

Survey of finiteness results for hyperkähler manifolds

Ljudmila Kamenova ¹

Abstract

This paper is a survey of finiteness results in hyperkähler geometry. We review some classical theorems by Sullivan, Kollár-Matsusaka, Huybrechts, as well as theorems in the recent literature by Charles, Sawon, and joint results of the author with Verbitsky. We also strengthen a finiteness theorem of the author. These are extended notes of the author's talk during the closing conference of the Simons Semester in the Banach Center in Będlewo, Poland.

Contents

1	Introduction	1
2	Basic results in hyperkähler geometry	3
3	Some classical results	5
4	Some recent results	7

1 Introduction

In each given complex dimension $2n$ there are only a few known examples of compact hyperkähler manifolds up to deformation, namely, the Hilbert scheme of n points on a K3 surface S , the generalized Kummer variety $K^n(A)$ and the exceptional examples O_6 (for $n = 3$) and O_{10} (for $n = 5$) given by O'Grady. A natural question to ask is if there are only finitely many compact hyperkähler manifolds in any given dimension, up to deformation. This paper surveys some of the known results in this direction.

Based on Kollár-Matsusaka's finiteness theorem, Daniel Huybrechts proved the following theorem in [Hu3].

Theorem 1.1: (Huybrechts, [Hu3]) If the second integral cohomology group $H^2(\mathbb{Z})$ and the homogeneous polynomial of degree $2n - 2$ on $H^2(\mathbb{Z})$ defined by the first Pontryagin class are given, then there exist at most finitely many diffeomorphism types of compact hyperkähler manifolds of complex dimension $2n$ realizing this structure.

Huybrechts [Hu3] also showed that fixing the Beauville-Bogomolov-Fujiki form (i.e., giving the abelian group $H^2(\mathbb{Z})$ a ring structure) is equivalent to fixing the the first Pontryagin class.

¹This work was partially supported by grant 346300 for IMPAN from the Simons Foundation and the matching 2015-2019 Polish MNiSW fund

If the diffeomorphic structure on a manifold M is given, then there are also finitely many deformation types of hyperkähler metrics on M .

Theorem 1.2: (Huybrechts, [Hu3]) Let M be a fixed compact manifold. Then there exist at most finitely many different deformation types of irreducible holomorphic symplectic complex structures on M .

Using Theorem 1.2, the author and Misha Verbitsky established the following finiteness results in [KV] about hyperkähler fibrations.

Theorem 1.3: (Kamenova-Verbitsky, [KV]) Let M be a fixed compact manifold of complex dimension $2n$ and $b_2(M) \geq 7$. Then there are only finitely many deformation types of hyperkähler Lagrangian fibrations $M \rightarrow \mathbb{C}P^n$.

François Charles has the following boundedness result for families of hyperkähler varieties up to deformation. This sharpens Kollár-Matsusaka's finiteness result for hyperkähler manifolds, because he replaces the assumption that L is ample with the weaker assumption $q(L) > 0$, but one must still bound the top self-intersection of L .

Theorem 1.4: (Charles, [Ch]) Let n and r be two positive integers. Then there exists a scheme S of finite type over \mathbb{C} , and a projective morphism $\mathcal{M} \rightarrow S$ such that if M is a complex hyperkähler variety of dimension $2n$ and L is a line bundle on M with $L^{2n} = r$ and $q(L) > 0$, where q is the Beauville-Bogomolov form, then there exists a complex point s of S such that \mathcal{M}_s is birational to M .

For a hyperkähler manifold M , the Fujiki constant and the discriminant of the Beauville-Bogomolov-Fujiki form q are topological invariants. If we fix them instead of fixing the whole intersection form q , we can ask for finiteness given much less data than in Theorem 1.1. Here we establish the following result, which generalizes our theorem in [K] previously stated for Lagrangian fibrations.

Theorem 1.5: There are at most finitely many deformation classes of hyperkähler manifolds with a fixed Fujiki constant c and a given discriminant of the Beauville-Bogomolov-Fujiki lattice (Λ, q) .

In our original proof we used a primitive vector v with $q(v) = 0$ coming from the line bundle L associated to the Lagrangian fibration. However, we do not necessarily need v to come from a Lagrangian fibration.

