectic manifold X and an isomorphism $\varphi : H^2(X, \mathbb{Z}) \cong \Gamma$ compatible with q_X and q_Γ. The *period* of (X, φ) is by definition the one-dimensional subspace $\varphi(H^{2,0}(X)) \subset \Gamma_\mathbb{C}$ considered as a point in the projective space $\mathbb{P}(\Gamma_\mathbb{C})$. If $\mathcal{X} \to Def(X)$ is the universal deformation of $\mathcal{X}_0 = X$, then a marking φ of X naturally defines markings φ_t of all the fibres $\mathcal{X}_t$. Thus we can define the *period map*

$$\mathcal{P} : Def(X) \to \mathbb{P}(\Gamma_\mathbb{C})$$

as the map that takes t to the period of $(\mathcal{X}_t, \varphi_t)$. Note $\mathcal{P}$ is holomorphic. Its tangent map is given by the contraction

$$H^1(X, \mathcal{T}_X) \to Hom(H^{2,0}(X), H^{1,1}(X)) \subset Hom(H^{2,0}(X), H^2(X, \mathbb{C})/H^{2,0}(X)).$$

The holomorphic two-form σ on X satisfies $q_X(\sigma) = 0$ and $q_X(\sigma + \bar{\sigma}) > 0$. Hence the image of $\mathcal{P}$ is contained in the *period domain* $Q \subset \mathbb{P}(\Gamma_{\mathbb{C}})$ defined as $\{x \in \mathbb{P}(\Gamma_{\mathbb{C}}) | q_\Gamma(x) = 0, \quad q_\Gamma(x + \bar{x}) > 0\}$, which is an open (in the analytic topology) subset of the non-singular quadric defined by q_Γ.

Beauville proved in [2] the Local Torelli Theorem: *For any marked irreducible symplectic manifold (X, φ) the period map $\mathcal{P} : Def(X) \to Q$ is a local isomorphism.*

2.2.2 Examples

Hilbert schemes of K3 surfaces. *If S is a K3 surface, then $Hilb^n(S)$ is irreducible symplectic (cf. [2])*

Strictly speaking, $Hilb^n(S)$ is a scheme only if S is algebraic. In general, it is just a complex space. Using that S is smooth, compact, connected, and of dimension two, one shows that $Hilb^n(S)$ is a smooth compact connected manifold of dimension $2n$. By results of Varouchas [21] the Hilbert scheme is Kähler if the underlying surface is Kähler which is the case for K3 surfaces. Beauville then concludes that for any K3 surface S the Hilbert scheme $Hilb^n(S)$ is irreducible symplectic by showing that $Hilb^n(S)$ admits a unique (up to scalars) everywhere non-degenerate holomorphic two-form and that it is simply connected.

It is interesting to note that for $n > 1$ one has $b_2(Hilb^n(S)) = 23$. Moreover, the second cohomology $H^2(Hilb^n(S), \mathbb{Z})$ endowed with the natural quadratic form q_X is isomorphic to the lattice $H^2(S, \mathbb{Z}) \oplus (-2(n-1) \cdot \mathbb{Z})$.

Generalized Kummer varieties. *If A is a two-dimensional torus, then $\mathrm{K}^{n+1}(A)$ is irreducible symplectic (cf. [2]).*

The generalized Kummer variety $\mathrm{K}^{n+1}(A)$ is by definition the fibre over $0 \in A$ of the natural morphism $Hilb^{n+1}(A) \to S^{n+1}(A) \xrightarrow{\Sigma} A$, where Σ is the summation and $0 \in A$ is the zero-point of the torus. $Hilb^{n+1}(A)$ itself also admits an everywhere non-degenerate two-form, but neither is this two-form unique nor is $Hilb^{n+1}(A)$ simply connected. Both conditions are satisfied for $\mathrm{K}^{n+1}(A)$. The second Betti number of $\mathrm{K}^{n+1}(A)$ is 7 (cf. [2]).

The examples provided by the Hilbert schemes of K3 surfaces and by the generalized Kummer varieties are the two standard series of examples of irreducible symplectic manifolds. Thus in any real dimension $4n$ we have at least two different compact real manifolds admitting irreducible hyper-Kähler metrics. They are not diffeomorphic (in fact, not even homeomorphic), because their second Betti numbers are different.

2.3 Fibrations

Definition 2.4. *By* abelian fibration *on a $2n$-dimensional irreducible holomorphic symplectic manifold X we mean the structure of a fibration over $\mathbb{P}^n$ whose generic fibre is a smooth abelian variety of dimension n.*

This is a higher dimensional analogue of elliptic fibrations on K3 surfaces. At first sight, this definition may appear to be unnecessarily restrictive. For example, maybe we should allow the base to be a more general n-fold than $\mathbb{P}^n$, or to have dimension different to n. However, this is the only fibration structure that can exist on an irreducible holomorphic symplectic manifold, because of the following result by Matsushita [12]:

Theorem 2.3. *For projective symplectic manifold X, let $f : X \to B$ be a proper surjective morphism such that the generic fibre F is connected. Assume that B is smooth and $0 < \dim B < \dim X$. Then*

(1) F is an abelian variety up to a finite unramified cover,

(2) B is n-dimensional and has the same Hodge numbers as $\mathbb{P}^n$,

(3) the fibration is Lagrangian with respect to the holomorphic symplectic form.

In particular, if X is 4-dimensional, we can use the Castelnuovo-Enriques classification of surfaces to deduce that the generic fibre is an abelian surface and the base is $\mathbb{P}^2$.

