Non-hyperbolicity of holomorphic symplectic varieties

Ljudmila Kamenova and Christian Lehn

Abstract. We prove non-hyperbolicity of primitive symplectic varieties with $b_2 \geq 5$ that satisfy the rational SYZ conjecture. If in addition $b_2 \geq 7$, we establish that the Kobayashi pseudometric vanishes identically. This in particular applies to all currently known examples of irreducible symplectic manifolds and thereby completes the results by Kamenova–Lu–Verbitsky. The key new contribution is that a projective primitive symplectic variety with a Lagrangian fibration has vanishing Kobayashi pseudometric. The proof uses ergodicity, birational contractions, and cycle spaces.

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1. Introduction

The Kobayashi pseudometric on a complex variety is the maximal pseudometric such that any holomorphic map from the Poincaré disk to the variety is distance-decreasing. It is a fundamental object and of great interest in complex geometry. A variety is called *Kobayashi hyperbolic* if this pseudometric is a genuine metric, i.e. if it is nondegenerate. Kobayashi's conjectures [Kob76, Problem F.2, p. 405] predict that for Calabi–Yau varieties, the opposite is the case: this pseudometric vanishes identically.

In this article, we study non-hyperbolicity and vanishing of the Kobayashi pseudometric of compact Kähler holomorphic symplectic varieties. While Verbitsky [Ver15,

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2020 *Mathematics Subject Classification.* 32Q45, 53C26 (Primary); 14B07, 14J10, 32S45 (Secondary).

Key words and phrases. hyperkähler manifold, symplectic variety, hyperbolicity, Kobayashi pseudometric, locally trivial deformation, birational contraction, Lagrangian fibration, ergodicity, cycle space.

Ver17] has shown that any irreducible symplectic manifold with second Betti number $b_2 \geq 5$ is non-hyperbolic (building on Campana's result that any twistor family of irreducible symplectic manifolds contains at least one non-hyperbolic member [Cam92, Theorem 1]), a stronger statement has been shown by Kamenova–Lu–Verbitsky [KLV14] under some additional geometric assumptions. More precisely, they prove that irreducible symplectic manifolds with second Betti number ≥ 13 satisfying the hyperkähler version of the SYZ Conjecture (see Conjecture 2.14) have vanishing Kobayashi pseudometric, see [KLV13, Theorem 1.2]. Their strategy is to deform to a variety admitting two transversal Lagrangian fibrations and then use ergodicity and the upper semi-continuity of the Kobayashi diameter [KLV14, Corollary 1.23] to transport the result to any other manifold of the same deformation type.

The purpose of this article is to improve the Kamenova–Lu–Verbitsky bound on the second Betti number in order to obtain the vanishing of the Kobayashi pseudometric for all known examples of irreducible symplectic manifolds. Our key discovery is that for the pseudometric to vanish it is already enough to have *one* Lagrangian fibration instead of two, see Theorem 5.6 for a precise statement. With this at hand, we can prove our main result (see Theorem 5.3 for a slightly stronger statement).

Theorem 1.1. *Let X be a primitive symplectic variety. Suppose that every primitive symplectic variety which is a locally trivial deformation of X satisfies the rational SYZ conjecture. Then the following hold.*

(1) *If $b_2(X) \geq 5$, then X is non-hyperbolic.*
(2) *If $b_2(X) \geq 7$, then the Kobayashi pseudometric d_X vanishes identically.*

Notice that our results are valid for singular symplectic varieties as well, see Section 2 for the precise definitions. In fact, singular varieties are the natural context for our arguments. The proof of Theorem 5.6 for example crucially needs to pass through the singular world, even if you start with a smooth variety. For smooth varieties, the main result, Theorem 1.1, could be proven modifying the arguments slightly so as to (mostly) avoid singularities, but formulating and proving it for primitive symplectic varieties leads to greater clarity.

In view of the decomposition theorem [BGL22, Theorem A], see also [HP19, Theorem 1.5], it is natural to ask whether the vanishing of the Kobayashi pseudometric holds for any compact Kähler symplectic variety. The following result is an easy consequence of the decomposition theorem and also justifies why we may restrict our attention to primitive (or even irreducible) symplectic varieties.

Proposition 1.2. *If the Kobayashi pseudometric vanishes for every irreducible symplectic variety, then the same holds true for any compact Kähler symplectic variety.*

This result is proven as Proposition 5.9. As every irreducible symplectic variety is primitive symplectic, it would in particular be sufficient to get rid of the assumptions on b_2 and on the validity of the SYZ conjecture in Theorem 1.1. Removing these hypotheses would however require a new idea.

1.1. **Outline of the argument.** As in [KLV14], the idea is to first prove the vanishing of the Kobayashi pseudometric for a special class of primitive symplectic varieties and then, after having obtained this "initial" vanishing statement, deduce the Kobayashi conjecture for all primitive symplectic varieties of the same (locally trivial) deformation type.

While Kamenova–Lu–Verbitsky used symplectic varieties admitting two transversal Lagrangian fibrations, we show that already a single Lagrangian fibration is sufficient. Given two transversal fibrations, the vanishing statement is an obvious consequence of the triangle inequality for the Kobayashi pseudometric. The drawback is that assuring the existence of two fibrations increases the second Betti number (although we suspect that the approach in [KLV14] can be pushed to get better bounds). Improving their result to just one fibration is the main new contribution of this work and occupies the large part of the article. We will elaborate on this part below, but let us first explain how to conclude the proof of the main result.

Assuming the SYZ conjecture, the existence of Lagrangian fibrations reduces to a lattice theoretic question, which by Meyer's theorem has a positive answer for a lattice of rank at least 5. Incidentally, also the ergodicity properties of periods require the hypothesis $b_2 \geq 5$. From there we follow the argument of Kamenova–Lu–Verbitsky with some minor modifications due to the presence of singularities. Ergodicity is then used to transport the vanishing of the Kobayashi metric from varieties admitting Lagrangian fibrations to all varieties of the same locally trivial deformation type using the aforementioned upper semi-continuity of the Kobayashi diameter [KLV14, Corollary 1.23]. The semi-continuity was proven for families of smooth varieties, so at this point the existence of simultaneous resolutions in locally trivial families proven in [BGL22, Corollary 2.27] comes in handy.