In [Saw2], Sawon proved a finiteness result for Lagrangian fibrations with several assumptions on the fibration. We'll formulate Sawon's theorem in section 4. In [K] we gave the following generalization of his result.

Theorem 1.6: (Kamenova, [K]) Consider a Lagrangian fibration $\pi : M \rightarrow \mathbb{C}P^n$ such that there is a line bundle P on M with $q(P) > 0$ and with a given P -degree d on the general fiber F of π , i.e., $P^n \cdot F = d$. Then there are at most finitely many deformation classes of hyperkähler manifolds M as above, i.e., they form a bounded family.

2 Basic results in hyperkähler geometry

Definition 2.1: A **hyperkähler manifold** is a compact simply connected Kähler holomorphic symplectic manifold. A hyperkähler manifold M is called **irreducible** if $H^{2,0}(M) = \mathbb{C}$.

According to Bogomolov's decomposition theorem, [Bo], any hyperkähler manifold admits a finite covering which is a product of finitely many irreducible hyperkähler manifolds. From now on we shall assume that all hyperkähler manifolds are irreducible.

Remark 2.2: In the compact case the following two notions are equivalent: a holomorphic symplectic Kähler manifold and a manifold with a *hyperkähler structure*, that is, a triple of complex structures satisfying the quaternionic relations and parallel with respect to the Levi-Civita connection. This equivalence follows from Yau's solution of Calabi's conjecture ([Bes]). Throughout this paper we assume compactness and we use the complex algebraic point of view.

Definition 2.3: Let M be a compact complex manifold and $\text{Diff}^0(M)$ the connected component of the identity of its diffeomorphism group. Denote by Comp the space of complex structures on M , equipped with a structure of Fréchet manifold. The *Teichmüller space* of M is the quotient $\text{Teich} := \text{Comp} / \text{Diff}^0(M)$. For a hyperkähler manifold M , the Teichmüller space is finite-dimensional ([Cat]). Let $\text{Diff}^+(M)$ be the group of orientable diffeomorphisms of a complex manifold M . The *mapping class group*

$$\Gamma := \text{Diff}^+(M) / \text{Diff}^0(M)$$

acts naturally on Teich . For $I \in \text{Teich}$, let Γ_I be the subgroup of Γ which fixes the connected component Teich_I of the complex structure I . The *monodromy group* is the image of Γ_I in $\text{Aut } H^2(M, \mathbb{Z})$.

On $H^2(M, \mathbb{Z})$ there is a natural primitive integral quadratic form, called the **Beauville-Bogomolov-Fujiki form**, or **BBF form** for shortness. The easiest way to define it is via the Fujiki relation below. For the classical definition we refer the reader to [Bea] and [Hu2].

Theorem 2.4: (Fujiki, [F]) Let $\eta \in H^2(M, \mathbb{Z})$ and $\dim M = 2n$, where M is a hyperkähler manifold. Then $\int_M \eta^{2n} = c \cdot q(\eta, \eta)^n$, for a primitive integral

quadratic form q on $H^2(M, \mathbb{Z})$, where $c > 0$ is a constant depending on the topological type of M . The constant c in Fujiki's formula is called the **Fujiki constant**. ■

Remark 2.5: The form q has signature $(3, b_2 - 3)$. It is negative definite on primitive forms and positive definite on the space $\langle \Omega, \bar{\Omega}, \omega \rangle$, where Ω is the holomorphic symplectic form and ω is a Kähler form (see [V1], Theorem 6.1 and [Hu2], Corollary 23.9).

Definition 2.6: Let $\eta \in H^{1,1}(M)$ be a real $(1,1)$ -class on a hyperkähler manifold M . We say that η is **parabolic** if $q(\eta, \eta) = 0$. A line bundle L is called **parabolic** if the class $c_1(L)$ is parabolic.

Remark 2.7: If L is a parabolic class and $P \in H^2(M)$ is any class, then after we substitute $\eta = P + tL$ into Fujiki's formula in Theorem 2.4, and compare the coefficients of t^n on both sides, we obtain $\binom{2n}{n} P^n L^n = c 2^n q(P, L)^n$.