Both Examples $Hilb^n(S)$ and $K^{n+1}(A)$, the Hilbert scheme of points on a K3 surface and the generalized Kummer variety, are abelian fibrations when the underlying K3 surface S or complex tori A, respectively, is an elliptic surface. For example, if $f : S \to \mathbb{P}^1$ is the fibration on S, we get an induced fibration

$$f^{[n]} : Hilb^n(S) \to \mathrm{Sym}^n\mathbb{P}^1 \cong \mathbb{P}^n$$

on $Hilb^n(S)$. The fibres in this case are products of n elliptic curves: special n-dimensional abelian varieties. A similar thing happens for the generalized Kummer variety.

In Chapter 5 we prove that the opposite also holds: if a hyper-Kähler manifold of dimension 4 admits a fibration with fibers that are products of elliptic curves, then it is deformation equivalent to a Hilbert scheme of points on a K3 surface.

Chapter 3

Degenerations of 2-dimensional Tori

In this section we classify the possible degenerate fibers which can occur in a semistable degeneration of two-dimensional tori under the assumption that the canonical bundle of the total space of the family is trivial.

3.1 Basic Tools

Let $\pi : X \to \Delta$ be a proper map of a Kähler manifold X onto the unit disk $\Delta = \{t \in \mathbb{C} :| t |< 1\}$, such that the fibers X_t are nonsingular compact complex manifolds for every $t \neq 0$. We call π a *degeneration* and the fiber $X_0 = \pi^{-1}(0)$ - the *degenerate fiber.*

Definition 3.1. *A map $\psi : Y \to \Delta$ is called a modification of a degeneration π if there exists a birational map $f : X \to Y$ such that $\psi = \pi \circ f$ and ψ is an isomorphism outside of the degenerate fiber.*

A degeneration is called *semistable* if the degenerate fiber is a reduced divisor with normal crossings. Not every degeneration can be modified to a semistable one.

Nonetheless, it is possible to reduce any degeneration to a semistable one after a base change according to Mumford's theorem ([9]).

Definition 3.2. *The polyhedron* $\Pi(V)$ *of a variety with normal crossings* $V = V_1 + \cdots + V_n$*,* $dimV_i = d$ *is the polyhedron whose vertices correspond to the irreducible components* V_i *and the vertices* $V_{i_1}, \cdots, V_{i_k}$ *form a* $(k-1)-$*simplex if* $V_{i_1} \cap \cdots \cap V_{i_k} \neq 0$*.*

Let $\pi : X \longrightarrow \Delta$ be a semistable degeneration of surfaces whose degenerate fiber is $X_0 = V_1 + V_2 + \cdots + V_n$. If $D \in Pic(X)$ and V is a component of the fiber, then let $D_V = i^*(D) = D \cdot V$, where $i : V \hookrightarrow X$ is the inclusion. For $D, D' \in Pic(X)$ the intersection index on V is defined: $D \cdot D' \cdot V = D_V \cdot D'_V$. We will state some results from [10].

Lemma 3.1. *([10]) Let* $C = V_i \cap V_j$ *be a double curve of a semistable degeneration of surfaces. Then* $(C^2)_{V_i} + (C^2)_{V_j} = -T_C$*, where* T_C *is the number of triple points of the fiber* X_0 *on* C*.*

Proof. Note that C is a union of non-singular curves since X_0 is a divisor with normal crossings. We have $V_i \cdot V_j \cdot X_0 = 0$, because $X_0 \sim X_t$. On the other hand,

$$V_i \cdot V_j \cdot X_0 = V_i \cdot V_j \cdot (V_i + V_j + \sum_{k \neq i,j} V_k) = V_iV_jV_i + V_iV_jV_j + T_C = (C^2)_{V_j} + (C^2)_{V_i} + T_C.$$

□

Let $\chi(V) = h^0(\mathcal{O}_V) - h^1(\mathcal{O}_V) + h^2(\mathcal{O}_V) = p_g - q + 1$ be the Euler characteristic of the structure sheaf $\mathcal{O}_V$ of the algebraic surface V.

Lemma 3.2. *([10]) Let* T *be the number of all triple points of* π*, then*

$$\chi(X_t) = \sum_{i=1}^{n} \chi(V_i) - \sum_{i<j} \chi(C_{i,j}) + T,$$

where $C_{i,j} = V_i \cap V_j$

Remark 3.1. *([10]) For a variety with normal crossing X_0 there is a natural mixed Hodge structure with weight filtration W and $W_0H^m(X_0) \cong H^m(\Pi(X_0))$.*

Theorem 3.1. *(Kulikov [10], Persson [15]) Let $\pi : X \to \Delta$ be a semistable Kähler degeneration of surfaces, then*

$$h^1(X_t) = \sum_{i=1}^{n} h^1(V_i) - \sum_{i<j} h^1(C_{i,j}) + 2h^1(\Pi) + ckh^1,$$

$$p_g(X_t) = \sum_{i=1}^{n} p_g(V_i) + h^2(\Pi) + \frac{1}{2}ckh^1,$$

where ckh^1 = dim Coker $(\oplus H^1(V_i) \to \oplus H^1(C_{i,j}))$, d is the number of double curves of the fiber X_0 and Π is its polyhedron.

Lemma 3.3. *A surface V is ruled or $\mathbb{CP}^2$ if and only if $H^0(V, nK_V) = 0$ for every $n > 0$.*

3.2 Main Theorem

In [19, 20] K. Ueno studies degenerations of normally polarized abelian surfaces which are of the first kind. A degeneration is said to be of the first kind if it corresponds to an inner point of the Siegel upper half plane. Ueno doesn't impose any condition on the total space of the fibration. Here we assume that the total space has a trivial canonical bundle and we consider general degenerations. In order to classify the possible degenerations, we don't refer to Ueno's list.

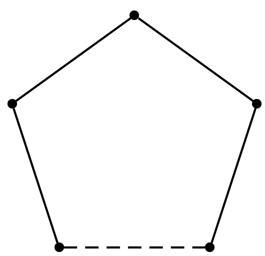
We prove the following theorem which is analogous to the classification theorems in [10] which Kulikov gives for K3-surfaces and Enriques surfaces.