Coming back to the "initial" vanishing statement for varieties admitting a Lagrangian fibration, let us illustrate our strategy with the following simple example.

Example 1.3. Let $f : S \longrightarrow \mathbb{P}^1$ be an elliptic K3 surface with a section $E \subset S$. Then S is chain connected by subvarieties with vanishing Kobayashi metric, hence $d_S \equiv 0$. We are however looking for a different interpretation of this argument as in higher dimensions, we do not want to assume our fibrations to have sections. Instead, we divide the problem in two. First, we will contract E and thus obtain a birational map $\pi : S \longrightarrow \bar{S}$. Let us observe that now all (images of) fibers of f meet in a single point. Hence, $\bar{S}$ is chain connected by an irreducible family of cycles with vanishing Kobayashi pseudometric, in particular, $d_{\bar{S}} \equiv 0$. As a second step, we remark that in this situation, the problem is invariant under birational maps, and thus conclude also $d_S \equiv 0$. This second point of view on the argument generalizes to higher dimensions.

Even though the above example is very simple, the general strategy is rather similar to the one illustrated in the example. First, we show that given a (rational) Lagrangian fibration either there is a second one that is transversal to it, or our variety has nontrivial divisorial contractions. In the latter case, the ultimate goal is to show that the given fibration ceases to be almost holomorphic (see Definition 4.2) on some birational model. Then we use cycle spaces and Campana's theorem on almost holomorphic maps to conclude that the resulting rational Lagrangian fibration on the contracted variety is chain connected by its fibers (as is the singular K3 surface $\bar{S}$ in Example 1.3). As the Kobayashi pseudometric vanishes when restricted to the fibers, we infer the vanishing of the Kobayashi pseudometric of our symplectic variety.

1.2. **Organization of the article.** In Section 2, we recall definitions of (singular) symplectic varieties, the Beauville–Bogomolov–Fujiki (or BBF for short) form on the second cohomology and its properties as well as some background on Lagrangian fibrations. None of the material is new, we however carefully compile the results that lead to the proof of Matushita's theorem for primitive symplectic varieties, see Theorem 2.8, and we discuss the relation between the different versions of the SYZ conjecture in Section 2.3. Section 3 is of preliminary nature as well and provides basic notions concerning hyperbolicity and properties of the Kobayashi pseudometric. The purpose of Section 4 is to explain Campana's theorem on almost holomorphic maps and to adapt a result from [GLR13] on almost holomorphic Lagrangian fibrations to primitive symplectic varieties, see Theorem 4.6.

The new contributions of this article are contained in Section 5. Here, our main result, Theorem 5.3 is proven. As explained before, we assume the second Betti number to be at least seven. Unlike in the smooth case, there are examples of singular symplectic varieties (even $\mathbb{Q}$-factorial, terminal ones) with $b_2(X)$ strictly smaller than seven. We illustrate some of these in Section 6.

Conventions. A *variety* will be a reduced complex Hausdorff space which is countable at infinity[1]. An *algebraic variety* over a field k is a reduced scheme that is separated and of finite type over k. A resolution of singularities of a variety X is a proper, bimeromorphic morphism $\pi : Y \longrightarrow X$ such that Y is a smooth variety. We denote by Ω_X^p the sheaf of holomorphic p-forms on X and by $\Omega_X^{[p]}$ its double dual, the sheaf of *reflexive* (holomorphic) p-forms.

Acknowledgments. We would like to thank Ben Bakker for helpful conversations around the Kobayashi pesudometric, Ariyan Javanpeykar for pointing out Example 3.5 to us, Giovanni Mongardi for Example 6.2, Steven Lu for the reference on length functions on complex spaces, and Claire Voisin for pointing out a confusion in one of the arguments.

Ljudmila Kamenova was partially supported by a grant from the Simons Foundation/SFARI (522730, LK) 2017-2024. Christian Lehn was supported by the DFG research grant Le 3093/3-2 and the SMWK research grant SAXAG.

[1] That is, it is a countable union of compact subspaces. This property is also known as σ-compactness.

2. Symplectic Varieties

This section is a brief reminder on holomorphic symplectic varieties. Let us begin by recalling the notion of an irreducible symplectic manifold.

Definition 2.1. An *irreducible symplectic manifold* is a connected compact complex Kähler manifold M satisfying $\pi_1(M) = 0$ and $H^{2,0}(M) = \mathbb{C}\sigma$ where σ is a holomorphic symplectic form.

These manifolds are sometimes referred to as *compact hyperkähler manifolds*, which is an equivalent concept. Indeed, in every Kähler class on an irreducible symplectic manifold there is a unique hyperkähler metric (i.e. with holonomy equal to $\mathrm{Sp}(n)$) by Yau's theorem. Conversely, a compact hyperkähler manifold is irreducible symplectic for a $\mathbb{P}^1$ worth of complex structures.

Let us now come to *singular* symplectic varieties.

Definition 2.2. A *primitive symplectic variety* is a normal compact Kähler variety X with rational singularities such that $H^1(X, \mathcal{O}_X) = 0$ and $H^0(X, \Omega_X^{[2]}) = \mathbb{C}\sigma$ for a symplectic[2] form σ.

An *irreducible symplectic variety* is a normal compact Kähler variety X with rational singularities such that for each finite, quasi-étale[3] cover $\pi : X' \longrightarrow X$ the algebra $\Gamma(X', \Omega_{X'}^{[\bullet]})$ of global reflexive holomorphic forms is generated by a symplectic form $\sigma' \in \Gamma(X', \Omega_{X'}^{[2]})$.

The notion of an irreducible symplectic variety is due to Greb–Kebekus–Peternell, see [GKP16, Definition 8.16] where we just replaced the projectivity assumption by the requirement for X to be compact Kähler. Irreducible symplectic is more restrictive than primitive symplectic and serves a different purpose: irreducible symplectic varieties are one of the three fundamental building blocks in the decomposition theorem (see [HP19, BGL22] and references therein) whereas for primitive symplectic varieties, moduli theory essentially works as in the smooth case (see [BL21, BL22]).

