Algebraically hyperbolic manifolds have finite automorphism groups

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Abstract. A projective manifold M is algebraically hyperbolic if there exists a positive constant A such that the degree of any curve of genus g on M is bounded from above by $A(g-1)$. A classical result is that Kobayashi hyperbolicity implies algebraic hyperbolicity. It is known that Kobayashi hyperbolic manifolds have finite automorphism groups. Here we prove that, more generally, algebraically hyperbolic projective manifolds have finite automorphism groups.

1   Introduction

J.-P. Demailly introduced an algebraic analogue for the analytic notions of hyperbolicity in [De]. Using the classical Gauss-Bonnet formula he showed that every compact complex Kobayashi hyperbolic manifold is also algebraically hyperbolic. Demailly conjectured that for complex projective manifolds the two notions, Kobayashi hyperbolicity and algebraic hyperbolicity, coincide.

Kobayashi proved that Kobayashi hyperbolic manifolds have only finite order automorphisms, [Ko]. In [KV] L. Kamenova and M. Verbitsky proved that algebraically hyperbolic projective hyperkähler manifolds have finite groups of automorphisms. The general expectation is that all projective hyperkähler manifolds are algebraically non-hyperbolic.

Theorem 1.1: (Kamenova, Verbitsky) Let M be an algebraically hyperbolic projective hyperkähler manifold. Then M has finite group of automorphisms.

Here we generalize the result above and show that all algebraically hyperbolic projective manifolds have finite groups of automorphisms.

Theorem 1.2: The group $\mathrm{Aut}(M)$ of automorphisms of an algebraically hyperbolic manifold M is finite.

The idea behind the proof of our main theorem is the following. We consider the induced action of the automorphism group $\mathrm{Aut}(M)$ on the cohomology $H^{1,1}(M, \mathbb{R})$. From the Hodge-Riemann relations it follows that the image of $\mathrm{Aut}(M)$ in $GL(H^{1,1}(M, \mathbb{R}))$ has to preserve some rational Kähler class. Furthermore, we consider the case when the image of $\mathrm{Aut}(M)$ in $\mathrm{Aut}(\mathrm{Pic}^0(M))$ is infinite. Using the Albanese map we produce a subvariety of M that admits

self-isogenies of arbitrarily high order, thus giving curves of constant genus and arbitrary high degree in this subvariety, i.e., we obtain an algebraically non-hyperbolic subvariety. This contradiction implies that the image of $\mathrm{Aut}(M)$ in $\mathrm{Aut}(\mathrm{Pic}^0(M))$ is finite. If we assume that $\mathrm{Aut}(M)$ is infinite, there would be an infinite subgroup Γ acting trivially on $\mathrm{Aut}(\mathrm{Pic}(M))$. It fixes an ample line bundle $L \in \mathrm{Pic}(M)$. The action of Γ on $\mathbb{P}H^0(M, L)^*$ preserves the image of the projective embedding $M \hookrightarrow \mathbb{P}H^0(M, L)^*$. Let $\overline{\Gamma}$ be the Zariski closure of Γ in $PGL(H^0(M, L)^*)$. Then the orbits of $\overline{\Gamma}$ are positive dimensional, which is a contradiction, because $\mathrm{Aut}(M)$ is discrete when M is algebraically hyperbolic.

2   Brody curves, Kobayashi and algebraic hyperbolicity

In this section let M be a compact complex manifold. We introduce some basic hyperbolicity notions that can be found in [Ko] and [De]. Brody introduced the Brody curves in [Br].

2.1   Kobayashi pseudometric and Brody curves

Definition 2.1: A *pseudometric* on M is a function $d: M \times M \longrightarrow \mathbb{R}^{\geqslant 0}$ which is symmetric: $d(x, y) = d(y, x)$ and satisfies the triangle inequality $d(x, y) + d(y, z) \geqslant d(x, z)$.

