MAT536 HW9

Due 12/8 in class. Each problem is worth 10 points

Problem 1. (a) Let X be an affine variety, M a $k[X]$-module, and $\mathcal{F}$ and $\mathcal{O}_X$-module. Show that $Hom_{k[X]}(M, \Gamma(X, \mathcal{F})) \cong Hom_{\mathcal{O}_X}(\tilde{M}, \mathcal{F})$;

(b) If X is an affine variety and M and N are $k[X]$-modules, then $\widetilde{M \otimes_{k[X]} N} = \tilde{M} \otimes_{\mathcal{O}_X} \tilde{N}$;

(c) If $f \in k[X]$, then $\tilde{M}|_{D(f)} = \widetilde{M_f}$;

(d) If $f : X \to Y$ is a morphism of varieties and $\mathcal{G}$ is a (quasi-)coherent $\mathcal{O}_Y$-module, then $f^*\mathcal{G}$ is a (quasi-)coherent $\mathcal{O}_X$-module.

Problem 2. (a) Let X be a ringed space, $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{O}_X$-modules. Prove that the assignment: $U \to Hom_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U)$ defines an $\mathcal{O}_X$-module. It is denoted by $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$.

(b) Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Show that $\mathcal{L}^{-1} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$ is also invertible and that $\mathcal{L}^{-1} \otimes_{\mathcal{O}_X} \mathcal{L} \cong \mathcal{O}_X$.

Problem 3. Let X be a scheme of characterstic $p > 0$, $F : X \to X$ the Frobenius morphism, and $\mathcal{L}$ an invertible $\mathcal{O}_X$-module. Show that $F^*\mathcal{L} \cong \mathcal{L}^{\otimes p}$.

Problem 4. A morphism $f : X \to Y$ of varieties is called *affine* if for every open affine set $V \subset Y$, the inverse image $f^{-1}(V)$ is also affine. f is called *finite* if it is affine and $k[f^{-1}(V)]$ is a finitely generated $k[V]$-module for all open affine $V \subset Y$. Let $Y = \bigcup V_i$ be an open affine covering of Y such that $f^{-1}(V_i)$ is affine for every i. Show that f is affine. If, moreover, $k[f^{-1}(V_i)]$ is a finitely generated $k[V_i]$-module for all i then f is finite.

Problem 5. (a) Let X be a projective variety and $f : X \to Y = Spec - m(k)$ the unique morphism to a point. Show that $f^* : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is an isomorphism.

(b) Find a projective variety X and a birational morphism $f : X \to Y$ such that $f_*\mathcal{O}_X$ is not locally free on Y.