2.1. The Bogomolov–Beauville–Fujiki form.

Given a primitive symplectic variety X, there is the *Beauville–Bogomolov–Fujiki (BBF) form*

$$q_X : H^2(X, \mathbb{Z}) \longrightarrow \mathbb{Z}$$

which is a quadratic form that generalizes the intersection pairing for K3 surfaces. As in the smooth case, it carries a lot of information about the variety in question. We refer to [BL22, Section 5] for the explicit formula defining it, several basic results (such as the proof that it is actually an *integral* quadratic form), and for references to many earlier partial results about this form. Here, we content ourselves to listing its most important properties.

[2] A reflexive 2-form is called *symplectic* if its restriction to the regular part is.

[3] Recall that *quasi-étale* means étale in codimension one.

Lemma 2.3. *The BBF form q_X on a primitive symplectic variety X has the following properties.*

(1) *It is invariant under locally trivial deformation.*
(2) *It is nondegenerate of signature $(3, b_2(X) - 3)$.*
(3) *On the real space underlying $H^{2,0}(X) \oplus H^{0,2}(X)$, the form is positive definite.*
(4) *The orthogonal complement of $(H^{2,0}(X) \oplus H^{0,2}(X))$ equals $H^{1,1}(X)$.*
(5) *The Fujiki relation holds, i.e. there is a constant $c \in \mathbb{Z}$ which is invariant under locally trivial deformation such that*
$$\int_X \alpha^{2n} = c \cdot q_X(\alpha)^n$$
for any $\alpha \in H^2(X, \mathbb{Z})$.

Proof. We again refer to [BL22, Section 5] and references therein, see in particular Lemma 5.3 and Lemma 5.7. □

Notice that the restriction of the form q_X to $H^{1,1}(X, \mathbb{R})$ has signature $(1, b_2(X) - 3)$ because of Lemma 2.3 (3). Therefore, the cone

$$\{\alpha \in H^{1,1}(X, \mathbb{R}) \mid q_X(\alpha) > 0\}$$

has two connected components.

Definition 2.4. The positive cone $C_X \subset H^{1,1}(X, \mathbb{R})$ of a primitive symplectic variety X is the connected component of the cone $\{\alpha \in H^{1,1}(X, \mathbb{R}) \mid q_X(\alpha) > 0\}$ containing the Kähler cone $\mathcal{K}_X$ of X.

Definition 2.5. For a primitive symplectic variety X, let $\operatorname{Pic}(X)_{\mathbb{R}}$ be the real Picard group $\operatorname{Pic}(X) \otimes \mathbb{R}$. Inside $\operatorname{Pic}(X)_{\mathbb{R}}$, we consider the following cones:

- The *ample cone* $\operatorname{Amp}(X)$ of X is the cone generated by all ample (integral) Cartier divisors on X.
- The *nef cone* $\operatorname{Nef}(X)$ of X is the intersection of the closure of the Kähler cone $\mathcal{K}_X$ with the real Picard group $\operatorname{Pic}(X)_{\mathbb{R}}$.
- The *movable cone* $\operatorname{Mov}(X)$ of X is the cone generated by all movable line bundles (i.e. whose linear system is nonempty and has no fixed part) in $\operatorname{Pic}(X)_{\mathbb{R}}$. We denote by $\overline{\operatorname{Mov}}(X)$ its closure.

Note that in general, the movable cone is neither open, nor closed. Also, our definition of the nef cone is slightly nonstandard, for usually it is defined as the closure of the ample cone. If X is projective, both definitions coincide. If however the ample cone is zero, there can still be nontrivial nef line bundles. As an example, take a primitive symplectic variety of Picard rank one admitting a Lagrangian fibration.

Definition 2.6. Let $N_1(X)_{\mathbb{R}}$ denote the space of 1-cycles (with real coefficients) modulo numerical equivalence. We furthermore define the cone $\operatorname{NE}(X) \subset N_1(X)_{\mathbb{R}}$ to be the cone generated by the classes of effective 1-cycles and let $\overline{\operatorname{NE}}(X)$ denote its closure. The cone $\overline{\operatorname{NE}}(X)$ is called the *Mori cone* of X.

2.2. **Lagrangian Fibrations.** A subvariety Y of a symplectic manifold (X, σ) is *Lagrangian* if $\dim Y = \frac{1}{2} \dim X$ and the restriction of σ to the regular locus Y^{reg} vanishes. This is equivalent to saying that the pullback of σ to a resolution of singularities of Y vanishes. For singular X, one can extend this notion in an obvious way to subvarieties not contained in the singular locus X^{sing}. However, as by definition all our symplectic varieties have rational singularities, we can do better. Thanks to [KS21, Theorem 1.10], there is a functorial pullback for reflexive differentials. Hence, we can define Lagrangian subvarieties in full generality.

Definition 2.7. Let (X, σ) be a symplectic variety. A subvariety $Y \subset X$ is called *Lagrangian* if $\dim Y = \frac{1}{2} \dim X$ and the Kebekus–Schnell pullback of σ to a resolution of singularities of Y vanishes.

The following theorem is due to Matsushita in the smooth case, see [Mat99, Mat01, Mat00, Mat03], respectively Hwang [Hwa08] for the last statement. Subsequently, results for singular varieties were obtained by Matsushita [Mat15] and Schwald [Sch20]. We summarize their results and include a sketch of a proof, also because some of the results hold in greater generality than originally stated.

Theorem 2.8. *Let X be a primitive symplectic variety of dimension $2n$ and let $f : X \to B$ be a surjective holomorphic map with connected fibers to a normal Kähler variety B with $0 < \dim B < 2n$. Then the following holds.*

(1) *The base B is a projective variety with Picard rank $\varrho(B) = 1$, in particular, B is projective and has $\mathbb{Q}$-factorial, log-terminal singularities. Furthermore, $\dim B = n$.*
(2) *The morphism f is equidimensional and each irreducible component of each fiber of f endowed with the reduced structure is a Lagrangian subvariety. The singular locus X^{sing} does not surject onto B.*
(3) *All smooth fibers are abelian varieties.*
(4) *If, in addition, X is irreducible symplectic, then B is Fano. Moreover, if B is smooth, then $B \cong \mathbb{P}^n$.*

Proof. The argument in [Mat03] shows that B is actually projective by first showing that it has log terminal, hence rational singularities, and then that it is Moishezon. As in [Mat01] one shows that the general fiber of f is Lagrangian (and hence a complex torus of dimension n) so that $\dim B = n$. With the argument of [Mat99], one deduces that B is $\mathbb{Q}$-factorial of Picard rank one.