Notice that hyperkähler manifolds have a very restricted fibration structure.

Theorem 2.8: (Matsushita, [Ma1]). Let $\pi : M \rightarrow B$ be a surjective holomorphic map with connected fibers from a hyperkähler manifold M to a base B , with $0 < \dim B < \dim M$. Then $\dim B = \frac{1}{2} \dim M$, and the fibers of π are holomorphic Lagrangian (i.e., the symplectic form vanishes when restricted to the fibers).

Such a map is called a **holomorphic Lagrangian fibration**.

Remark 2.9: D. Matsushita ([Ma2]) proved that if the base of π is smooth and M is projective, B has the same rational cohomology as $\mathbb{C}P^n$. Later J.-M. Hwang ([Hw]) proved that under the same assumptions $B \cong \mathbb{C}P^n$.

Definition 2.10: A line bundle L is called **semiample** if L^N is generated by its holomorphic sections which have no common zeros.

Remark 2.11: From semiamplicity it trivially follows that L is nef, however a nef bundle is not necessarily semiample. Let $\pi : M \rightarrow B$ be a holomorphic Lagrangian fibration and let ω_B be a Kähler class on B . Then $\eta := \pi^* \omega_B$ is semiample and parabolic. By Matsushita's theorem, the converse is also true: if L is semiample and parabolic, L induces a Lagrangian fibration.

Conjecture 2.12: (Hyperkähler SYZ conjecture) Let L be a parabolic nef line bundle on a hyperkähler manifold. Then L is semiample.

Remark 2.13: The SYZ conjecture can be seen as a hyperkähler version of the "abundance conjecture" (see e.g. [DPS], 2.7.2). It was stated by many authors such as Tyurin, Bogomolov, Hassett-Tschinkel, Huybrechts, Sawon, Verbitsky,

etc. For more details on the SYZ conjecture one might look into [Saw1] and [V2].

Remark 2.14: As a corollary to the SYZ conjecture we see that any hyperkähler manifold M with $b_2(M) \geq 5$ can be deformed to one that admits a Lagrangian fibration. This is one of the reasons to study Lagrangian fibrations on hyperkähler manifolds. Indeed, if $b_2(M) \geq 5$, by Meyer’s theorem there is an isotropic vector v . One can deform M to a hyperkähler manifold M' whose Picard group is generated by a primitive line bundle L corresponding to v , i.e., $q(c_1(L)) = 0$. The positive cone of M' coincides with the Kähler cone, and therefore, either L or L^* is nef. By the SYZ conjecture, M' admits a Lagrangian fibration. This is the argument of Proposition 4.3 in Sawon’s paper [Saw1].

Relatively recently Matsushita conjectured that the fibration structure of a hyperkähler manifold is even more restricted than the structure given by Matsushita’s theorem. The following conjecture was introduced to the author in private communications with J. Sawon, D. Matsushita and J.-M. Hwang in 2013/14.

Conjecture 2.15: (Matsushita’s conjecture) Every holomorphic Lagrangian fibration $\pi : M \rightarrow \mathbb{C}P^n$ is either locally isotrivial or the fibers vary maximally in the moduli space of Abelian varieties \mathcal{A}_n .

B. van Geemen and C. Voisin recently proved the following weaker version of Matsushita’s conjecture. Here ρ is the rank of the Picard group.

Theorem 2.16: (B. van Geemen, C. Voisin, [vGV]) Let X be a projective hyperkähler manifold of dimension $2n$ admitting a Lagrangian fibration $f : X \rightarrow B$. Assume $b_{2,tr}(X) = b_2(X) - \rho(X) \geq 5$. Then a very general deformation (X', f', B') of the triple (X, f, B) satisfies Matsushita’s conjecture.

3 Some classical results

There is a general classical result of Dennis Sullivan about finiteness of simply connected Kähler manifolds. It follows from the \mathbb{Q} -formality of Kähler manifolds combined with the description of the rational homotopy type of Kähler manifolds in terms of a differential graded algebra. This algebra encodes the cohomology with its ring structure as a primary invariant.