Theorem 3.2. *Let $\pi : X \to \Delta$ be a semistable Kähler degeneration of two-dimensional* *tori such that K_X is trivial. Then the degenerate fiber X_0 is one of the following four types:*

(i) $X_0 = V_1$ is a nonsingular torus;

(ii) $X_0 = V_1 + V_2 + \cdots + V_n, n > 1,$ all V_i are elliptic ruled surfaces, the double curves $C_{1,2}, \cdots, C_{n-1,n}$ are elliptic curves and the polyhedron Π is a simple path.

(iii) $X_0 = V_1 + V_2 + \cdots + V_n, n > 1,$ all V_i are elliptic ruled surfaces, the double curves $C_{1,2}, \cdots, C_{n-1,n}, C_{n,1}$ are elliptic curves and the polyhedron Π is a cycle.



(iv) $X_0 = V_1 + V_2 + \cdots + V_n, n > 1,$ all V_i are rational surfaces, and all the double curves $C_{i,j}$ are rational. The polyhedron Π is a triangulation of the real 2-dimensional torus T^2.

In the first case the monodromy M is trivial, i.e. $N = \log M = 0$. In the second and the third cases $N^2 = 0$. And, in the fourth case the monodromy is of maximal rank.

Proof. Case (i) is when X_0 has a single component.

Let $n > 1$. The fibers X_t and X_0 are linearly equivalent and in addition $X_0 = V_1 + \cdots + V_n \sim 0$, hence by the adjunction formula,

$$K_{V_i} = K_X \otimes [V_i] \mid_{V_i} = O_{V_i}(\sum_{j \neq i} -V_j) = -\sum_{j \neq i} C_{i,j}$$

because K_X is trivial. Then K_{V_i} is anti-effective, and thus all of V_i are ruled surfaces

(Lemma 3.3). Consider a double curve $C_{i,j}$ on V_i. We have:

$$2g(C_{i,j}) - 2 = (K_{V_i} + C_{i,j}, C_{i,j})_{V_i} = -\sum_{k \neq i,j} (C_{i,k}, C_{i,j})_{V_i} = -T_{C_{i,j}},$$

where $T_{C_{i,j}}$ is the number of triple points of X_0 on $C_{i,j}$. Since $T_{C_{i,j}} \geq 0$ and $g(C_{i,j}) \geq 0$, there are two possibilities:

(A) $g(C_{i,j}) = 0$ and $T_{C_{i,j}} = 2$, so $C_{i,j}$ is a rational curve and there are exactly two triple points on $C_{i,j}$.

(B) $g(C_{i,j}) = 1$ and $T_{C_{i,j}} = 0$, so $C_{i,j}$ is an elliptic curve and $C_{i,j}$ does not intersect any other double curve.

In the case (A) we see that $C_{i,j}$ intersects some other double curves which must be rational as well and also contains two triple points. Thus every V_i is a ruled surface and the set of double curves on V_i consists of a disjoint union of a finite union of elliptic curves and a finite number of cycles of rational curves.

Let $V = V_{i_0}$ be one of the components, let $\phi : V \to \bar{V}$ be a morphism onto the minimal model $\bar{V}$ (ϕ is a composition of monoidal transforms) and let L be an exceptional curve on V such that $L \cong \mathbb{P}^1, (L^2)_V = -1$ and L is blown down to a point by the morphism ϕ. Then $(L, K_V)_V = -1$, so $(L, \sum_{j \neq i_0} C_{i_0,j})_V = 1$. Thus, either L intersects only one of the connected components of the divisor $\sum_{j \neq i_0} C_{i_0,j}$ or L coincides with one of $C_{i_0,j}$. It follows that the number of connected components of the divisor $\sum_{j \neq i_0} \phi_* C_{i_0,j}$ equals the number of connected components of the divisor $\sum_{j \neq i_0} C_{i_0,j}$ since $K_{\bar{V}} = \phi_* K_V$.

In Lemma 2.18 from [10] Kulikov gives a list of possible components of an effective divisor linearly equivalent to $-K_{\bar{V}}$, where $\bar{V}$ is either a minimal ruled surface or $\mathbb{CP}^2$.

Since the reduced divisor

$$\sum_{j \neq i_0} \phi_* C_{i_0,j} \sim -K_{\bar{V}},$$

we have the following possibilities for V:

(a) V is a rational surface and $\sum_{j \neq i_0} C_{i_0,j}$ is a cycle of rational curves;

(b) V is a rational or an elliptic ruled surface and $\sum_{j \neq i_0} C_{i_0,j} = C$ is a single elliptic curve;

(c) V is a ruled elliptic surface and $\sum_{j \neq i_0} C_{i_0,j} = C_1 + C_2$ consists of two disjoint elliptic curves.

Case 1: One of V_i is of type (a). Then the double curves on the components adjacent to V_i also form a cycle, hence the components adjasent to V_i are also of type (a). Since X_0 is connected, it follows that all V_i are rational surfaces and their double curves form cycles. Therefore, the polyhedron Π is a triangulation of a compact real surface without a boundary. There is no boundary, because there are exactly two triple points on each double curve.

Since V_i and $C_{i,j}$ are rational, $p_g(V_i) = 0, h^1(V_i) = 0, h^1(C_{i,j}) = 0$ and from the second equality, $ckh^1 = 0$ (see [15]). Then the first formula in Theorem 3.1 says that $h^1(\Pi) = 2$ and the second formula says that $h^2(\Pi) = p_g(X_t) = 1$. We also know that $h^0(\Pi) = 1$ (from Remark 3.1). There is only one real surface without boundary with these cohomology numbers, namely the torus T^2. In this case the degenerate fiber falls into type (*iv*) in the statement of the theorem.