For (2), let $\rho : Y \longrightarrow X$ be a resolution of singularities. By [Kol86, Theorem 2.1] respectively [Sai90, Theorem 2.3, Remark 2.9] in the analytic case, the derived direct images $R^i(f \circ \rho)_* \omega_Y$ are torsion free for $i \geq 0$. As X has rational singularities, we have $R\rho_* \omega_Y = \omega_X$ so that the $R^i f_* \omega_X$ are also torsion free. From there, the proof of equidimensionality and Lagrangeness is essentially the same as [Mat00, Theorem 1].

To see that X^{sing} does not dominate the base, we adapt the proof of Matsushita's "Theorem of Matreshka", see [Mat15, Theorem 3.1]. The crucial point is that the singular locus of a symplectic variety is a Poisson subvariety (for the Poisson structure induced on X by the symplectic form), see [Kal06, Theorem 2.3] and also [BL22, Theorem 3.4] for an adaption to the complex analytic setting. We consider the following diagram

$$\begin{tikzcd} X_1 \arrow[r] \arrow[d, "f_1"'] & X \arrow[d, "f"] \\ B_1 \arrow[r] & B \end{tikzcd}$$

where we denote by X_1 the normalization of X^{sing}, by B_1 the normalization of $f(X^{\mathrm{sing}})$. Then f being Lagrangian implies that pullbacks of functions in $\mathcal{O}_B$ Poisson commute. As X_1 is a Poisson subvariety, the Poisson structure is compatible with restriction and hence also the f_1-pullbacks of functions in $\mathcal{O}_{B_1}$ Poisson-commute. Hence, coordinate functions around a smooth point of B_1 give $\dim B_1$ many linearly independent Hamiltonian vector fields whose action preserves the fibers of f_1. Therefore, the fibers of f_1 have dimension at least $\dim B_1$. In particular, $\dim B_1 < \dim B$, which implies the claim.

From the classical theory of integrable systems, it follows that the smooth fibers are complex tori. The projectivity statement in (3) follows as in the smooth case by Voisin's argument, see [Cam06, Proposition 2.1].

(4): This is essentially identical to the proof of [Mat03, Lemma 2.2]. For the existence of a singular Kähler–Einstein metric on X that is smooth on the regular part X^{reg}, we refer to [EGZ09, Theorem A] and [Pău08, Corollary 1.1]. The last statement has been proven by Hwang ([Hwa08]) if the total space X is smooth and projective building on work of Matsushita ([Mat05]). For singular projective X, this is due to Matsushita [Mat15], see also [Sch20]. Finally, Greb and the second named author treated the case for smooth Kähler total space X in [GL14]. Their argument however works literally the same if X is singular and Kähler. □

Definition 2.9. Let X be a primitive symplectic variety. A map $f : X \longrightarrow B$ as in Theorem 2.8 is called a *(holomorphic) Lagrangian fibration*. A *rational Lagrangian fibration* is a meromorphic map $f : X \dashrightarrow B$ to a normal Kähler variety B such that f has connected fibers[4] and its general fiber is a Lagrangian subvariety of X. We say that a (rational) Lagrangian fibration f is induced by a line bundle L if f is the map associated to $L^{\otimes n}$ for $n \gg 0$.

Remark 2.10. Note that for a rational Lagrangian fibration $f : X \dashrightarrow B$ to be induced by a line bundle L, we do not require L to be movable. The linear system could have a stable base locus. For an elliptic K3 surface $f : S \longrightarrow \mathbb{P}^1$ with a section as in Example 1.3, we denote ℓ the class of a fiber and σ the class of a section. Then f is induced by ℓ, but also by $\ell + \sigma$. The latter has σ as stable base locus.

[4]Recall that a *fiber* of a meromorphic map $f : X \dashrightarrow B$ is the Zariski closure of a fiber of the restriction of f to its domain of definition. In particular, fibers are always compact if X is.

Remark 2.11. It is not even true that every rational Lagrangian fibration is induced by a line bundle at all.

(1) One obvious potential obstruction is non-$\mathbb{Q}$-factoriality. Let $f : X \dashrightarrow B$ be a dominant meromorphic map. Recall that the pullback of a line bundle M on B along a rational map $f : X \dashrightarrow B$ is defined by taking a resolution of indeterminacy

$$\begin{tikzcd} & \tilde{X} \arrow[dl, "\pi"'] \arrow[dr, "\tilde{f}"] & \\ X \arrow[rr, dashed, "f"] & & B \end{tikzcd}$$

and putting

$$f^*M := \left(\pi_*\tilde{f}^*M\right)^{\vee\vee}. \tag{2.1}$$

In general, f^*M is only a reflexive rank one sheaf. If X is $\mathbb{Q}$-factorial, taking a (reflexive) tensor power of this construction gives a line bundle on X. It would be interesting to have an explicit example of a rational Lagrangian fibration $f : X \dashrightarrow B$ where non-$\mathbb{Q}$-factoriality of X obstructs the pullback of an ample line bundle on B to be a ($\mathbb{Q}$-)line bundle.

(2) Even in the $\mathbb{Q}$-factorial case, the line bundle L given by pullback of an ample $A \in \mathrm{Pic}(B)$ along $f : X \dashrightarrow B$ in the sense of (2.1) need not always induce the fibration in the sense of Definition 2.9. By reflexivity, it is clear that f is the map associated to the linear system $f^*|A| = |L|$ (this last equality follows from f having connected fibers), but multiples of L might induce a different map. Consider for example the singular elliptic K3 surface $\bar{S} \dashrightarrow \mathbb{P}^1$ from Example 1.3. We discussed that $f^*\mathcal{O}(1)$ is ample in that case.

