Finiteness of Lagrangian fibrations with fixed invariants

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Abstract
We prove finiteness of the deformation classes of hyperkähler Lagrangian fibrations in any fixed dimension with fixed Fujiki constant and discriminant of the Beauville-Bogomolov-Fujiki lattice. We also prove there are only finitely many deformation classes of hyperkähler Lagrangian fibrations with an ample line bundle of a given degree on the general fiber of the fibration.

Contents

1 Introduction

For a hyperkähler manifold M, the Fujiki constant and the discriminant of the Beauville-Bogomolov-Fujiki lattice are topological invariants. It is very natural to fix them and ask for finiteness of hyperkähler manifolds with these invariants. In this paper we establish finiteness of Lagrangian fibrations of hyperkähler manifolds with fixed topological invariants as above.

Theorem 1.1: There are at most finitely many deformation classes of Lagrangian fibrations $\pi : M \to \mathbb{C}P^n$ with a fixed Fujiki constant c and a given discriminant of the Beauville-Bogomolov-Fujiki lattice (Λ, q).

Francois Charles has the following boundedness result for families of hyperkähler varieties up to deformation. He drops the assumption that L is ample in Kollár-Matsusaka's theorem applied for hyperkähler manifolds and replaces it with the assumption that $q(L) > 0$.

Theorem 1.2: (Charles, [Ch]) Let n and r be two positive integers. Then there exists a scheme S of finite type over $\mathbb{C}$, and a projective morphism $\mathcal{M} \longrightarrow S$ such that if M is a complex hyperkähler variety of dimension $2n$ and L is a line bundle on M with $c_1(L)^{2n} = r$ and $q(L) > 0$, where q is the Beauville-Bogomolov form, then there exists a complex point s of S such that $\mathcal{M}_s$ is birational to M.

In our case, there is a natural line bundle L associated to the Lagrangian fibration. Using Fujiki's formula, it is a straightforward observation that $q(L) = 0$, while F. Charles deals with the case when $q(L) > 0$ (in which case M is projective by a result of D. Huybrechts: Theorem 3.11 in [Hu1]).

In the proof of our main theorems we use F. Charles' finiteness result applied to an ample line bundle with minimal positive square of the Beauville-Bogomolov-Fujiki form. Since we are interested in a finiteness result up to deformation equivalence, one can obtain an ample line bundle after deforming a given Lagrangian fibration to a projective one. We also use lattice theory estimates applied to the Beauville-Bogomolov-Fujiki form.

In [Saw], Sawon proved a finiteness theorem for Lagrangian fibrations with a lot of natural assumptions on the fibration, such as existence of a section, fixed polarization type of a very ample line bundle, semi-simple degenerations as the general singular fibers, and a maximal variation of the fibers. We give the precise statement of Sawon's theorem in 2.4. Due to a very recent progress of B. van Geemen and C. Voisin ([vGV]) towards Matsushita's conjecture, the last condition of Sawon's theorem can be modified to only exclude isotrivial fibrations. Using the techniques in our proofs, one can also drop most of the other conditions in Sawon's theorem. We prove the following generalization.

Theorem 1.3: Consider a Lagrangian fibration $\pi : M \longrightarrow \mathbb{C}P^n$ such that there is a line bundle P on M with $q(P) > 0$ and with a given P-degree d on the general fiber F of π, i.e., $P^n \cdot F = d$. Then there are at most finitely many deformation classes of hyperkähler manifolds M as above, i.e., they form a bounded family.

For completeness of the exposition, we also mention Huybrechts' classical finiteness results.

Theorem 1.4: (Huybrechts, [Hu3]) If the second integral cohomology $H^2(\mathbb{Z})$ and the homogeneous polynomial of degree $2n-2$ on $H^2(Z)$ defined by the first Pontrjagin class are given, then there exist at most finitely many diffeomorphism types of compact hyperkähler manifolds of real dimension $4n$ realizing this structure.

Theorem 1.5: (Huybrechts, [Hu3]) Let M be a fixed compact manifold. Then there exist at most finitely many different deformation types of irreducible holomorphic symplectic complex structures on M.

Using Theorem 1.5, the author and Misha Verbitsky established the following finiteness results in [KV].