Theorem 3.1: (Sullivan, [Sull]) The diffeomorphism type of a simply connected Kähler manifold (of complex dimension greater than 2) is finitely determined by its integral cohomology ring and its Pontryagin classes.

Daniel Huybrechts noticed that for a hyperkähler manifold M much less of the topology needs to be fixed in order to determine the diffeomorphism type

up to finite ambiguity. The first Pontryagin class $p_1(M) \in H^4(M, \mathbb{Z})$ gives rise to a homogeneous polynomial $\tilde{p}_1 : H^2(M, \mathbb{Z}) \rightarrow \mathbb{Z}$ of degree $2n - 2$.

Theorem 3.2: (Huybrechts, [Hu3]) If the second integral cohomology group $H^2(\mathbb{Z})$ and the homogeneous polynomial of degree $2n - 2$ on $H^2(\mathbb{Z})$ defined by the first Pontryagin class are given, then there exist at most finitely many diffeomorphism types of compact hyperkähler manifolds of complex dimension $2n$ realizing this structure.

Here Huybrechts fixes $H^2(\mathbb{Z})$ as an abelian group. The first Pontryagin class, or equivalently, the Beauville-Bogomolov-Fujiki form, give the ring structure of the second cohomology $H^2(\mathbb{Z})$. The proof of this theorem uses Kollár-Matsusaka's finiteness result:

Theorem 3.3: (Kollár-Matsusaka, [KM]) There are finitely many deformation types of projective manifolds M of dimension d that admit an ample line bundle L with fixed intersection numbers L^d and $K_M \cdot L^{d-1} \in \mathbb{Z}$.

In particular, for Calabi-Yau manifolds M of dimension d it is enough to fix the top intersection L^d . In order to remove the fixed polarization, Huybrechts shows that any hyperkähler manifold with fixed $H^2(\mathbb{Z})$ and \tilde{p}_1 can be deformed to one that admits a polarization L with bounded L^d , using Hitchin-Sawon's formulas in [HS]. Here is a version of Huybrechts's finiteness theorem where the BBF form is given in place of \tilde{p}_1 .

Theorem 3.4: (Huybrechts, [Hu3]) There are only finitely many deformation types of hyperkähler manifolds M of fixed dimension such that the lattice $(H^2(M, \mathbb{Z}), q_M)$ is isomorphic to a given one.

Once the diffeomorphism type of M is fixed, Huybrechts also shows that there are finitely many deformation types of hyperkähler metrics g on M .

Theorem 3.5: (Huybrechts, [Hu3]) Let M be a fixed compact manifolds. Then there exist at most finitely many deformation types of hyperkähler structures on M .

The idea of the proof is the following. Let $\Lambda = (H^2(M, \mathbb{Z}), q)$ and \mathfrak{M}_Λ be the coarse moduli space of marked hyperkähler manifolds (M, φ) . Fix a primitive positive element $v \in H^2(M, \mathbb{Z})$, i.e., $q(v) > 0$. An isomorphism, or a marking, $\varphi : (H^2(M, \mathbb{Z}), q) \rightarrow \Lambda$ determines a point $(M, \varphi) \in \mathfrak{M}_\Lambda$. By Huybrechts's proectivity criterion [Hu1], since $q(v) > 0$, M is deformation equivalent to a projective manifold M' with an ample line bundle L corresponding to v . By Fujiki's formula in Theorem 2.4, $L^{2n} = c \cdot q(L)^n = c \cdot q(v)^n$. Fujiki's constant c depends only on the topological type of M , which is fixed. Since v is also fixed, $q(v)$ is determined, and therefore the top intersection L^{2n} is known and one applies Kollár-Matsusaka's Theorem 3.3.

4 Some recent results

Together with Misha Verbitsky in [KV] we were interested in studying finiteness questions about hyperkähler manifolds that admit Lagrangian fibrations. As we pointed out in Remark 2.14, if the SYZ conjecture holds, then any hyperkähler manifold (with $b_2 \geq 5$) can be deformed to one that admits a Lagrangian fibration, and therefore it would be enough to study Lagrangian fibrations for questions concerning deformation classes of hyperkähler varieties. By Huybrechts's Theorem 3.5, for a fixed compact manifold there are at most finitely many deformation types of hyperkähler structures on it. We proved that for each hyperkähler structure there are only finitely many ways in which it fibers over a smooth base.