Remark 2.2: Let $\mathfrak{D}$ be a set of pseudometrics on M. Then $d_{\max}(x, y) := \sup_{d \in \mathfrak{D}} d(x, y)$ is also a pseudometric.

Definition 2.3: The *Kobayashi pseudometric* d_M on a complex manifold M is $d_M := d_{\max}$ for the set $\mathfrak{D}$ of all pseudometrics on M such that any holomorphic map from the Poincaré disk to M is distance-decreasing.

Remark 2.4: The definition above is equivalent to the standard definition that the Kobayashi distance between points $x, y \in M$ is the infimum of the Poincaré distance over all sets of Poincaré disks connecting x to y.

Definition 2.5: A manifold M is called *Kobayashi hyperbolic* if the Kobayashi pseudometric d_M is non-degenerate.

Definition 2.6: Let M be a complex Hermitian manifold. A *Brody curve* is a non-constant holomorphic map $f: \mathbb{C} \longrightarrow M$ such that $|df| \leqslant C$ for some constant C. Here $|df|$ is understood as an operator norm of $df: T_z\mathbb{C} \longrightarrow TM$, where $\mathbb{C}$ is equipped with the standard Euclidean metric.

Definition 2.7: Let (Δ_r, g_r) be a disk of radius r in $\mathbb{C}$ with the Poincaré metric g_r, rescaled in such a way that the unit tangent vector to 0 has length 1. A

Brody map to a Hermitian complex manifold M is a map $f : \Delta_r \longrightarrow M$ such that $|df| \leqslant 1$ (here the operator norm is taken with respect to the Poincaré metric on Δ_r) and $|df|(z) = 1$ at $z = 0$.

Lemma 2.8: Let $f_r : \Delta_r \longrightarrow M$ be a sequence of Brody maps with $r \longrightarrow \infty$. Then $\{f_r\}$ has a subsequence which converges uniformly to a Brody curve f satisfying $|df|(z) = 1$ at $z = 0$.

Proof. Let $r_1 < r_2$. The identity map $\tau : (\Delta_{r_1}, g_1) \longrightarrow (\Delta_{r_2}, g_2)$ is 1-Lipschitz. Indeed, it is 1-Lipschitz with respect to the usual Poincaré metric: $\tau^*(r_2^{-2} g_2) \leqslant r_1^{-2} g_1$. Since $r_1 < r_2$, this gives $\tau^* g_2 \leqslant g_1$. Restricted on any disk Δ_R, the family $\{f_r, r > R\}$ is a normal family (since it is Lipschitz), hence it has a subsequence which converges uniformly to a Lipschitz map. Since a uniform limit of holomorphic maps is holomorphic, the family $\left\{ f_r \big|_{\Delta_R}, r > R \right\}$ converges to a holomorhic map on $\varphi_R : \Delta_R \longrightarrow M$. The map φ_R is Lipschitz with respect to all metrics g_r, $r > R$. Since $\lim\limits_{r \longrightarrow \infty} g_r$ is the standard Euclidean metric g_∞, the map φ_R is Lipschitz with respect to g_∞. The sequence $\lim\limits_{R \longrightarrow \infty} \varphi_R$ converges to a holomorphic Lipschitz map $\mathbb{C} \longrightarrow M$. Since all f_r and φ_R satisfy $|d\varphi_R|(z) = 1$ at $z = 0$, the same is true for the limit. ∎

A classical result of Brody states that a compact complex manifold M is Kobayashi hyperbolic if and only if there are no non-constant holomorphic maps from $\mathbb{C}$ to M.

Theorem 2.9: (Brody's lemma) Let M be a compact complex manifold which is not Kobayashi hyperbolic. Then M contains a Brody curve.