Theorem 1.6: (Kamenova-Verbitsky, [KV]) Let M be a fixed compact manifold. Then there are only finitely many deformation types of hyperkähler Lagrangian fibrations $(M, I) \longrightarrow \mathbb{C}P^n$, for all complex structures I on M.

In the main theorem of this paper we prove finiteness of deformation classes of the total space M of the Lagrangian fibration $M \longrightarrow \mathbb{C}P^n$ with fixed dimension, Fujiki constant and discriminant of the Beauville-Bogomolov-Fujiki lattice. As a corollary of Theorem 1.6 one also obtains finiteness of the deformation classes of the Lagrangian fibration $M \longrightarrow \mathbb{C}P^n$.

2 Hyperkähler geometry: preliminary results

2.1 Basic definitions

Definition 2.1: A **hyperkähler manifold** is a compact Kähler holomorphic symplectic manifold.

Definition 2.2: A hyperkähler manifold M is called **simple** if $H^1(M, \mathbb{C}) = 0$ and $H^{2,0}(M) = \mathbb{C}$.

Theorem 2.3: (Bogomolov's Decomposition Theorem, [Bo], [Bes]). Any hyperkähler manifold admits a finite covering, which is a product of a torus and a finite collection of simple hyperkähler manifolds. ■

Remark 2.4: From now on, we assume that all hyperkähler manifolds are simple.

Remark 2.5: The following two notions are equivalent: a holomorphic symplectic Kähler manifold and a manifold with a *hyperkähler structure*, that is, a triple of complex structures satisfying the quaternionic relations and parallel with respect to the Levi-Civita connection. In the compact case the equivalence between these two notions is provided by Yau's solution of Calabi's conjecture ([Bes]). In this paper we assume compactness and we use the complex algebraic point of view.

Definition 2.6: Let M be a compact complex manifold and $\operatorname{Diff}^0(M)$ the connected component of the identity of its diffeomorphism group. Denote by Comp the space of complex structures on M, equipped with a structure of Fréchet manifold. The *Teichmüller space* of M is the quotient $\operatorname{Teich} := \operatorname{Comp} / \operatorname{Diff}^0(M)$. For a hyperkähler manifold M, the Teichmüller space is finite-dimensional ([Cat]). Let $\operatorname{Diff}^+(M)$ be the group of orientable diffeomorphisms of a complex manifold M. The *mapping class group*

$$\Gamma := \operatorname{Diff}^+(M) / \operatorname{Diff}^0(M)$$

acts naturally on Teich. For $I \in$ Teich, let Γ_I be the subgroup of Γ which fixes the connected component of complex structure I. The *monodromy group* is the image of Γ_I in $\operatorname{Aut} H^2(M, \mathbb{Z})$.

2.2 The Beauville-Bogomolov-Fujiki form

Theorem 2.7: (Fujiki, [F]) Let $\eta \in H^2(M)$, and $\dim M = 2n$, where M is hyperkähler. Then $\int_M \eta^{2n} = c \cdot q(\eta, \eta)^n$, for some integral quadratic form q on $H^2(M)$, where $c > 0$ is a constant depending on the topological type of M. The constant c in Fujiki's formula is called the **Fujiki constant**. ■

Definition 2.8: This form is called the **Beauville-Bogomolov-Fujiki form**.

Remark 2.9: The form q has signature $(3, b_2 - 3)$. It is negative definite on primitive forms, and positive definite on the space $\langle \Omega, \overline{\Omega}, \omega \rangle$ where Ω is the holomorphic symplectic form and ω is a Kähler form (see e. g. [V], Theorem 6.1, or [Hu2], Corollary 23.9).

Definition 2.10: Let $[\eta] \in H^{1,1}(M)$ be a real (1,1)-class on a hyperkähler manifold M. We say that $[\eta]$ is **parabolic** if $q([\eta], [\eta]) = 0$. A line bundle L is called **parabolic** if the class $c_1(L)$ is parabolic.

Remark 2.11: If L is a parabolic class and $P \in H^2(M)$ is any class, then after we substitute $\eta = P + tL$ into Fujiki's formula in Theorem 2.7, and compare the coefficients of t^n on both sides, we obtain $\binom{2n}{n} P^n L^n = c 2^n q(P, L)^n$.