Observe that for a *holomorphic* Lagrangian fibration $f\colon X \to B$, the pullback of an ample line bundle A on B satisfies $q_X(f^*A) = 0$. This is a direct consequence of the Fujiki relation Lemma 2.3 (5). The example of the elliptic K3 surface with a section shows that it is better to impose this condition. Then it turns out that (in the projective case), rational Lagrangian fibrations are not that far apart from holomorphic ones.

Lemma 2.12. *Let X be a projective primitive symplectic variety with $b_2(X) \geq 5$. Let L be a movable line bundle on X inducing a rational Lagrangian fibration and satisfying $q_X(L) = 0$. Then there exist a birational map $\phi : X \dashrightarrow X'$ to a primitive symplectic variety X' with $\mathbb{Q}$-factorial terminal singularities and a holomorphic Lagrangian fibration $f' : X' \longrightarrow B$ such that the birational transform of L is the pullback of an ample line bundle on B.*

Proof. By taking a $\mathbb{Q}$-factorial terminalization of X, see [BCHM10, Corollary 1.4.3], and pulling back the line bundle, we may assume that X itself has $\mathbb{Q}$-factorial terminal singularities. By [LMP22b, Theorem 1.2], there is a rational polyhedral fundamental domain for the action of the group of birational automorphisms of X on $\mathrm{Mov}^+(X) :=$

$\overline{\mathrm{Mov}}(X) \cap \mathrm{Pic}(X)_{\mathbb{Q}}$. From the proof, we deduce that there is a rational polyhedral cone $C^+ \subset \mathrm{Mov}^+(X)$ containing L and being contained in the nef cone of a birational model X' of X. As both X' and X have $\mathbb{Q}$-factorial terminal singularities, they are isomorphic in codimension one and the pullback L' of L to X' is still isotropic for the BBF-form on X'. By assumption, the Kodaira–Iitaka dimension $\kappa(L)$ of L is $n := \dim X/2$, hence so is $\kappa(L')$. Being $q_{X'}$-isotropic, also the numerical Kodaira dimension of L' is equal to n, so L' is nef and abundant, and the claim follows from Kawamata's theorem [Kaw85, Theorem 6.1], see also [Fuj11, Theorem 1.1]. □

Remark 2.13. If we have the MMP (that is, termination of flips) at our disposal, we can argue differently in the first part of Lemma 2.12. Indeed, if f is induced by a linear system $|D|$ of a Cartier divisor D on X, one can obtain X' as in the definition by running a log-MMP for (X, Δ) where Δ is a general element in $|D|$. Note that flips terminate if X is smooth by [LP16] or more generally if X has hyperquotient singularities by [LMP22a]. In these cases, we can in particular drop the assumption $b_2(X) \geq 5$.

It is also likely that we can drop the projectivity assumption in Lemma 2.12. For Kähler irreducible symplectic manifolds for example, it also follows from the fact that the birational Kähler cone coincides with the (closure of the) fundamental exceptional chamber, see [Mar11, Theorem 1.5].

2.3. **The SYZ conjecture.** The SYZ conjecture is one of the most important conjectures about primitive symplectic varieties and is wide open in general. Note however that it is known in all known smooth examples, see Remark 2.15 below. Before we state it, let us recall that given a Lagrangian fibration $f\colon X \to B$, the pullback of an ample class A on B satisfies $q_X(f^*A) = 0$ and induces f in the sense of Definition 2.9.

Conjecture 2.14 (SYZ)**.** If L is a nef line bundle on a primitive symplectic variety X with $q_X(L) = 0$, then L induces a holomorphic Lagrangian fibration.

Remark 2.15. In the smooth case, this conjecture is known for deformations of $\mathrm{K3}^{[n]}$ (Bayer–Macrì [BM14, Theorem 1.5]; Markman [Mar14, Theorems 1.3 and 6.3]), for deformations of $K_n(A)$ (Yoshioka [Yos16, Proposition 3.38]), and for deformations of the O'Grady examples OG_6 (Mongardi–Rapagnetta [MR21, Corollary 1.3 and 7.3]) and OG_{10} (Mongardi–Onorati [MO22, Theorem 2.2]).

We will also need a rational version of the SYZ conjecture.

Conjecture 2.16 (Rational SYZ)**.** Let X be a primitive symplectic variety. If a nontrivial line bundle L on X satisfies $q_X(L) = 0$, then L induces a rational Lagrangian fibration $f : X \dashrightarrow B$.

Suppose that L is a nontrivial line bundle on X with BBF-square zero and that the rational SYZ conjecture holds on X so that L that induces a rational Lagrangian fibration. As mentioned in Remark 2.10, the line bundle L need not be movable. Let us take $m \in \mathbb{N}$ sufficiently large that

$$\left|L^{\otimes m}\right| = |M| + F \tag{2.2}$$

where M is a movable divisor and F is the stable base locus of L. We claim that F is negative, more precisely, we have the following result. Recall that a *prime exceptional* divisor on a primitive symplectic variety is a prime Cartier divisor with negative BBF-square.

Lemma 2.17. *Assume that X is a projective primitive symplectic variety with $\mathbb{Q}$-factorial terminal singularities satisfying Conjecture 2.16. In the situation of (2.2), we have that M is a movable line bundle of BBF-square zero and the BBF-intersection matrix of the irreducible components of F is negative definite. In particular, F is a sum of prime exceptional divisors.*

Proof. By assumption, M is movable of Kodaira dimension $\kappa(M) = n := \dim X/2$. By the Fujiki relation (see Lemma 2.3) again, the numerical Kodaira dimension $\nu(L)$ is either equal to n or $2n$, depending on whether $q_X(M) = 0$ or > 0. If $q_X(M) > 0$, by [LMP22b, Proposition 5.8] it would lie in the interior of the birational ample cone, hence be big and nef on some $\mathbb{Q}$-factorial terminal model X' so that in induces a birational contraction. This contradicts (2.2) and we infer $\nu(M) = n$ respectively $q_X(M) = 0$. Consequently, $-q_X(F) = 2q_X(M, F) \geq 0$ as M is movable. If $F \neq 0$, then this term must be different from zero for otherwise F, M would span an isotropic plane in the hyperbolic space $\mathrm{Pic}(X)$ which is absurd.