Case 2: All of the V_i have types (b) or (c). Then X_0 has no triple points ($T = 0$) and thus Π is 1-dimensional, so $h^2(\Pi) = 0$.

Let the number of rational surfaces be r. For a ruled elliptic surface V_i the Euler characteristic $\chi(V_i)$ is 0, while for a rational surface $\chi(V_i) = 1$. Also, $\chi(C_{i,j}) = 0$ for an elliptic curve $C_{i,j}$. Therefore, after we apply Lemma 3.2, we get $r = \chi(X_t) = 0$. In other words, there are no rational surfaces in the fiber X_0.

Since for an elliptic ruled surface V_i we have $p_g(V_i) = 0$, then from the second formula in Theorem 3.1 we get $ckh^1 = 2$. Now we substitute it in the first formula in this theorem and use that $h^1(V_i) = 2, h^1(C_{i,j}) = 2$ to obtain that

$$n = 1 + d - h^1(\Pi),$$

where n is the number of components and d is the number of double curves.

If $h^1(\Pi) = 0$, then $n = d + 1$ and Π is a tree. Moreover, on each component V_i there are at most two double curves, therefore Π is the simple path described in case (*ii*) of the theorem.

If $h^1(\Pi) = 1$, then $n = d$ and there is one loop in the graph, hence Π is the simple cycle from case (*iii*) (because there are at most two edges coming out of every vertex).

If $h^1(\Pi) \geq 2$, then there will be at least one vertex in which there are at least three edges meeting, which is a contradiction.

The claims about the monodromy follow from the fact that $N = 0$ if and only if $h^2(\Pi) = 0$ and $ckh^1 = 0$; and $N^2 = 0$ if and only if $h^2(\Pi) = 0$ (see Theorem 2.7 in the paper [10]). $\square$

Chapter 4

Deformation of Hilbert Schemes

In this chapter we answer the following question:

Question 1. *Let $X = Hilb^2(S)$ for a K3 sufrace S. Assume that there is a fibration structure $f : X \to \mathbb{P}^2$ of X. Does a general deformation of X preserving the fiber structure still remain isomorphic to $Hilb^2(S')$ for a K3 surface S' ?*

The answer to the question above is negative. In general, a deformation of X preserving the fiber structure is no longer $Hilb^2(S')$ for a K3 surface S'.

Proof. Let $D = f^* c_1(\mathcal{O}_{\mathbb{P}^2}(1)) \in H^{1,1}(Hilb^2(S))_{\mathbb{Z}}$ be the pull-back of the hyperplane class and $E \in H^2(Hilb^2(S), \mathbb{Z})$ be the exceptional divisor.

Lemma 4.1. *Denote the Kuranishi space of Hilbert schemes of two points on K3 surfaces by $\mathcal{K}_H$. The analytic subvariety of $\mathcal{K}_H$ for which $D \in H^{1,1}(Hilb^2(S))_{\mathbb{Z}}$ is of type $(1,1)$ is of complex codimension at most 1.*

Proof. $H^2(Hilb^2(S), \mathbb{Z}) = H^2(S, \mathbb{Z}) \oplus E$.

The Hodge numbers of $H^2(Hilb^2(S), \mathbb{Z})$ are $(1, 21, 1)$, and let $\sigma \in H^{2,0}(Hilb^2(S))$. The only condition on a first order deformation of $Hilb^2(S)$ that D remains in $H^{1,1}$ is that the "deformation" of σ has intersection zero with D.

Therefore, inside the 21-dimensional space $\mathcal{K}_H$ there is at least a 20-dimensional space of deformations which are fibrations over $\mathbb{P}^2$. $\square$

Lemma 4.2. *For every K3 surface S, the analytic subvariety of the moduli space of Kähler K3 surfaces S' such that $Hilb^2(S') \cong Hilb^2(S)$, is discrete.*

Proof. Since E is an integral class, there are countably many possibilities for E. From E we can reconstruct its orthogonal complement $H^2(S, \mathbb{Z})$ and by Torelli's theorem, we can reconstruct S, ([1]). Hence, we get countably many possibilities for S. $\square$

Denote the Kuranishi space of K3 surfaces by $\mathcal{K}$. Consider the analytic subvariety $\mathcal{I} = \{([S'], [X']) \in \mathcal{K} \times \mathcal{K}_H | S'$ is a deformation of S, X' is a deformation of $X = Hilb^2(S)$, X' is a fibration over $\mathbb{P}^2$ and $X' \cong Hilb^2(S')\}$.

There are two maps: $\pi_1 : \mathcal{I} \to \mathcal{K}$ and $\pi_2 : \mathcal{I} \to \mathcal{K}_H$. The map π_2 has 0-dimensional fibers by Lemma 2.

Proposition 4.1. *The map $\pi_1 : \mathcal{I} \to \mathcal{K}$ is nowhere submersive. More precisely, if S' is a K3 surface with $H^{1,1}(S')_{\mathbb{Z}} = (0)$, then S' is not in the image of π_1, i.e., $Hilb^2(S')$ is not a fibration over $\mathbb{P}^2$.*

Proof. A general K3 surface S' doesn't have any divisors, i.e., $H^{1,1}(S')_{\mathbb{Z}} = (0)$. We are going to prove the proposition by way of contradiction. Assume there exists a surjective holomorphic map $g : Hilb^2(S') \to \mathbb{P}^2$. Then the divisors D and E are proportional, because $H^{1,1}(Hilb^2(S'))_{\mathbb{Z}} = H^{1,1}(S')_{\mathbb{Z}} \oplus E = (0) \oplus E = E$.

In particular, $D \cup D \cup D \in H^{(3,3)}(Hilb^2(S'))_{\mathbb{Z}})$ is proportional to $E \cup E \cup E$.