Proof: Let us equip M with a Hermitian metric h. If $|df|(0) \leqslant C$ for any holomorphic map $(\Delta_1, g_1) \longrightarrow M$, then the Kobayashi pseudometric satisfies $d_K \geqslant C^{-1} h$, and M is Kobayashi hyperbolic. If this quantity is non-bounded, we can always rescale the disc to obtain a map $f_r : (\Delta_r, g_r) \longrightarrow M$ with $r = |df|(0)$, and then $|df_r|(0) = 1$. Then Brody's lemma follows from Lemma 2.8 and the following lemma.

Lemma 2.10: Let M be a compact Hermitian manifold, and $\psi_r : (\Delta_r, g_r) \longrightarrow M$ a sequence of holomorphic maps satisfying $|d\psi_r|(0) \geqslant 1$, $r \longrightarrow \infty$. Then there exists a sequence of Brody maps $f_s : (\Delta_s, g_s) \longrightarrow M$, with $s \longrightarrow \infty$.

Proof. We need to construct a sequence of Brody maps, which are 1-Lipschitz maps $f_s : (\Delta_s, g_s) \longrightarrow M$, with $|df_s|(0) = 1$. The identity map

$$\Psi_{r-\varepsilon, r} : (\Delta_{r-\varepsilon}, g_{r-\varepsilon}) \longrightarrow (\Delta_r, g_r)$$

is 1-Lipschitz, and satisfies

$$\lim_{z \longrightarrow \partial \Delta_{r-\varepsilon}} |d\Psi_{r-\varepsilon, r}|(z) = 0.$$

Let $u := r - \varepsilon$ and $\widetilde{f_u} := \Psi_{r-\varepsilon,r} \circ \psi_r$ be a restriction of f_r to the disk $(\Delta_{r-\varepsilon}, g_{r-\varepsilon})$. Then f_u is also Lipschitz and $|d\widetilde{f_u}|$ reaches maximum at a point z_u somewhere inside the disk Δ_u. Applying an appropriate holomorphic isometry of Δ_u, we may assume that $|d\widetilde{f_u}|(z)$ takes maximum $C_u \geqslant 1$ for $z = 0$. Rescaling $\widetilde{f_u}$, and setting $s := C_u u$, we obtain a map $f_s : \Delta_s \longrightarrow M$ which is 1-Lipschitz and satisfies $|df_s| \leqslant 1$, $|df_s|(0) = 1$. ■

Definition 2.11: Let M be a projective manifold. We say that M is *algebraically hyperbolic* if there exists a constant $A > 0$ such that for any curve $C \subset M$ of geometric genus g one has $\deg C < A(g - 1)$.

Remark 2.12: Algebraically hyperbolic manifolds contain no elliptic nor rational curves.

Using Gauss-Bonnet in [De] Demailly proved that Kobayashi hyperbolicity implies algebraic hyperbolicity. Here we'll give a slightly different proof of this well known result. The converse implication, "algebraically hyperbolicity implies Kobayashi hyperbolicity", was conjectured by J.-P. Demailly who introduced the notion of algebraic hyperbolicity in [De].

Theorem 2.13: (Demailly, [De]) Kobayashi hyperbolicity implies algebraic hyperbolicity.

Proof. Let $C \subset M$ be a curve in a Kobayashi hyperbolic manifold. The genus 0 and 1 cases are ruled out by Brody's Lemma, Theorem 2.9. Then genus $g = g(C)$ is greater than 1, and its universal cover is the disk Δ. Denote by $\varphi_C : \Delta \longrightarrow M$ the universal covering map. The volume of C with respect to the Fubini-Study metric on M is $\deg C$, and its volume with respect to the Poincaré metric is $\int_C c_1(T^*C) = 2\pi(2g - 2)$. Therefore, $|\varphi_C| > \frac{\deg C}{2\pi(2g-2)}$ somewhere on Δ. Notice that M is not algebraically hyperbolic if and only if there is a sequence $\{C_i\}$ of curves in M with $\lim\limits_{i \to \infty} \frac{\deg C_i}{2\pi(2g(C_i)-2)} \longrightarrow \infty$. Suppose that M is not algebraically hyperbolic. This gives a sequence $\varphi_{C_i} : \Delta_1 \longrightarrow M$ with $|\varphi_{C_i}| > \frac{\deg C_i}{2\pi(2g(C_i)-2)}$ somewhere on Δ. Replacing Δ_1 by $\Delta_{1-\varepsilon}$ as above if necessary, we may assume that $|\varphi_{C_i}|$ reaches its maximum somewhere on Δ_1. Applying an isometry of Δ_1, we may assume that $|\varphi_{C_i}|$ reaches its maximum R_i in $0 \in \Delta_1$. Rescaling φ_{C_i} by R_i, we obtain a sequence of disks $\widetilde{\varphi}_{C_i}(z) = \varphi_{C_i}(z/R_i) : \Delta_{R_i} \longrightarrow M$, giving a Brody curve by Lemma 2.8. ■