Theorem 4.1: (Kamenova-Verbitsky, [KV]) Let M be a fixed compact manifold of complex dimension $2n$ with $b_2(M) \geq 7$. Then there are only finitely many deformation types of hyperkähler Lagrangian fibrations $p : M \rightarrow \mathbb{C}P^n$.

Let Teich be the Teichmüller space of M and $\Gamma_I < \Gamma$ as in Definition 2.3. If M admits a Lagrangian fibration, there is a natural nef line bundle $L = p^*\mathcal{O}(1)$ associated to the fibration. Consider the set

$$\text{Teich}_L = \{I \in \text{Teich} \mid L \in H^{1,1}((M, I), \mathbb{Z})\},$$

where L remains of type $(1, 1)$ on deformations of the complex structure. In [KV] we proved that there are finitely many orbits of the action of Γ_I on Teich_L using lattice theory methods and Nikulin-style techniques applied to the BBF form. If we take a general deformation of a Lagrangian fibration, the Picard group would be 1-dimensional and generated by the nef line bundle L . For each such pair (M, L) we prove that there is a unique deformation type of a fibration structure. Since there are finitely many orbits of Γ_I acting on Teich_L we conclude finiteness of the deformation types of Lagrangian fibrations.

Recently, François Charles sharpened Kollár-Matsusaka's theorem for hyperkähler varieties by replacing the ampleness assumption with a weaker one. Our results, Theorem 4.4 and Theorem 4.9, rely on his theorem.

Theorem 4.2: (Charles, [Ch]) Let n and r be two positive integers. Then there exists a scheme S of finite type over \mathbb{C} , and a projective morphism $\mathcal{M} \rightarrow S$ such that if M is a complex hyperkähler variety of dimension $2n$ and L is a line bundle on M with $c_1(L)^{2n} = r$ and $q(L) > 0$, where q is the Beauville-Bogomolov form, then there exists a complex point s of S such that \mathcal{M}_s is birational to M .

Consider a lattice Λ , i.e., a free \mathbb{Z} -module of finite rank together with a non-degenerate symmetric bilinear form q with values in \mathbb{Z} . If $\{e_i\}$ is a basis of Λ , the *discriminant* of Λ is $\text{discr}(\Lambda) = \det(q(e_i, e_j))$. Let us recall the following lemma from [K].

Lemma 4.3: Let (Λ, q) be an indefinite lattice and $v \in \Lambda$ be an isotropic non-zero vector. Then there exists a positive vector $w \in \Lambda$ such that $0 < q(w, v) \leq |\text{discr}(\Lambda)|$ and $0 < q(w, w) \leq 2|\text{discr}(\Lambda)|$.

Proof: Let w_0 be a vector with minimal positive intersection $q(w_0, v)$. Then by Lemma 3.7. in [KV], $q(w_0, v)$ divides $N = |\text{discr}(\Lambda)|$ (indeed, since v is primitive, we can complete $v_1 = v$ to a basis $\{v_1, \dots, v_r\}$ of Λ , and then $q(w_0, v)\mathbb{Z}$ is an ideal generated by $\{q(v, v_i)\}$, therefore the first column of the matrix $(q(v_j, v_i))$ is divisible by $q(w_0, v)$). Therefore, $0 < q(w_0, v) \leq N$. Let α be the smallest integer such that $q(w_0 + \alpha v, w_0 + \alpha v) > 0$. Since $q(v, v) = 0$, we have $q(w_0 + \alpha v, w_0 + \alpha v) = q(w_0, w_0) + 2\alpha q(w_0, v)$. Then $w = w_0 + \alpha v$ is a positive vector with $0 < q(w, v) = q(w_0, v) \leq N$. Notice that automatically $0 < q(w, w) = q(w_0 + \alpha v, w_0 + \alpha v) = q(w_0, w_0) + 2\alpha q(w_0, v) \leq 2N = 2|\text{discr}(\Lambda)|$. ■

One of our main results in [K] was the following finiteness theorem with the assumption of existence of a Lagrangian fibration. Here we have dropped that assumption and we obtain a more general result.