Consider the inclusion $\imath : E \hookrightarrow Hilb^2(S')$, then $E^{\cup 4} = \imath_*(c_1(N_{E/Hilb^2(S')})^{\cup 3})$. We want to compute this cup-product.

Take the commutative diagram:

$$\begin{array}{ccc} E & \hookrightarrow & \widetilde{S' \times S'} \\ \downarrow & & \downarrow \\ S' \cong \Delta & \hookrightarrow & S' \times S' \end{array}$$

By the properties of blow-up, $E = \mathbb{P}(TS')$, $N_{E/\widetilde{S'\times S'}} \cong \mathcal{O}(-1)$ and $N_{E/Hilb^2(S')} \cong \mathcal{O}(-2)$.

Then, $H^*(\mathbb{P}(TS')) = H^*(S')[\zeta]/(\zeta^2 + a_1\zeta + a_2)$, where $\zeta = c_1(\mathcal{O}_E(1))$.

Now we want to determine a_1 and a_2. There is an exact sequence:

$$0 \to \mathcal{O}(-1) \to \pi^*TS' \to Q \to 0,$$

where Q is a line bundle and $\pi : E \to S'$.

Applying Whitney sum gives us:

$$c(Q) = \frac{\pi^*c(TS')}{c(\mathcal{O}(-1))} = \frac{1 + \pi^*c_1(S') + \pi^*c_2(S')}{1-\zeta} = (1+\pi^*c_1(S')+\pi^*c_2(S'))(1+\zeta+\zeta^2+\dots)$$

Since Q is a line bundle, $c_2(Q) = 0$, therefore $\zeta^2 + \pi^*c_1(S')\zeta + \pi^*c_2(S') = 0$. For a K3-surface S', $c_1(S') = 0, c_2(S') = 24$, hence $\zeta^2 = -24[f]$, where $[f]$ is the class of a fiber of π.

We get $\imath_*E^{\cup 4} = (c_1(N_{E/Hilb^2(S')})^{\cup 3}) = (-2\zeta)^3 = (-8)\zeta(-24[f]) = 192[p]$, where $[p]$ is the class of a point.

However, $D^{\cup 3} = 0$, but $E^{\cup 4} \neq 0$, hence $E^{\cup 3} \neq 0$ as well and D and E cannot be proportional - a contradiction.

$\square$

Since S' determines $X' = Hilb^2(S')$, the map π_1 is injective. The dimension of $\mathcal{K}$ is 20 and since π_1 is not submersive, we get that $dim(\mathcal{I}) \leq 19$. Therefore, $dim(\pi_2(\mathcal{I})) \leq dim(\mathcal{I}) \leq 19$.

The locus of deformations of X that are fibrations over $\mathbb{P}^2$ has dimension at least 20 (by Lemma 1). So, a generic such deformation is not in the image of $\mathcal{I}$, i.e., it is not isomorphic to $Hilb^2(S')$ for any K3 surface S'.

$\square$

Chapter 5

A Special class of hyper-Kähler manifolds

5.1 Four-folds fibred by Jacobians

Here we assume that the irreducible holomorphic symplectic manifold X is fibred by abelian varieties, and that this fibration has a section. Ultimately, our goal is to relate X to the examples due to Beauville. Markushevich considers a special class of such fibrations and proves the following theorem:

Theorem 5.1. *[11] Suppose the irreducible holomorphic symplectic four-fold $\pi : X \to \mathbb{P}^2$ is fibred by Jacobians of genus-two curves, and that the fibration admits a section. Then X is birational to $Hilb^2(S)$ for some K3 surface S.*

Here we outline how the K3 surface S is constructed: it is actually the double cover of the dual plane $(\mathbb{P}^2)^\vee$ branched over a sextic B.

Let $Y \to \mathbb{P}^2$ be the family of genus-two curves. Each curve Y_t, for $t \in \mathbb{P}^2$, is hyperelliptic, being a double cover of $\mathbb{P}^1_t := \mathbb{P}(H^0(Y_t, \mathcal{K}_{Y_t}))$ branched over six points. Each of these lines $\mathbb{P}^1_t$ is canonically embedded in the dual plane $(\mathbb{P}^2)^\vee$.

The fibre X_t is the Jacobian of Y_t, and therefore its tangent space T_pX_t at any point $p \in X_t$ is $H^0(Y_t, \mathcal{K}_{Y_t})$. Using the holomorphic symplectic form, and the fact that the fibres of X are Lagrangian, we can identify T_pX_t with

$$(\pi^*\Omega^1_{\mathbb{P}^2})_p = (\Omega^1_{\mathbb{P}_2})_t.$$

The Euler sequence on $\mathbb{P}^2$ gives

$$0 \to \Omega^1_{\mathbb{P}^2} \to H^0(\mathbb{P}^2, \mathcal{O}(1)) \otimes \mathcal{O}(-1) \to \mathcal{O}_{\mathbb{P}^2} \to 0$$

and projectivizing we get the inclusion

$$\mathbb{P}(\Omega^1_{\mathbb{P}^2}) \hookrightarrow \mathbb{P}(H^0(\mathbb{P}^2, \mathcal{O}(1))) = (\mathbb{P}^2)^\vee$$

where the right hand side is the trivial bundle over $\mathbb{P}^2$ with fibre $(\mathbb{P}^2)^\vee$. Taking the fibre over $t \in \mathbb{P}^2$ proves the claim.