2.2 Algebraic Kobayashi metric

Definition 2.14: The Kobayashi pseudometric d_K is defined as the infimum of the path metric on connected chains of holomorphic disks equipped with the Poincaré metrics. Define the *algebraic Kobayashi pseudometric* d_A using the Poincaré metric on connected chains of algebraic curves instead of the disks.

Remark 2.15: This metric clearly satisfies $d_A \geqslant d_K$. In [DLS] Demailly, Lempert and Shiffman prove that $d_A = d_K$ for any quasi-projective manifold, i.e., one can compute the Kobayashi pseudodistance by means of chains of algebraic curves. Also, algebraic hyperbolicity follows if d_A is non-degenerate; this is proven by reverse-engineering the argument we used to prove that algebraic hyperbolicity is implied by Kobayashi hyperbolicity. It is not clear, however, if algebraic hyperbolicity implies the non-degeneracy of $d_A = d_K$.

Question 2.16: Is d_A always non-degenerate for algebraically hyperbolic manifolds?

Remark 2.17: Note that the group $\mathrm{Aut}(M)$ of holomorphic automorphisms of a complex manifold M with d_A non-degenerate is necessarily compact. Indeed, any holomorphic automorphism of M is an isometry of (M, d_A), and the group of isometries of a compact metric space is compact. On the other hand, a uniform limit of holomorphic maps is holomorphic, hence compactness of the isometry group implies compactness of $\mathrm{Aut}(M)$. Since an algebraically hyperbolic manifold has discrete group of holomorphic automorphisms by Proposition 2.19, the group $\mathrm{Aut}(M)$ is finite. Then an affirmative answer to Question 2.16 would imply the main statement of this paper: finiteness of $\mathrm{Aut}(M)$ for algebraically hyperbolic M.

Remark 2.18: For any g there are complex elliptic fibrations over $\mathbb{P}^1$ without multiple fibers where all horizontal curves (i.e., curves surjecting onto $\mathbb{P}^1$ under the projection) have genus greater than g (Remark 6.11 in [BT]). The Kobayashi pseudometric on such surfaces is trivial but there are no elliptic or rational curves connecting points in different fibers. This example illustrates the difference in obtaining the Kobayashi pseudometric on the surface from mapping complex surfaces and algebraic curves. We can connect any two points in the surface by the image of a complex line, as shown in [BL]. However, we need to take a limit over all chains of algebraic curves connecting points in different fibers in order to obtain the Kobayashi pseudometric using only algebraic curves.

2.3   Automorphisms of hyperbolic manifolds

In the rest of the section we summarize some facts about the automorphism group of hyperbolic manifolds.

Proposition 2.19: The group of automorphisms of an algebraically hyperbolic manifold M is discrete.