Theorem 4.4: There are at most finitely many deformation classes of hyperkähler manifolds M of dimension $2n$ with a fixed Fujiki constant c and a given discriminant of the Beauville-Bogomolov-Fujiki lattice $(H^2(X, \mathbb{Z}), q)$.

Proof: If $b_2(M) = 3$ or 4 , there are finitely many lattices of this rank with fixed discriminant (see Theorem 1.1, chapter 9 of [Cas]). For each one of these finitely many lattices the positive vector w with minimal positive intersection $q(w, w)$ has bounded self-intersection and we apply Charles's Theorem 4.2 directly.

Now assume that $b_2 \geq 5$. By Meyer's theorem on indefinite lattices of rank at least 5, there exists a non-trivial isotropic vector v ([Cas]). We apply Lemma 4.3 for $(\Lambda, q) = (H^2(X, \mathbb{Z}), q)$ and $v = v$. Then there exists a positive vector w such that $0 < q(w, w) \leq 2|\text{discr}(\Lambda)|$. By Fujiki's formula,

$$0 < w^{2n} = c \cdot q(w, w)^n \leq c \cdot (2|\text{discr}(\Lambda)|)^n,$$

i.e., w^{2n} is bounded by the given invariants. Deform M to a hyperkähler manifold M' with a line bundle L corresponding to the class w . For each top intersection $c_1(L)^{2n}$ in the interval $(0, (2|\text{discr}(\Lambda)|)^n]$, we obtain only finitely many deformation classes of M by Charles's Theorem 4.2. ■

Since the families of hyperkähler manifolds above form a bounded family, there are only finitely many choices of the second Betti number. The lattice $(H^2(M, \mathbb{Z}), q)$ encodes substantial information for hyperkähler manifolds and $b_2(M)$ is an important invariant, therefore we list separately the following direct corollary.

Corollary 4.5: In the assumptions of Theorem 4.4, i.e., given numbers $n \in \mathbb{Z}_+$, $c \in \mathbb{Q}$ and discriminant $d \in \mathbb{Z}$, the second Betti number $b_2(M)$ is bounded.

We would like to mention also the following theorem in the recent literature.

Theorem 4.6: (Sawon, [Saw2]) Fix positive integers n and d_1, \dots, d_n , with $d_1 | d_2 | \dots | d_n$. Consider Lagrangian fibrations $\pi : M \rightarrow \mathbb{C}P^n$ that satisfy:

- (1) $\pi : M \rightarrow \mathbb{C}P^n$ admits a global section;
- (2) there is a very ample line bundle on M which gives a polarization of type (d_1, \dots, d_n) when restricted to a generic smooth fiber M_t ;
- (3) over a generic point t of the discriminant locus the fiber M_t is a rank-one semi-stable degeneration of abelian varieties;
- (4) a neighbourhood U of a generic point $t \in \mathbb{C}P^n$ describes a maximal variation of abelian varieties.

Then there are finitely many such Lagrangian fibrations up to deformation.

Justin Sawon's proof is based on the following observations. The existence of a section implies that there is a distinguished point in each fiber. Together with the given type of a polarization it gives a natural classifying map $\varphi : \mathbb{C}P^n \setminus \Delta \rightarrow \mathcal{A}_{d_1, \dots, d_n}$, where Δ is the discriminant locus of π . Sawon extends φ to $\bar{\varphi} : \mathbb{C}P^n \setminus \Delta_0 \rightarrow \mathcal{A}_{d_1, \dots, d_n}^*$, where $\mathcal{A}_{d_1, \dots, d_n}^*$ is a partial compactification of $\mathcal{A}_{d_1, \dots, d_n}$ and $\Delta_0 \subset \Delta$ is of codimension 2 in $\mathbb{C}P^n$. He chooses an ample line bundle H on $\mathcal{A}_{d_1, \dots, d_n}^*$ and bounds $\deg(\bar{\varphi}^*H)$, which is non-zero by (4). Thus, $\bar{\varphi}$ belongs to finitely many families of rational maps $\mathbb{C}P^n \dashrightarrow \mathcal{A}_{d_1, \dots, d_n}^*$, and finiteness of Lagrangian fibrations as above follows.