The six branch points on $\mathbb{P}^1_t$ will vary holomorphically with t, so for a pencil of these lines in $(\mathbb{P}^2)^\vee$ it will cut out a sextic. A priori, different pencils could give different sextics; however, the six branch points on $\mathbb{P}^1_t$ will actually be the points of intersection of $\mathbb{P}^1_t$ with the curve B dual to the the degeneracy locus of the fibration X. This establishes that B is a sextic, and the double cover of $(\mathbb{P}^2)^\vee$ branched over B is therefore a K3 surface S. Moreover, pulling-back the line $\mathbb{P}^1_t$ from $(\mathbb{P}^2)^\vee$ will give us a curve in S isomorphic to Y_t, as both curves are double covers of $\mathbb{P}^1_t$ branched over the same six points.

We have thus realized the base $\mathbb{P}^2$ as a linear system of curves on a K3 surface. Indeed X is isomorphic to the moduli space of rank-one torsion sheaves on S, which moreover is birational to $Hilb^2(S)$ (see [17]).

5.2 Main Theorem

We consider hyper-Kähler manifolds of complex dimension 4. The special kind of fibrations we consider are the ones fibred by elliptic abelian varieties. We use techniques which are different from the ones that Markushevich uses and we obtain the following result:

Theorem 5.2. *Let $p : X \to \mathbb{P}^2$ be a hyper-Kähler fibration with general fiber a product of two elliptic curves. Assume that the fibration admits a section τ and that the general singular fiber is semi-stable. Then X is deformation equivalent to $Hilb^2(S)$ for a K3 sufrace S.*

Proof. Take the open subset $U \subset \mathbb{P}^2$ over which the fibers of p are smooth. U is algebraic. Indeed, p is a proper morphism, the singular locus of the morphism is closed and the image of a Zariski closed subset of X is Zariski closed subset in $\mathbb{P}^2$. Therefore, the complement of the image of the singular locus, which is U, is an algebraic open set.

The fibers over U are of the form $E_t^1 \times E_t^2$. We can form the fibration $\tilde{Y} \to U$ with fibers of the form $E_t^1 \cup E_t^2$, where the two elliptic curves are glued along the section τ. Since $\tilde{Y}$ is not normal, we can take the normalization $\tilde{Y}^{nor} \to \tilde{Y} \to U$. Using Stein factorization, the morphism $\tilde{Y}^{nor} \to U$ factors through a smooth proper morphism with connected fibers and an étale morphism of degree 2: $\tilde{Y}^{nor} \to V \to U$.

According to [6], Section 6.3., there is a unique normal variety $\bar{V}$, a finite degree-2 morphism $f : \bar{V} \to \mathbb{P}^2$ and a fiber diagram:

$$\begin{array}{ccc} V & \xrightarrow{2:1} & U \\ \curvearrowright & & \curvearrowright \\ \bar{V} & \xrightarrow{2:1} & \mathbb{P}^2 \end{array}$$

Indeed, let's cover $\mathbb{P}^2 = \bigcup E_i$ with open affine sets. Then we have the following maps between the rings of functions: $\mathbb{C}(E_i) \to \mathbb{C}(U) \to \mathbb{C}(V)$. Take the integral closure of $\mathbb{C}(E_i)$ inside the fraction field of $\mathbb{C}(V)$. It corresponds to $\bar{E}_i$. We can glue all of them together to get a variety $\bar{V}$ ([7]) which is normal by construction. $\bar{V}$ is unique, because it is such that for every normal T and dominant morphism $T \to \mathbb{P}^2$, the set of lifts: $T \dashrightarrow \bar{V} \to \mathbb{P}^2$ is naturally in bijection with the set of factorizations: $\mathbb{C}(\mathbb{P}^2) \to \mathbb{C}(\bar{V}) \dashrightarrow \mathbb{C}(T)$.

Since the general fiber of X is semi-stable, the fibration $\tilde{Y}^{nor} \to V$ extends to a minimal family of elliptic curves $\pi : \mathcal{E} \to \bar{V}$ with a general fiber being semi-stable. The singular fibers are from Kodaira's list of degenerations of elliptic curves. In codimension one there are no multiple fibers. However, in codimension two there might be multiple fibers, the fiber dimension can jump or $\mathcal{E}$ might not be even defined. We are only interested in codimension one. Notice that $\mathcal{E}$ is a Néron model [3] (the fibers are abelian varieties). There is an induced section σ of the fibration.

Since the map $f : \bar{V} \to \mathbb{P}^2$ is 2:1, there is an involution i acting on $\bar{V}$ which interchanges the sheets of the fibers. The involution is well defined on V and from the fiber diagram above it is well defined on $\bar{V}$ as well, because with $i^{-1}V$ we can construct the same fiber diagram since the maps to $\mathbb{P}^2$ are the same. Therefore there will be an involution on $\bar{V}$ compatible with the involution on V. Denote the branched locus of f by D and $f^{-1}(D) = \tilde{D}$. Let G be the discriminant locus of $\pi : \mathcal{E} \to \bar{V}$. Note that the intersection $G \cap \tilde{D}$ consists of finitely many points. Indeed, if it wasn't true, then G and $\tilde{D}$ would have a whole component in common. The fibers above this component would be very degenerate (they will be products of two degenerate elliptic curves). However, we assumed that in codimension one the fibers of the original fibration have at worst simple normal crossing singularities, so this cannot happen.

The section σ induces a section $(\sigma, \sigma \circ i)$ of the map:

$$pr_{\bar{V}} : \mathcal{E} \times_{\bar{V}} i^*\mathcal{E} \to \bar{V}$$

and the involution $i : \circlearrowleft \bar{V}$ induces an involution $i_{\mathcal{E}}$ on $\mathcal{E} \times_{\bar{V}} i^*\mathcal{E}$.