    Proof. The group of automorphisms G of a projective manifold M is a complex Lie group. If its connected component G^0 is non-trivial, this gives a holomorphic map $\varphi : \ G^0 \longrightarrow M$, depending on the choice of a point. Any connected complex algebraic Lie group is an extension of an affine group and an abelian variety. Affine algebraic groups are rational varieties, hence the orbit

of an affine algebraic group is unirational and covered by rational curves. An abelian variety is not algebraically hyperbolic, because an abelian variety of dimension n admits a self-isogeny of order m^n mapping a curve C of genus g to a curve of genus g and degree $m^n \deg C$. The same argument shows that any positive-dimensional orbit of a compact complex commutative Lie group is not algebraically hyperbolic. Notice that the image of an algebraically non-hyperbolic manifold is also not algebraically hyperbolic. Therefore, M cannot be algebraically hyperbolic. ∎

An even stronger statement is true for Kobayashi hyperbolic manifolds.

Proposition 2.20: (Kobayashi, [Ko]) The group of automorphisms of a compact Kobayashi hyperbolic manifold M is finite.

Proof. The group G of automorphisms of M is closed in its group of isometries under the Kobayashi pseudometric. The group of isometries of a compact metric space is compact, hence G has only finitely many connected components. Finally, $\dim G^0 = 0$ as shown in Proposition 2.19, i.e., G is discrete and compact, therefore finite. ∎

3   Main Results

In this section M is a compact complex projective manifold. For the classical notions of the Picard scheme and the Albanese variety we refer the readers to the survey manuscript [Kl].

Proposition 3.1: Let M be a complex projective manifold, $\dim_{\mathbb{C}} M = n$. Suppose that the image of $\mathrm{Aut}(M)$ in $GL(H^{1,1}(M, \mathbb{R}))$ does not preserve any rational Kähler class. Then M is not algebraically hyperbolic.

Proof. Assume M is algebraically hyperbolic. Let ω be a rational Kähler class which has an infinite $\mathrm{Aut}(M)$-orbit. Replacing ω by $N\omega$, from the $(1,1)$-theorem we may assume that ω is a class of a hyperplane section. Then $\omega^{n-1} = [C]$ is the fundamental class of a smooth complex curve $C \subset M$ from Bertini's theorem. Let $f_i(\omega)$ be in the orbit of ω, which is infinite by assumption. Then

$$\deg_{\omega} f_i(C) = \int_M \omega \wedge (f_i(\omega))^{n-1} = \int_M f_i^{-1}(\omega) \wedge \omega^{n-1}.$$

Since the genus of $f_i(C)$ is constant, from algebraic hyperbolicity we obtain $\int_M f_i(\omega) \wedge \omega^{n-1} < A$ for some constant $A > 0$.

Let $|\cdot|$ denote the positive-definite Hodge-Riemann metric on $H^{p,q}(M, \mathbb{R})$, that is, the metric defined as $\eta, \eta' \longrightarrow -(-1)^l (\sqrt{-1})^{p-q} \int_M \eta \wedge \overline{\eta'} \wedge \omega^{n-p-q}$, where η, η' are (p,q)-forms which belong to a weight l representation of $\mathfrak{sl}(2)$ associated with the Lefschetz $\mathfrak{sl}(2)$-action. Replace the sequence $\frac{f_i(\omega)}{|f_i(\omega)|}$ of points in the

sphere $S \subset H^{1,1}(M, \mathbb{R})$ by a converging subsequence, and let R be its limit. Since the sequence $f_i(\omega)$ is infinite, distinct and integral, one has $\lim\limits_{i \longrightarrow \infty} |f_i(\omega)| = \infty$. Then $\int_M f_i(\omega) \wedge \omega^{n-1} < A$ implies that $\int_M R \wedge \omega^{n-1} = 0$. By the Hodge-Riemann relations [GH], this gives $\int_M R \wedge R \wedge \omega^{n-2} = -|R|^2 = -1$, hence

$$\lim_i \frac{\int_M f_i(\omega) \wedge f_i(\omega) \wedge \omega^{n-2}}{|f_i(\omega)|^2} = -1.$$

This is a contradiction, because $f_i(\omega)$ is Kähler, hence $\int_M f_i(\omega) \wedge f_i(\omega) \wedge \omega^{n-2} > 0$. ∎

Proposition 3.2: Let M be a complex projective manifold. Suppose that $\mathrm{Aut}(M)$ is infinite, but the image of $\mathrm{Aut}(M)$ in $\mathrm{Aut}(\mathrm{Pic}(M))$ is finite. Then M is not algebraically hyperbolic.