Remark 4.7: Notice that as a corollary of Matsushita's conjecture, part (4) of Sawon's Theorem simply excludes locally isotrivial fibrations. We need to apply only the deformational version (see van Geemen-Voisin's Theorem 2.16) of Matsushita's conjecture to Sawon's theorem in order to remove the seemingly restrictive assumption (4).

Remark 4.8: If there is a section $\sigma : \mathbb{C}P^n \rightarrow M$, this means that $\sigma(\mathbb{C}P^n)$ is a Lagrangian subvariety in M . Finding Lagrangian subvarieties in a hyperkähler manifold is itself a very interesting problem (for example, see [HT]).

Using the methods above combined with F. Charles' Theorem 4.2, in [K] we generalized Sawon's Theorem 4.6.

Theorem 4.9: Consider a Lagrangian fibration $\pi : M \rightarrow \mathbb{C}P^n$ such that there is a line bundle P on M with $q(P) > 0$ and with a given P -degree d on the general fiber F of π , i.e., $P^n \cdot F = d$. Then there are at most finitely many deformation classes of hyperkähler manifolds M as above, i.e., they form a bounded family.

Proof: Let $L = \pi^*\mathcal{O}(1)$ be a nef parabolic class ($q(L) = 0$) coming from the Lagrangian fibration. The fundamental class of the general fiber F of π is $[F] = L^n$. By assumption, $P^n \cdot L^n = d$ is fixed. Define $v = L/m$, where $m \in \mathbb{Z}_{>0}$ is the divisibility of L , and therefore v is a primitive class. Since P is in the interior of the of the positive cone \mathcal{C} and v is on the boundary of \mathcal{C} , it follows that $q(P, v) > 0$ (Corollary 7.2 in [BHPV]). Now we shall follow the proof of Lemma 4.3. Let k be the smallest integer such that $q(P + kv) > 0$. Then $q(P + kv) \leq 2q(P, v)$ and by applying Fujiki's formula twice (as in Theorem 2.4 and Remark 2.7), we obtain:

$$(P + kv)^{2n} = c \cdot q(P + kv)^n \leq c 2^n q(P, v)^n = \binom{2n}{n} P^n \cdot v^n =$$

$$\binom{2n}{n} \frac{P^n \cdot L^n}{m^n} = \binom{2n}{n} \frac{d}{m^n} \leq \binom{2n}{n} d.$$

By Charles's Theorem 4.2 there is a bounded family of such M , which implies finiteness of deformations of M . ■

Remark 4.10: In Theorem 4.9 we proved finiteness of deformation classes of the total space M of the Lagrangian fibration. However, in Theorem 4.1 the author together with Misha Verbitsky proved that for a fixed compact manifold M there are only finitely many deformation types of hyperkähler Lagrangian fibrations with total space M provided that $b_2(M) \geq 7$. For all known examples of hyperkähler manifolds one has $b_2(M) \geq 7$ and it is suspected that this is always the case.

Acknowledgments. The author is very grateful to the organizers of the Simons Semester in Poland for their kind invitation and to the Banach Center in Będlewo for their excellent hospitality and conference. She would like to thank François Charles, Misha Verbitsky, Dennis Sullivan, Ulrike Rieß and Michel Brion for interesting conversations and suggestions.

References

- [BHPV] Barth, W., Hulek, K., Peters, C., Van de Ven, A., *Compact complex surfaces. 2nd enlarged ed.*, EMG **3**, Springer, Berlin (2004).
- [Bea] Beauville, A. *Varieties Kähleriennes dont la première classe de Chern est nulle.* J. Diff. Geom. **18** (1983) 755-782.
- [Bes] Besse, A., *Einstein Manifolds*, Springer-Verlag, New York (1987)
- [Bo] Bogomolov, F. A., *On the decomposition of Kähler manifolds with trivial canonical class*, Math. USSR-Sb. **22** (1974) 580 - 583.
- [Cas] Cassels, J. W. S., *Rational quadratic forms*, LMS **13**, Academic Press Inc., London, 1978.
- [Cat] Catanese, F., *A Superficial Working Guide to Deformations and Moduli*, Advanced Lectures in Mathematics, Volume XXVI Handbook of Moduli, Volume III, 161 - 216.