2.3 The SYZ conjecture and Matsushita's conjecture

Theorem 2.12: (Matsushita, [Ma1]). Let $\pi : M \to B$ be a surjective holomorphic map from a hyperkähler manifold M to a base B, with $0 < \dim B < \dim M$. Then $\dim B = 1/2 \dim M$, and the fibers of π are holomorphic Lagrangian (this means that the symplectic form vanishes on the fibers).

Definition 2.13: Such a map is called **a holomorphic Lagrangian fibration**.

Remark 2.14: The base of π is conjectured to be rational. J.-M. Hwang ([Hw]) proved that $B \cong \mathbb{C}P^n$, if it is smooth. D. Matsushita ([Ma2]) proved that it has the same rational cohomology as $\mathbb{C}P^n$, if it is smooth.

Definition 2.15: A line bundle L is called **semiample** if L^N is generated by its holomorphic sections which have no common zeros.

Remark 2.16: From semiampleness it trivially follows that L is nef. Indeed, let $\pi : M \to \mathbb{P}H^0(L^N)^*$ be the standard map. Since the sections of L have no common zeros, π is holomorphic. Then $L \cong \pi^*\mathcal{O}(1)$, and the curvature of L is the pullback of a Kähler form on $\mathbb{C}P^n$. However, a nef bundle is not necessarily semiample (see e.g. [DPS1, Example 1.7]).

Remark 2.17: Let $\pi : M \to B$ be a holomorphic Lagrangian fibration, and ω_B a Kähler class on B. Then $\eta := \pi^* \omega_B$ is semiample and parabolic. The converse is also true, by Matsushita's theorem: if L is semiample and parabolic, L induces a Lagrangian fibration.

Conjecture 2.18: (**Hyperkähler SYZ conjecture**) Let L be a parabolic nef line bundle on a hyperkähler manifold. Then L is semiample.

Remark 2.19: The SYZ conjecture can be seen as a hyperkähler version of the "abundance conjecture" (see e.g. [DPS2], 2.7.2).

Conjecture 2.20: (**Matsushita's conjecture**) Every holomorphic Lagrangian fibration $\pi : M \to \mathbb{C}P^n$ is either locally isotrivial or the fibers vary maximally in the moduli space of Abelian varieties $\mathcal{A}_n$.

Remark 2.21: This conjecture was introduced to the author in private communictions with J. Sawon, D. Matsushita and J.-M. Hwang.

B. van Geemen and C. Voisin recently proved a weeker version of Matsushita's conjecture.

Theorem 2.22: (B. van Geemen, C. Voisin, [vGV]) Let X be a projective hyperkähler manifold of dimension $2n$ admitting a Lagrangian fibration $f : X \to B$, where B is smooth. Assume $b_{2,tr}(X) = b_2(X) - \rho(X) \geqslant 5$. Then a very general deformation (X', f', B') of the triple (X, f, B) satisfies Matsushita's conjecture.

2.4 Charles' and Sawon's finiteness theorems

Our main results in this paper rely on the following theorem.

Theorem 2.23: (Charles, [Ch]) Let n and r be two positive integers. Then there exists a scheme S of finite type over $\mathbb{C}$, and a projective morphism $\mathcal{M} \longrightarrow S$ such that if M is a complex hyperkähler variety of dimension $2n$ and L is a line bundle on M with $c_1(L)^{2n} = r$ and $q(L) > 0$, where q is the Beauville-Bogomolov form, then there exists a complex point s of S such that $\mathcal{M}_s$ is birational to M.

We would also like to mention the following theorem in the recent literature.

Theorem 2.24: (Sawon, [Saw]) Fix positive integers n and $d_1, \cdots, d_n$, with $d_1 | d_2 | \cdots | d_n$. Consider Lagrangian fibrations $\pi : M \to \mathbb{C}P^n$ that satisfy:

(1) $\pi : M \to \mathbb{C}P^n$ admits a global section,

(2) there is a very ample line bundle on M which gives a polarization of type $(d_1, \cdots, d_n)$ when restricted to a generic smooth fibre M_t,

(3) over a generic point t of the discriminant locus the fibre M_t is a rank-one semi-stable degeneration of abelian varieties,

(4) a neighbourhood U of a generic point $t \in \mathbb{C}P^n$ describes a maximal variation of abelian varieties.