Suppose that $F \neq 0$ and that there were some $0 < F' \leq F$ with $q_X(F') \geq 0$. We will use that no multiple of F, hence no linear combination of the components of F can move inside its linear system. This immediately excludes the case that $q_X(F') > 0$ because an effective divisor of positive square on a primitive symplectic variety is big, so in particular, multiples have many sections. The remaining case $q_X(F') = 0$ is ruled out by the hypothesis that the rational SYZ conjecture holds on X. $\square$

Given that we use the rational SYZ conjecture only in the last step of the proof, it is likely that this assumption is unnecessary in the above lemma. As we need the SYZ conjecture for our main result, it is however not harmful to impose it here.

The following lemma is not essential for our main result. It nevertheless seems worthwhile clarifying the relation of the rational and the holomorphic version of the SYZ conjecture.

Lemma 2.18. *Let X be a projective primitive symplectic variety of dimension $2n$ with $\mathbb{Q}$-factorial singularities. If $b_2(X) \geq 5$ or X is smooth, then the following statements are equivalent.*

(1) *All primitive symplectic varieties locally trivially deformation equivalent to X satisfy the SYZ conjecture.*

(2) *All primitive symplectic varieties locally trivially deformation equivalent to X satisfy the rational SYZ conjecture.*

Proof. Assume that (2) holds and let L be a nontrivial nef line bundle on X with $q_X(L) = 0$. By assumption, global sections of L give rise to a rational Lagrangian fibration $f : X \dashrightarrow B$. By a standard argument, the Fujiki relation (Lemma 2.3 (5)) implies that $L^{n+1} = 0$ in cohomology while $L^n \neq 0$. If X is projective, it follows from Kawamata's semi-ampleness theorem [Kaw85, Theorem 6.1] that f is regular (as L is nef and abundant). For nonprojective X, we use [Nak87, Theorem 5.5] instead.

Suppose now that (1) holds and let L be a nontrivial line bundle on X with $q_X(L) = 0$. Let (X_t, L_t) be a locally trivial deformation of the pair (X, L) corresponding to a general point t of the Hodge locus of L inside the local Kuranishi space of X. Then $\mathrm{Pic}(X_t)$ is generated by L_t, see [BL22, Proposition 5.5 and Corollary 5.9]. By a forthcoming work of Bakker and the second-named author, the positive cone of X_t equals the Kähler cone in this case. In particular, the line bundle L_t is nef[5] and by (2) the bundle L_t is semi-ample and thus induces a holomorphic Lagrangian fibration on X_t. By semi-continuity of $h^0(X_t, L_t^{\otimes n})$, also for the point $t = 0$ corresponding to X we obtain that $L_0 = L$ has Kodaira dimension n and thus defines a rational Lagrangian fibration. □

Note that we have actually proven (2)⇒(1) without any assumptions on b_2 or singularities and without resorting to deformations. The smoothness hypothesis can be relaxed to having quotient singularities with $\operatorname{codim}_X X^{\mathrm{sing}} \geq 4$ by [Men20, Corollary 5.6]. We believe that the codimension assumption can be dropped if one copies Menet's argument, replacing arbitrary deformations by locally trivial ones.

3. Hyperbolicity

Here we recall some classical hyperbolicity notions that can be found in [Kob76] and [Bro78].

Definition 3.1. Let X be a complex variety. The *Kobayashi pseudometric* on X is the maximal pseudometric d_X such that all holomorphic maps $f\colon (D, \rho) \to (X, d_X)$ are distance decreasing where (D, ρ) is the disk with the Poincaré metric. A variety X is *Kobayashi hyperbolic* if d_X is a metric.

One immediately sees that the complex line $\mathbb{C}$ is not Kobayashi hyperbolic. In fact, the Kobayashi pseudometric of $\mathbb{C}$ vanishes identically. Therefore, the existence of an entire curve (that is, a non-constant holomorphic map from the complex line) implies Kobayashi non-hyperbolicity. The converse holds for compact manifolds.

Theorem 3.2 (Brody)**.**

[5] Recall from Definition 2.5 that this means that $c_1(L)$ is in the closure of the Kähler cone. Hence, this is not a trivial statement.

(1) *Let X be a compact complex space. Then X is Kobayashi non-hyperbolic if and only is there exists an entire curve $\mathbb{C} \to X$.*
(2) *The Kobayashi non-hyperbolicity property is preserved on taking limits.*

Proof. The first statement is due to Brody in the smooth case, see [Bro78, Theorem 4.1], and the argument essentially goes through in the singular case, see e.g. Lang's book [Lan87, III. § 2, Theorem 2.1]. The second statement is [Bro78, Theorem 3.1] in the smooth case. In the singular case, all that is needed is the notion of a length function on a complex space as in [Lan87, Chap. 0]. Using compactness, one can argue that a limit of disks with increasing radii gives a Brody curve, see e.g. [BKV20, Lemma 2.8]. □

A variety admitting no entire curve is sometimes called *Brody hyperbolic*. Brody's theorem thus says that for compact complex varieties, Brody hyperbolicity coincides with Kobayashi hyperbolicity.

Remark 3.3.

(1) Note that Kobayashi hyperbolicity always implies Brody hyperbolicity, also without the hypothesis of compactness. Indeed, as soon as there is an entire curve, the Kobayashi distance between points in its image has to be zero.
(2) While Brody's theorem tells us that the limit of non-hyperbolic compact manifolds is non-hyperbolic, the limit of *hyperbolic* compact manifolds can be either hyperbolic or non-hyperbolic. For an example of hyperbolic manifolds specializing to a non-hyperbolic one, we consider a generic family of degree d hypersurfaces in $\mathbb{P}^n$. For big enough d, these are hyperbolic by the main theorem of [Bro17]. They specialize however to the Fermat hypersurface given by the polynomial $x_0^d + x_1^d + \ldots + x_n^d$ which contains a line as soon as $n \geq 3$.
(3) If we allow singular fibers, it is even easier to obtain a (Brody, hence also Kobayashi) non-hyperbolic variety as the limit of hyperbolic ones. Take for example a family of genus 2 curves that degenerate to a nodal elliptic curve. There, the general fiber is hyperbolic, while the special fiber is not.
(4) If we drop the compactness requirement, then Brody's theorem fails, see [Kob98, Example 3.6.6] for an example of a Kobayashi non-hyperbolic manifold with no entire curves.
(5) Non-hyperbolicity is closed in families of singular varieties by Theorem 3.2(2).