Proof. From the assumption in the proposition we obtain that an infinite subgroup $\Gamma \subset \mathrm{Aut}(M)$ acts trivially on $\mathrm{Pic}(M)$. Then it fixes a very ample line bundle $L \in \mathrm{Pic}(M)$. Therefore, Γ acts on $\mathbb{P}H^0(M, L)^*$ preserving the image of the projective embedding $M \longrightarrow \mathbb{P}H^0(M, L)^*$. Let $G = \overline{\Gamma}$ be the Zariski closure of Γ in $PGL(H^0(M, L)^*)$. Since M is Zariski closed and Γ-invariant, it is G-invariant. Since Γ acts on M with infinite orbits, the orbits of the G-action on M are positive-dimensional. This is impossible, because $\mathrm{Aut}(M)$ is discrete as shown in Proposition 2.19. ∎

Remark 3.3: Notice that if the image $\mathfrak{G}$ of $\mathrm{Aut}(M)$ in $GL(H^{1,1}(M, \mathbb{R}))$ does not preserve any rational Kähler classes, then $\mathfrak{G}$ is infinite. Otherwise, take a finite orbit of a Kähler class and notice that its geometric center is an $\mathrm{Aut}(M)$-invariant Kähler class, because the convex hull of a set of Kähler classes lies inside the Kähler cone.

Remark 3.4: When M is projective and the image $\mathfrak{G}$ of $\mathrm{Aut}(M)$ in $GL(H^{1,1}(M, \mathbb{R}))$ is finite, we could take a $\mathfrak{G}$-orbit of a rational Kähler class, and its geometric center is also rational. Therefore, any projective manifold M with $\mathfrak{G}$ finite admits an $\mathrm{Aut}(M)$-invariant integer Kähler class.

Proposition 3.5: Let M be a complex projective manifold. Suppose that the image of $\mathrm{Aut}(M)$ in $GL(H^{1,1}(M, \mathbb{R}))$ is finite, but its image in $\mathrm{Aut}(\mathrm{Pic}^0(M))$ is infinite. Then M is not algebraically hyperbolic.

Proof. Consider the orbit $\mathrm{Aut}(M) \cdot \alpha$ of a Kähler class α. Its geometric center gives an $\mathrm{Aut}(M)$-invariant Kähler class ω, because the convex envelope of a set of Kähler classes lies in the Kähler cone, as in Remark 3.3.
The Albanese manifold $\mathrm{Alb}(M) = H^0(M, \Omega_M^1)^*/H_1(M, \mathbb{Z})$ admits a natural $\mathrm{Aut}(M)$-invariant flat Kähler metric induced by the Hodge-Riemann form on $H^1(M, \mathbb{C})$. Since $\mathrm{Aut}(M)$ acts on $\mathrm{Alb}(M)$ by isometries, the image $\mathfrak{T}$ of $\mathrm{Aut}(M)$ in $\mathrm{Aut}(\mathrm{Alb}(M))$ is compact. The connected component of $\mathrm{Aut}(\mathrm{Alb}(M))$ is a

subgroup $\mathrm{Par}(\mathrm{Alb}(M)) = \mathrm{Alb}(M) \subset \mathrm{Aut}(\mathrm{Alb}(M))$ acting on the torus $\mathrm{Alb}(M)$ by parallel transport. Since $\mathfrak{T}$ is compact, its image in the discrete group $\frac{\mathrm{Aut}(\mathrm{Alb}(M))}{\mathrm{Par}(\mathrm{Alb}(M))}$ is finite. Let $\Gamma \subset \mathfrak{T}$ be the kernel of the map $\mathfrak{T} \longrightarrow \frac{\mathrm{Aut}(\mathrm{Alb}(M))}{\mathrm{Par}(\mathrm{Alb}(M))}$. This is a finite index subgroup in $\mathfrak{T}$. Therefore, Γ is infinite.