Then there are finitely many such Lagrangian fibrations up to deformation.

Remark 2.25: Notice that as a corollary of Matsushita's conjecture, part (4) of Sawon's Theorem simply excludes locally isotrivial fibrations. We need to apply only the deformational version (van Geemen-Voisin's Theorem 2.22) of Matsushita's conjecture to Sawon's theorem in order to remove the seemingly restrictive assumption (4).

Remark 2.26: We would like to point out that if there is a section $\sigma : \mathbb{C}P^n \longrightarrow M$, this means that $\sigma(\mathbb{C}P^n)$ would be a Lagrangian subvariety in M. Finding Lagrangian $\mathbb{C}P^n$'s in a hyperkähler manifold is itself a very interesting task (for example, see [HT]). Moreover, the Lagrangian $\sigma(\mathbb{C}P^n)$ would have to intersect the general fiber of π in one point.

3 Main results

Consider a lattice Λ, i.e., a free $\mathbb{Z}-$module of finite rank equipped with a non-degenerate symmetric bilinear from q with values in $\mathbb{Z}$. If $\{e_i\}$ is a basis of Λ, the *discriminant* of Λ is defined as $\mathrm{discr}(\Lambda) = \det(e_i \cdot e_j)$.

Lemma 3.1: Let (Λ, q) be an indefinite lattice and $v \in \Lambda$ be an isotropic non-zero vector. Then there exists a positive vector $w \in \Lambda$ such that $0 < q(w, v) \leqslant |\mathrm{discr}(\Lambda)|$ and $0 < q(w, w) \leqslant 2|\mathrm{discr}(\Lambda)|$.

Proof: Let w_0 be a vector with minimal positive intersection $q(w_0, v)$. Then by Lemma 3.7. in [KV], $q(w_0, v)$ divides $N = |\mathrm{discr}(\Lambda)|$. Therefore, $0 < q(w_0, v) \leqslant N$. If $q(w_0, w_0) > 0$, let α be the smallest integer such that $q(w_0 + \alpha v, w_0 + \alpha v) > 0$. Then we can take $w = w_0 + \alpha v$. Otherwise, if $q(w_0, w_0) \leqslant 0$, consider the vectors $\{w_0 + \alpha v\}$. Since $q(v, v) = 0$, the square of such a vector is: $q(w_0 + \alpha v, w_0 + \alpha v) = q(w_0, w_0) + 2\alpha q(w_0, v)$. Take α to be a positive integer such that $q(w_0 + \alpha v, w_0 + \alpha v) > 0$. Set $w = w_0 + \alpha v$. Then in both cases w is a positive vector with $0 < q(w, v) = q(w_0, v) \leqslant N$. Notice that automatically $0 < q(w, w) = q(w_0 + \alpha v, w_0 + \alpha v) = q(w_0, w_0) + 2\alpha q(w_0, v) \leqslant 2N = 2|\mathrm{discr}(\Lambda)|$. ■

We recall the following result from a paper of the author's together with Misha Verbitsky, Theorem 3.6 in [KV].

Theorem 3.2: Consider the action of the monodromy group Γ_I on $H^2(M,\mathbb{Z})$, and let $S \subset H^2(M,\mathbb{Z})$ be the set of all classes which are parabolic and primitive. Then there are only finitely many orbits of Γ_I on S.

Our main result is the following finiteness theorem.

Theorem 3.3: There are at most finitely many deformation classes of Lagrangian fibrations $\pi : M \to \mathbb{C}P^n$ with a fixed Fujiki constant c and a given discriminant of the Beauville-Bogomolov-Fujiki lattice (Λ, q).

Proof: As in Remark 2.17, Lagrangian fibrations correspond to parabolic semiample classes. Now consider $S \subset H^2(M,\mathbb{Z})$ defined above, the set of all classes which are parabolic and primitive, which is possibly larger than the set of parabolic semiample classes. By Theorem 3.2, there are only finitely many orbits of the monodromy group Γ_I on S.