Lemma 3.4. *All varieties are assumed to be compact. Let P be one of the properties "is non-hyperbolic" or "satisfies $d_X = 0$" where d_X is the Kobayashi pseudometric. Then the following hold.*

(1) *Holomorphic maps $f : X \longrightarrow Y$ are distance decreasing for the Kobayashi metric.*
(2) *If $f : X \longrightarrow Y$ is finite, then Y has property P if X does. For finite étale morphisms, the converse holds.*

(3) *Let X, Y be compact varieties and $f : X \dashrightarrow Y$ a dominant meromorphic map. If $d_X = 0$, then also $d_Y = 0$.*

(4) *If $X = X_1 \times \ldots \times X_n$ and X_i has property P for all $i = 1, \ldots, n$, then X has property P.*

(5) *If $f : Y \longrightarrow X$ is a bimeromorphic morphism onto a smooth variety X with $d_X \leq \varepsilon$ for some $\varepsilon \geq 0$, then also $d_Y \leq \varepsilon$.*

Proof. The first four items are standard. For the last item, we use that for a Zariski closed subset $V \subset X$ of codimension ≥ 2, we have

$$(d_X)|_{X \setminus V} = d_{X \setminus V}, \tag{3.1}$$

see [Kob98, Theorem 3.2.19]. □

Note that if in (4) the product X has vanishing Kobayashi metric, then so does every single X_i. However, the product of a hyperbolic and a non-hyperbolic compact manifold is non-hyperbolic, and therefore the statement of Lemma 3.4(3) is not "if and only if".

Example 3.5. Given a surjective morphism $f : X \longrightarrow B$ of complex varieties such that d_X vanishes, then also d_B and the Kobayashi pseudometric of the fibers vanish. The converse however is false due to the presence of multiple fibers, as the following example shows. Let C be a genus two curve and E an elliptic curve. Consider the μ_2 action on $C \times E$ given by the hyperelliptic involution ι on C and translation by 2-torsion point on E. As the action is free, its quotient $X := C \times E/\mu_2$ is smooth and the quotient morphism $\pi : C \times E \longrightarrow X$ is finite étale. In particular, d_X does not vanish by Lemma 3.4. However, the base and the fibers of the morphism $X \longrightarrow C/\iota \cong \mathbb{P}^1$ have vanishing Kobayashi distance. Indeed, if $\Sigma \subset C/\iota$ is the ramification locus, the fiber over a point in the complement of Σ is an elliptic curve (namely E) and the fibers over points of Σ are isomorphic to $\mathbb{P}^1$ with multiplicity 2.

Example 3.6. Let $C \subset \mathbb{P}^2 \subset \mathbb{P}^3$ be a curve of genus ≥ 2 and let $X \subset \mathbb{P}^3$ be the cone over C with vertex $v \notin \mathbb{P}^2$. Let $\pi : Y \longrightarrow X$ be the blow up in v. Then π is a resolution of singularities and the exceptional divisor is E is a section of a $\mathbb{P}^1$-bundle $f : Y \longrightarrow C$. As f is distance decreasing, we see that d_Y cannot vanish identically. On the other hand, X is rationally chain connected, hence $d_X \equiv 0$. This example shows that the vanishing of the Kobayashi pseudometric is not a birational invariant. Also, the quasi-projective variety $Y \setminus E \cong X \setminus \{v\}$ cannot have vanishing Kobayashi pseudometric. This is in stark contrast with the situation for smooth varieties where the Kobayashi pseudometric is determined by its restriction outside a codimension 2 subset, see [Kob98, Theorem 3.2.19].

One can still wonder whether the Kobayashi pseudometric is determined by its restriction outside a codimension 2 subset under some assumptions on the singularities. In concrete terms:

Question 3.7. *Let X be a complex variety with log-terminal (or, more generally, rational) singularities.*

(1) *Let $V \subset X$ be a Zariski closed subset of codimension ≥ 2. Is it true that* (3.1) *still holds?*
(2) *Is it true that if $\pi : Y \longrightarrow X$ is a resolution of singularities, then the vanishing of d_X implies the vanishing of d_Y?*

Note that a positive answer to (1) implies (2). A positive answer to (2) in full generality would simplify our argument. In our main result, we actually make heavy use of birational modifications, see Section 5. However, we are not affected by the above questions as we will have some stronger geometric input.

4. Almost holomorphic maps and Campana's theorem

This section surveys basic notions and results on almost holomorphic maps, the most important of which is undoubtedly Campana's theorem which allows us to produce almost holomorphic maps out of covering families of cycles, see Theorem 4.4. There are no new results, only Theorem 4.6 is a slight adaption from a result of [GLR13] on irreducible symplectic manifolds to the singular case. We begin by collecting basic results about cycle spaces.

4.1. **Cycle Spaces.** Let X be a compact complex space. We denote by $\mathscr{B}(X)$ Barlet's space of cycles on X, see [Bar75]. For a subspace $\mathfrak{F} \subset \mathscr{B}(X)$, we denote $(F_t)_{t\in\mathfrak{F}}$ the analytic family of cycles parametrized by $\mathfrak{F}$. Here, F_t is the cycle corresponding to $t \in \mathfrak{F}$. If F is a cycle on X, we denote by $|F| \subset X$ its support. We will usually drop the word analytic and just speak of a family of cycles.

If $(F_t)_{t\in\mathfrak{F}}$ is a family of cycles, we denote by

$$\Gamma_{\mathfrak{F}} := \{(t,x) \in \mathfrak{F} \times X \mid x \in |F_t|\} \subset \mathfrak{F} \times X$$

its graph, which is an analytic subset in $\mathfrak{F} \times X$ by [GPR94, Ch. VIII, Theorem 2.7]. We say that $\mathfrak{F}$ is a *covering* family of cycles if

$$\bigcup_{t\in\mathfrak{F}} |F_t| = X.$$

The actual definition of an analytic family of cycles is a bit involved, see [Bar75, définition fondamentale, p. 33], but we will not need it here. The Barlet space is the universal object classifying analytic families of cycles in the sense that every such family is obtained by pullback along a uniquely determined classifying map from the universal family of cycles. A very useful tool of how to obtain families of cycles is the following proposition taken from [GPR94, Ch. VIII, Proposition 2.20].