Consider the non-branched locus $\mathbb{P}^2 - D$ and its pre-image X_0 in X. Denote the induced fibration by $p_0 : X_0 \to \mathbb{P}^2 - D$. By construction,

$$X_0 = \mathcal{E} \times_{\bar{V}} i^*\mathcal{E}|_V / i_{\mathcal{E}} \tag{5.1}$$

Let $L \subset \mathbb{P}^2$ be a line intersecting D transversally at general points of D. Denote the pre-image of L in $\bar{V}$ by L_V. The map $f_V : L_V \to L$ is 2:1. We choose L general so that L doesn't intersect $\tilde{D} \cap G$, so every fiber of $\mathcal{E}_{L_V} \times_{L_V} i^*\mathcal{E}_{L_V} \to L_V$ over a point of $L \cap \tilde{D}$ is smooth. Now we pull back our construction to L_V and we have a rational morphism:

$$L_V \times_{\mathbb{P}^2} X \dashrightarrow \mathcal{E}_{L_V} \times_{L_V} i^*\mathcal{E}_{L_V}$$

which by Weil's extension theorem [3] is regular on the smooth locus of $L_V \times_{\mathbb{P}^2} X \to L_V$ (the fibers are abelian varieties).

Every section of $p : X \to \mathbb{P}^2$ is contained in the smooth locus of $X \to \mathbb{P}^2$, so it pulls-back to a curve in the smooth locus of $L_V \times_{\mathbb{P}^2} X \to L_V$ (via the section $f_V\tau$).

Since $\mathcal{K}_X = \mathcal{O}_X$, the relative sheaf $\omega_{X/\mathbb{P}^2} = p^*\mathcal{O}_{\mathbb{P}^2}(3)$ and therefore $\tau^*\omega_{X/\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(3)$.

Consider the relative tangent bundle with the natural isomorphism:

$$(f_V\tau)^* N_{f_V\tau(L_V)/L_V \times_{\mathbb{P}^2} X} = f_V^*(\tau^* T_{X/\mathbb{P}^2}) \xrightarrow{\sim} (\sigma, \sigma \circ i)^* T_{\mathcal{E} \times i^*\mathcal{E}/\bar{V}}|_{L_V}$$

For every point $p \in L_V \cap \tilde{D}$, the map is $L_1 \oplus L_2 \to L_1 \oplus L_2(p)$, where $L_1 \in T_{\Delta_{\mathcal{E}}}|_p$ and $L_2 \in N_{\Delta_{\mathcal{E}}/\mathcal{E}\times\mathcal{E}} \cong T_{\mathcal{E}}$. Outside the branched locus, the map is the identity.

From the above we get:

$$f_V^*(\tau^*\Lambda^2 T_{X/\mathbb{P}^2}) \to (\sigma, \sigma \circ i)^* \Lambda^2 T_{\mathcal{E} \times i^*\mathcal{E}/\bar{V}}|_{L_V},$$

or equivalently,

$$f_V^*\omega_{X/\mathbb{P}^2}\check{} \to (\sigma, \sigma\circ i)^*\omega_{\mathcal{E}\times i^*\mathcal{E}/\bar{V}}\check{}\,|_{L_V}$$

Also, we have:

$$\omega_{\mathcal{E}\times i^*\mathcal{E}/\bar{V}}|_{L_V} \cong f_V^*\omega_{X/\mathbb{P}^2}(-L_V\cap\tilde{D}) \cong f^*[\mathcal{O}_{\mathbb{P}^2}(3)(-\frac{1}{2}D)]|_{L_V}, \qquad (5.2)$$

because $\tilde{D}\to D$ is 2:1.

However, $\omega_{\mathcal{E}\times i^*\mathcal{E}/\bar{V}} = \omega_{\mathcal{E}/\bar{V}}\otimes\omega_{i^*\mathcal{E}/\bar{V}}$. Denote by $\bar{M}_{1,1}$ the coarse moduli space of marked elliptic curves. The singular locus of a family of elliptic curves maps to the boundary of $\bar{M}_{1,1}$. If we denote the pull-back of the boundary of $\bar{M}_{1,1}$ by δ, then $\omega_{\mathcal{E}/\bar{V}} = \frac{\delta}{12}$ and similarly, $\omega_{i^*\mathcal{E}/\bar{V}} = \frac{\delta}{12}$.

Therefore,

$$(\sigma,\sigma\circ i)^*\omega_{\mathcal{E}\times i^*\mathcal{E}/\bar{V}} \cong \mathcal{O}_{\bar{V}}\Big(\frac{G+i^{-1}G}{12}\Big) = \mathcal{O}_{\bar{V}}\Big(\frac{f^{-1}(f(G))}{12}\Big) = f^*\mathcal{O}_{\mathbb{P}^2}\Big(\frac{f(G)}{12}\Big) \qquad (5.3)$$

When we compare the isomorphisms (5.2) and (5.3), we get:

$$f^*\mathcal{O}_{\mathbb{P}^2}\Big(\frac{f(G)}{12}\Big)|_{L_V} \cong f^*[\mathcal{O}_{\mathbb{P}^2}(3)(-\frac{1}{2}D)]|_{L_V},$$

or equivalently,

$$f^*\mathcal{O}_{\mathbb{P}^2}\Big(\frac{D}{2}+\frac{f(G)}{12}\Big)|_{L_V} \cong f^*\mathcal{O}_{\mathbb{P}^2}(3)|_{L_V}$$

Comparing the degrees, we obtain the relation:

$$\frac{1}{2}deg(D)+\frac{1}{12}deg(G) = 3$$

The degrees of D and G are positive integers (otherwise we would have trivial fibrations), hence there are two possibilities: $(deg(D), deg(G)) = (2, 24)$ or $(4, 12)$.