- [Ch] Charles, F., *Birational boundedness for holomorphic symplectic varieties, Zarhin's trick for K3 surfaces, and the Tate conjecture*, arXiv:1407.0592 [math.AG].
- [DPS] Demailly, J.-P., Peternell, T., Schneider, M., *Pseudo-effective line bundles on compact Kähler manifolds*, Int. J. of Math. **6** (2001) 689 - 741.
- [F] Fujiki, A., *On the de Rham Cohomology Group of a Compact Kähler Symplectic Manifold*, Adv. Stud. Pure Math. **10** (1987) 105 - 165.
- [vGV] van Geemen, B., Voisin, C., *On a conjecture of Matsushita*, arXiv:1503.07323 [math.AG].
- [HT] Hassett, B., Tschinkel, Y., *Intersection numbers of extremal rays on holomorphic symplectic varieties*, Asian J. Math. **14** (2010) 303 - 322.
- [HS] Hitchin, N., Sawon, J., *Curvature and characteristic numbers of hyper-Kähler manifolds*, Duke Math. J. **106** (2001) 599 - 615.
- [Hu1] Huybrechts, D., *Compact hyper-Kähler manifolds: basic results*, Invent. Math. **135** (1999) 63 - 113. Erratum: Invent. Math. **152** (2003) 209 - 212.
- [Hu2] Huybrechts, D., *Compact hyperkähler manifolds, Calabi-Yau manifolds and related geometries*, Universitext, Springer-Verlag, Berlin, 2003, Lectures from the Summer School held in Nordfjordeid, June 2001, 161 - 225.
- [Hu3] Huybrechts, D., *Finiteness results for hyperkähler manifolds*, J. Reine Angew. Math. **558** (2003) 15 - 22.
- [Hw] Hwang, J.-M., *Base manifolds for fibrations of projective irreducible symplectic manifolds*, Invent. Math. **174** (2008) 625 - 644.
- [K] Kamenova, L., *Finiteness of Lagrangian fibrations with fixed invariants*, C. R. Math. Acad. Sci. Paris **354** (2016) 707 - 711.
- [KV] Kamenova, L., Verbitsky, M., *Families of Lagrangian fibrations on hyperkaehler manifolds*, Adv. Math. **260** (2014) 401 - 413.
- [KM] Kollár, J., Matsusaka, T., *Riemann-Roch type inequalities*, Am. J. Math. **105** (1983) 229 - 252.
- [Ma1] Matsushita, D., *On fibre space structures of a projective irreducible symplectic manifold*, Topology **38** (1999) 79 - 83. Addendum, Topology **40** (2001) 431 - 432.
- [Ma2] Matsushita, D., *Higher direct images of dualizing sheaves of Lagrangian fibrations*, Amer. J. Math. **127** (2005) 243 - 259.
- [Saw1] Sawon, J., *Abelian fibred holomorphic symplectic manifolds*, Turkish Jour. Math. **27** (2003) no. 1, 197 - 230, math.AG/0404362.
- [Saw2] Sawon, J., *A finiteness theorem for Lagrangian fibrations*, arXiv:1212.6470 [math.AG].
- [Sull] Sullivan, D., *Infinitesimal computations in topology*, Publ. Math. I.H.E.S. **47** (1977) 269 - 331.
- [V1] Verbitsky, M., *Cohomology of compact hyperkähler manifolds*. Ph.D. Thesis, arXiv:alg-geom/9501001, 87 pages.
- [V2] Verbitsky, M., *Hyperkahler SYZ conjecture and semipositive line bundles*, GAFA **19** (2010) 1481 - 1493.

LJUDMILA KAMENOVA
DEPARTMENT OF MATHEMATICS, 3-115
STONY BROOK UNIVERSITY
STONY BROOK, NY 11794-3651, USA,
kamenova@math.stonybrook.edu