Since Γ is infinite, its closure $T \subset \mathrm{Par}(\mathrm{Alb}\, M)$ is positive-dimensional. Take a smooth fiber $\mathrm{Alb}^{-1}(x)$ over $x \in \mathrm{Alb}(M)$. The general fiber of the real analytic map $\mathrm{Alb}^{-1}(T \cdot x) \xrightarrow{\pi} T \cdot x$ is smooth. Since all fibers of π exist in dense families, all fibers of π are smooth. Therefore, π is a locally trivial fibration with isomorphic fibers. Consider the Zariski closure T_1 of T, which is a compact commutative Lie group, that is, the compact complex torus. This gives an isotrivial fibration $\mathrm{Alb}^{-1}(X) \xrightarrow{\pi_1} X$, where X is an orbit of T_1.

Isotrivial fibrations over T_1 with fiber F are classified by $H^1(T_1, \mathrm{Aut}(F))$. The variety $F \subset M$ may be assumed to be an algebraically hyperbolic manifold with $\dim F < \dim M$. Using induction on dimension, we may assume that $\mathrm{Aut}(F)$ is finite.

The first cohomology of a torus with coefficients in a finite group A is the same as an A-valued local system. Therefore, it becomes trivial after an appropriate finite covering. Then π_1 becomes a trivial fibration after passing to a finite covering $Y \longrightarrow X$, giving a decomposition $\widetilde{\mathrm{Alb}^{-1}(X)} = F \times Y$. This manifold admits self-isogenies of arbitrary high order, giving curves of constant genus and arbitrary high degree in $\widetilde{\mathrm{Alb}^{-1}(X)}$ and its finite quotient $\mathrm{Alb}^{-1}(X)$. Therefore, $\mathrm{Alb}^{-1}(X) \subset M$ is not algebraically hyperbolic, and M is algebraically non-hyperbolic. ■

Using the above results we can prove the main result of this paper.

Theorem 3.6: The group $\mathrm{Aut}(M)$ of automorphisms of an algebraically hyperbolic manifold M is finite.

Proof. Assume $\mathrm{Aut}(M)$ is infinite. If the image $\mathfrak{G}$ of the group $\mathrm{Aut}(M)$ in $GL(H^{1,1}(M, \mathbb{R}))$ does not preserve any rational Kähler class, $\mathfrak{G}$ is infinite by Remark 3.4. Then we would get a contradiction by Proposition 3.1. If the image of $\mathrm{Aut}(M)$ in $\mathrm{Aut}(\mathrm{Pic}(M))$ is finite, we get a contradiction by Proposition 3.2, and if its image in $\mathrm{Aut}(\mathrm{Pic}(M))$ is infinite, we get a contradiction by Proposition 3.5. ■

To sum up the argument: we consider the image $\mathfrak{G}$ of the group $\mathrm{Aut}(M)$ in $GL(H^{1,1}(M, \mathbb{R}))$. If it is finite, we have an integer Kähler class which is $\mathfrak{G}$-invariant. If $\mathrm{Aut}(M)$ acts with finite image on the Albanese variety of M, this class gives an $\mathrm{Aut}(M)$-invariant polarization, hence $\mathrm{Aut}(M)$ is a linear algebraic group; its connected part is trivial because the orbits of a connected linear algebraic group are rational.

If $\mathfrak{G}$ is finite, but $\mathrm{Aut}(M)$ acts with infinite image on the Albanese variety of M, the closure of an orbit of $\mathrm{Aut}(M)$ contains a torus, also contradicting hyperbolicity.

Finally, if $\mathfrak{G}$ is infinite, we obtain curves of bounded genus and arbitrary degree using the action of $\mathfrak{G}$ on cohomology.

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