Case 1: $(deg(D), deg(G)) = (4, 12)$

Since $deg(D) = 4$, $\bar{V}$ is a del Pezzo surface (i.e., $K_{\bar{V}} < 0$). We want to show that $\mathcal{E}$ is rationally connected. Take two general points $p, q \in \mathcal{E}$. Then $f(\pi(p)), f(\pi(q))$ are two general points in $\mathbb{P}^2$. Then $f(\pi(p)) \in L_p$, where L_p is a tangent line to D at $f(\pi(p))$ and $f(\pi(q)) \in L_q$, where L_q is again a tangent line. Let $L_p \cap L_q = \{r\}$.

Take a tangent line L to D and pull it back to $\bar{V}$: $L_{\bar{V}} \doteq L \times_{\mathbb{P}^2} \bar{V}$. Its normalization is $\tilde{L}_{\bar{V}}$ and $\mathcal{E}|_{\tilde{L}_{\bar{V}}}$ is an elliptic fibration over $\mathbb{P}^1$ with 12 nodal fibers. The surface is rational, because it is deformation equivalent to $\mathbb{P}^2$ blown-up at 9 points in the base locus of a pencil of plane cubics ([1], section 5.12).

We can lift the lines L_p and L_q to $\mathcal{E}$ and get $\mathcal{E}|_{\tilde{L}_{p,\bar{V}}}$ and $\mathcal{E}|_{\tilde{L}_{p,\bar{V}}}$ which are rational surfaces. We can connect any two points on a rational surface with a rational curve. Connect p to $\tilde{r}$ and q to $\tilde{r}$, where $\tilde{r} \in (f\pi)^{-1}(r)$, so p is rationally chain connected to q.

The point $\tilde{r}$ is a smooth point of $\mathcal{E}$. In characteristic 0, we can smooth the nodal rational curve if we have a general pair p, q and if $\tilde{r}$ is smooth. Therefore, $\mathcal{E}$ is rationally connected.

Fix a section s of $i^*\mathcal{E} \to \bar{V}$. Then we get a section $\tilde{s}$ of $\mathcal{E} \times_{\bar{V}} i^*\mathcal{E} \to \mathcal{E}$. Since we have a finite morphism $\mathcal{E} \times_{\bar{V}} i^*\mathcal{E} \to X$, we get a finite morphism $\mathcal{E} \to X$.

The image of a finite morphism from a rationally connected variety to a hyper-Kähler manifold is of dimension at most $\frac{1}{2}dim(X)$, because we have a $(2,0)$ form on X and there are no holomorphic $(2,0)$ or $(1,0)$ forms on a rational variety. However, $dim(\mathcal{E}) = 3$ and it is bigger than $\frac{1}{2}dim(X) = 2$ - a contradiction.

We ruled out the first case and the only remaining case is:

Case 2: $(deg(D), deg(G)) = (2, 24)$

Then $\bar{V} \cong \mathbb{P}^1 \times \mathbb{P}^1$.

Take a tangent line L to the conic and pull it back to $\bar{V}$: $L_{\bar{V}} \doteq L \times_{\mathbb{P}^2} \bar{V}$. Its normalization $\tilde{L}_{\bar{V}}$ is reducible and consists of two copies of $\mathbb{P}^1$, say $\tilde{L}_{\bar{V},1}$ and $\tilde{L}_{\bar{V},2}$.

Case 2.1:
$\mathcal{E}|_{\tilde{L}_{\bar{V},k}}$ is an elliptic fibration over $\mathbb{P}^1$ with 12 nodal fibers, $k = 1, 2$. Then we repeat the same argument as in Case 1 in order to exclude this case.

Case 2.2:
$\mathcal{E}|_{\tilde{L}_{\bar{V},1}}$ is an elliptic fibration over $\mathbb{P}^1$ with 24 nodal fibers and $\mathcal{E}|_{\tilde{L}_{\bar{V},2}}$ is an elliptic fibration over $\mathbb{P}^1$ with no singular fibers. Then $\mathcal{E}|_{\tilde{L}_{\bar{V},2}}$ is the trivial fibration and therefore, $\mathcal{E}$ is the pull back of an elliptic fibration on $\mathbb{P}^1$ through the projection on this factor. And, since the elliptic fibration on $\mathbb{P}^1$ has 24 nodal fibers, it is an elliptic K3 surface $S \to \mathbb{P}^1$.

After considering all the cases, we see that:

$$\mathcal{E} \times_{(\mathbb{P}^1 \times \mathbb{P}^1)} i^*\mathcal{E} = (S \times \mathbb{P}^1) \times_{(\mathbb{P}^1 \times \mathbb{P}^1)} (\mathbb{P}^1 \times S).$$

We want to prove that $(S \times \mathbb{P}^1) \times_{(\mathbb{P}^1 \times \mathbb{P}^1)} (\mathbb{P}^1 \times S) \cong S \times S$. Indeed, we have the following commutative fiber diagram:

$$\begin{array}{ccc} S \times S & \longrightarrow & \mathbb{P}^1 \times S \\ \downarrow & \searrow & \downarrow \\ S \times \mathbb{P}^1 & \longrightarrow & \mathbb{P}^1 \times \mathbb{P}^1 \end{array}$$

Therefore,

$$\mathcal{E} \times_{(\mathbb{P}^1 \times \mathbb{P}^1)} i^*\mathcal{E} = (S \times \mathbb{P}^1) \times_{(\mathbb{P}^1 \times \mathbb{P}^1)} (\mathbb{P}^1 \times S) \cong S \times S.$$

But X is birational to the desingularization of $\mathcal{E} \times_{(\mathbb{P}^1 \times \mathbb{P}^1)} i^*\mathcal{E}/\tilde{i}$ (by (5.1)) which is deformation equivalent to $S \times S/\mathbb{Z}_2$. Therefore, X is deformation equivalent to the desingularization of $S \times S/\mathbb{Z}_2$ which is $Hilb^2(S)$ by Fogarty's theorem (see Chapter 2, Section 2.1). With this we finish the proof of our main theorem.

$\square$

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