FINITENESS FOR HYPERKÄHLER MANIFOLDS OF COMPLEX DIMENSION 4

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ABSTRACT. In this paper we prove that there at most finitely many deformation types of hyperKähler manifolds of complex dimension 4 for a given topological structure.

1. PRELIMINARIES

Compact hyperkähler manifolds are higher-dimensional analogues of K3 surfaces. One of the most importaint problems in the theory is: What are the possible deformation (or diffeomorphism) types of irreducible symplectic manifolds? Once the topological type of X is fixed, we want to know how many deformation types of hyperKähler metrics g, or equivalently, of irreducible holomorphic complex structures, do exist on X. One can try to prove that there are finitely many deformation types of hyperKähler manifolds. We do this for complex dimension 4.

Our proof relies on the finiteness results from Hyubrechts [7]. We follow some of the arguments in [2] and [4].

First we will give some definition and state the results which we will use. For more information about properties of hyperKähler manifolds, we refer the reader to [6].

Definition 1.1. *A complex manifold X is called irreducible symplectic (or hyperKähler) if i) X is compact and Kähler, ii) X is simply connected, and iii) $H^0(X, \Omega_X^2)$ is spanned by an everywhere non-degenerate two-form σ.*

From here on we consider a hyperKähler manifold X of complex dimension $2n$.

In [1] Beauville endows $H^2(X, \mathbb{Z})$ with a natural non-degenerate quadratic form q_X, which is primitive integral form of index $(3, b_2(X) - 3)$.

The first Pontrjagin class $p_1(X) \in H^4(X, \mathbb{Z})$ defines a homogeneous polynomial of degree $2n - 2$ on $H^2(X, \mathbb{Z})$ by $\alpha \to \int_X \alpha^{2n-2} p_1(X)$. In [7] Huybrechts applies a finiteness result of Kollár and Matsusaka to hyperKähler manifolds and obtains the following:

Theorem 1.1 (Huybrechts, [7]). *If the second integral cohomology H^2 and the homogeneous polynomial of degree $2n - 2$ defined by the first Pontrjagin class are given, then there exist at most finitely many deformation types of compact hyperKähler manifolds of dimension $4n$ realizing this structure.*

Let C be a polynomial in the Chern classes of degree $4r$. Fujiki [3] defines the following invariant:

$$N(C) = \int_X C u^{2n-2r} / (\int_X u^{2n})^{\frac{n-r}{n}},$$

which is independent of $u \in H^2$ with $\int_X u^{2n} \neq 0$.

Theorem 1.2 (Hitchin & Sawon, [5]). *We have the following formula for $N(c_2)$:*

$$\frac{((2n)!)^{n-1} N(c_2)^n}{(24n(2n-2)!)^n} = \sqrt{\hat{A}}[X].$$

By $\sqrt{\hat{A}}[X]$ we mean the multiplicative sequence of Pontrjagin classes defined by the power series

$$\left(\frac{\sqrt{z}/2}{\sinh \sqrt{z}/2} \right)^{1/2}.$$

As an observation we would like to mension the following:

Remark 1.1. *There are no hyperKählers with $b_2 = 3$ for any dimension.*

Proof. Let X be a hyperKähler manifold. Assume that $dim_{\mathbb{C}} X = 2n$ and $b_2 = 3$. Take $A \subset H^*(X, \mathbb{Z})$ to be the algebra generated by $H^2(X, \mathbb{Z})$. Then, according to [9], $A = Sym^* H^2(X, \mathbb{Z})/I$, where I is the ideal generated by $\{v^{n+1}/q_X(v) = 0, v \in H^2(X, \mathbb{Z})\}$.

But q_X is of type $(3, b_2 - 3) = (3, 0)$, so it is possitive definite, hence $I = 0$, so A is infinite dimensional. We obtain a contradiction, because the cohomology ring is finite dimensional. □

2. Main theorem

Here we prove the following:

Theorem 2.1. *Let X be a fixed compact topological manifold of complex dimension 4. Then there are at most finitely many deformation types of hyperKähler structures on X.*

Proof. We have to prove that there are finitely many choices for c_2, or equivalently, for the polynomial defined by the first Ponrjagin number p_1, and that there are finitely many possibilities for the lattice $(H^2(X, \mathbb{Z}), q_X)$. Then by Theorem 1.1 the statement will follow.

Step I: c_2^2 is bounded.

Guan in [4] has obtained bounds for b_2, b_3 :

$$3 \leq b_2 \leq 23, 0 \leq b_3 \leq 68.$$

From [5] we obtain:

$$c_2^2 = \frac{720\hat{A}_2 + c_4}{3} = 736 + 4b_2 - b_3,$$

so it is bounded.

Step II: $\det q_X$ is bounded.

We will follow some arguments from [2] and [4]. Let $Q = q_X^{-1} \in Sym^2 H^2(X, \mathbb{Q}) =: H^{(4)}$ be the dual of the Beauville's form. We use the same notation for the corresponding element in $H^4(X, \mathbb{Q})$.

Take an orthonormal basis $\{e_i\}$ of $H^2(M, \mathbb{C})$ for the quadratic form q_X over the complex numbers. Then $Q = \sum_1^{b_2} e_i^2$. By [3] we have that $\int_X u^4 = cq_X(u)^2$, where c is a constant depending only on the topological type of X. Since $q_X(e_i) = 1$, we have that for $i \neq j$, $\int_X (e_i + e_j)^4 = 4c = \int_X (e_i - e_j)^4$ and since $\int_X e_i^4 = c$, it follows that $\int_X e_i^2 e_j^2 = c/3$. Thus

$$\int_X Q^2 = \int_X (\sum e_i^2)^2 = b_2(b_2 - 1)\frac{c}{3} + b_2 c = \frac{b_2(b_2 + 2)}{3} c.$$

So, since the cup-product and the constant c depend only on the topological type of X and b_2 is bounded, then $\int_X Q^2$ is bounded, hence $\int_X q_X^2$ and $\det q_X$ are bounded.

Step III: There are finitely many possibilities for the lattice $(H^2(X, \mathbb{Z}), q_X)$.

First notice that $\text{rank} H^2 = b_2 \leq 23$ and $\det q_X$ is bounded.

One can easily see that given integers r and d, there are only finitely many distinct indefinite bilinear form spaces over $\mathbb{Z}$ with rank r and determinant d. It follows from Minkovski's theorem ([8]).

Step IV: Since c_2^2 is bounded and there are finitely many possibilities for the lattice $H^2(X, \mathbb{Z})$, there are finitely many choices for c_2 (and p_1) also.

Indeed, since c_2 is a Pontrjagin class, then its projection in $H^{(4)}$ is a multiple of Q and the orthogonal complement to $H^{(4)}$ in $H^4(X)$ consists of primitive forms ([2]).

We have $c_2 = \lambda Q + \beta_1 e_1 + \ldots \beta_s e_s$, where the cup-product is possitive on $\{e_1, \ldots, e_s\}$, where $\{e_1, \ldots, e_s\}$ is a basis for $H^{(4)\perp} \cap H^{(2,2)}$. Also we have $Q^2 > 0$, so we have finitely many choices for c_2.

With this the proof of the theorem is completed.

□

It remains an open question whether there are finitely many deformation types of hyperKähler manifolds for a fixed dimension. A next step for understanding 4−dimensional hyperKählers is to classify their possible deformation types.

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