Let L be a nef parabolic class ($q(L) = 0$) coming from the Lagrangian fibration. Deform the Lagrangian fibration preserving the fibration structure, i.e., preserving the class of L to a projective hyperkähler Lagrangian fibration. Since we are interested in finiteness results up to deformation, we are going to work in the projective setting. By Huybrechts result (Theorem 3.11 in [Hu1]), there exists a line bundle with positive square. Apply Lemma 3.1 for $(\Lambda, q) = (H^2(X,\mathbb{Z}), q)$ and $v = L$. There exists a positive vector w with $0 < q(w,v) \leqslant |\mathrm{discr}(\Lambda)| = N$. We could choose w to be a vector with the smallest positive square $q(w,w) > 0$. From the lemma we see that $0 < q(w,w) \leqslant 2|\mathrm{discr}(\Lambda)|$, which is bounded since we consider a fixed discriminant.

Now we can apply F. Charles' Theorem 2.23 to the case when the first Chern class is w, in which case, by Fujiki's formula, $0 < r = w^{2n} = c \cdot q(w,w)^n \leqslant c \cdot (2|\mathrm{discr}(\Lambda)|)^n$ is bounded. For each r in this interval we obtain only finitely many deformation classes of the total space M. ■

Since the families of hyperkähler manifolds as above form a bounded family, there are only finitely many choices of the second Betti number which plays an important role in studying the geometry of hyperkähler manifolds. We obtain the following.

Corollary 3.4: In the assumptions of Theorem 3.3, the second Betti number $b_2(M)$ is bounded.

Using similar methods as above together with F. Charles' Theorem 2.23, we generalize Sawon's Theorem 2.24 by dropping most of the assumptions.

Theorem 3.5: Consider a Lagrangian fibration $\pi : M \longrightarrow \mathbb{C}P^n$ such that there is a line bundle P on M with $q(P) > 0$ and with a given P-degree d on the general fiber F of π, i.e., $P^n \cdot F = d$. Then there are at most finitely many deformation classes of hyperkähler manifolds M as above, i.e., they form a bounded family.

Proof: Let L be a nef parabolic class ($q(L) = 0$) coming from the Lagrangian fibration (e.g., as the pullback of a hyperplane class on $\mathbb{C}P^n$). The fundamental class $[F]$ of the general fiber of π is proportional to L^n. We can fix the constant multiple in such a way that $[F] = L^n$. By assumption, $P^n \cdot L^n = d$ is fixed. Consider the classes $\{P - kL\}$ for $k \in \mathbb{Z}$. We would like to bound the top degree of one of these classes, and apply F. Charles' theorem. Choose an integer $k \geqslant \frac{q(P,P)-2q(P,L)}{2q(P,L)}$. For such k we have the following estimate, where c is the Fujiki constant: $(P-kL)^{2n} = c \cdot q(P-kL, P-kL)^n = c(q(P,P)-2kq(P,L))^n \leqslant c(2q(P,L))^n = 2^n c \cdot q(P,L)^n = \binom{2n}{n} P^n \cdot L^n = \binom{2n}{n} d$. Here we applied Fujiki's formula twice (as in Theorem 2.7 and Remark 2.11). In order to apply Theorem 2.23, we also need $q(P - kL, P - kL) > 0$, i.e., $q(P,P) - 2kq(P,L) > 0$. Combining with the previous restriction on k, we have to choose

$$k \in \Big[\frac{q(P,P) - 2q(P,L)}{2q(P,L)}, \frac{q(P,P)}{2q(P,L)}\Big).$$

Since P is in the interior of the of the positive cone $\mathcal{C}$ and L is on the boundary of $\mathcal{C}$, it follows that $q(P,L) > 0$ (Corollary 7.2 in [BHPV]). The interval above is well-defined, because $q(P,L) > 0$. For such a choice of the integer k, the top intersection of $P - kL$ is bounded and $q(P - kL) > 0$. We apply F. Charles' Theorem 2.23 to obtain a bounded family of such M, which implies finiteness of deformations of M. ■

Remark 3.6: In Theorem 3.3 and Theorem 3.5 we prove finiteness of deformation classes of the total space M of the Lagrangian fibration. However, in Theorem 1.6 the author together with Misha Verbitsky prove that for a fixed compact manifold M there are only finitely many deformation types of hyperkähler Lagrangian fibrations with total space M.

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