Proposition 4.1. *Let X and S be irreducible compact complex spaces. Then there is a one-to-one correspondence between*

(1) *meromorphic maps $S \dashrightarrow \mathscr{B}(X)$; and*

(2) *pure-dimensional, S-proper cycles F on $S \times X$.*

□

4.2. **Almost holomorphic maps.**

Definition 4.2. For a meromorphic map $f : X \dashrightarrow B$ and a subset $U \subset B$, we denote by $f^{-1}(U)$ the set of points from the domain of definition of f that map to U. The *fiber* of f over $b \in B$ is the closure of $f^{-1}(b)$. A dominant meromorphic map $f : X \dashrightarrow B$ between compact complex varieties is called *almost holomorphic* if there is a dense open subset $U \subset B$ such that $f|_{f^{-1}(U)} : f^{-1}(U) \longrightarrow U$ is holomorphic and proper.

Note that being almost holomorphic can also be phrased by saying that the fibers of f are pairwise disjoint over a dense open set in the target.

An important theorem due to Campana allows us to produce many almost holomorphic maps out of (covering) families of cycles. We need to introduce some terminology in order to formulate it. Let X be a compact Kähler space[6] and suppose we are given a family $\{F_t\}_{t\in\mathfrak{F}}$ of cycles where $\mathfrak{F} \subset \mathscr{B}(X)$ is a closed subspace of the Barlet space. Then, one can define an equivalence relation on X as follows.

Definition 4.3. Two points $x, y \in X$ are $\mathfrak{F}$-*equivalent* (or simply *equivalent* if the family $\mathfrak{F}$ is clear from the context) if they can be connected by a chain of cycles in $\mathfrak{F}$ or if $x = y$. By definition, being *connected by a chain of cycles* in $\mathfrak{F} = \{F_t\}_{t\in\mathfrak{F}}$ means that there exist finitely many points $x = x_1, x_2, \ldots, x_{n+1} = y$ and $t_1, \ldots, t_n \in \mathfrak{F}$ such that $x_i, x_{i+1} \in |F_{t_i}|$ for all $i = 1, \ldots, n$. We write $x \sim_{\mathfrak{F}} y$ (or simply $x \sim y$) to express that x and y are equivalent.

It is clear that $\mathfrak{F}$-equivalence is an equivalence relation. Observe that every $x \in X$ can be connected to itself by a chain of cycles in $\mathfrak{F}$ if and only if the family is covering. We are now able to state Campana's theorem, see [Cam81, Théorème 1].

Theorem 4.4 (Campana). *Let X be a compact complex space which is globally and locally irreducible. Let $\mathfrak{F} \subset \mathscr{B}(X)$ be a closed subspace, let $(F_t)_{t\in\mathfrak{F}}$ be the corresponding family of cycles, and assume that for a general point $t \in \mathfrak{F}$ the cycle F_t is integral. Then there is an almost holomorphic map $f : X \dashrightarrow B$ such that general fibers of f are equivalence classes for the relation of $\mathfrak{F}$-equivalence.*

Note that the statement of the theorem is trivial in case $\mathfrak{F}$ is not a covering family of cycles. In Campana's original result, the subspace $\mathfrak{F}$ was assumed to be irreducible, but this assumption can be removed, see [Cam04, Theorem 1.1]. An algebraic version of Campana's theorem has been obtained by Kollár [Kol87, Theorem 2.6], see also Chapter 5 of [Deb01].

Remark 4.5. The space B from the theorem is constructed in [Cam81] as a subspace of the Barlet space. Therefore, it can be chosen Kähler (respectively Fujiki class) if X

[6]Actually, Fujiki class is sufficient here.

is Kähler (respectively Fujiki class), see [Var86, Théorème 2] or [Var89, Theorem 4'] in the Kähler case and [Cam80, Corollaire 3] for spaces of Fujiki class.

4.3. **Almost holomorphic Lagrangian fibrations.** The following theorem is a slight adaption of [GLR13, Lemma 6.5] for primitive symplectic varieties. Some special attention has to be paid to $\mathbb{Q}$-factoriality and to "horizontal" singularities. We include a sketch of the argument for convenience.

Theorem 4.6. *Let X be a projective primitive symplectic variety, B a projective variety, and $f : X \dashrightarrow B$ be a dominant almost holomorphic map with $0 < \dim B < \dim X = 2n$. Then there is a diagram*

$$\begin{tikzcd} X \arrow[r, dashed] \arrow[d] & X' \arrow[d] \\ B \arrow[r, dashed] & B' \end{tikzcd}$$

where the horizontal maps are birational and X' is a primitive symplectic variety. In particular, $\dim B = n$ and f is a rational Lagrangian fibration.

Proof. Replacing X by a $\mathbb{Q}$-factorialization, see [BCHM10, Corollary 1.4.3], we may assume that X itself has only $\mathbb{Q}$-factorial singularities. In this case, we may define $D := f^*A$ for some very ample Cartier divisor A on B. We choose a rational number $\delta > 0$ small enough such that the pair (X, Δ) with $\Delta := \delta D$ is klt. Note that this is always possible, as a primitive symplectic variety has canonical singularities. We choose a dense open $U \subset B$ such that $f|_{f^{-1}(U)} : f^{-1}(U) \longrightarrow U$ is holomorphic and proper. Then we resolve the indeterminacy of the linear series $|d\Delta|$ for a certain multiple of the boundary divisor, see [GLR13, 6.3.1], by a proper modification $p : \widetilde{X} \longrightarrow X$ which is the identity over $f^{-1}(U)$. One considers the canonical bundle formula for the pair:

$$K_{\widetilde{X}} + p_*^{-1}\Delta = p^*(K_X + \Delta) + F - E$$

where F, E are effective divisors supported on the exceptional locus of p, in particular, in the complement of U. Moreover, $\lfloor E \rfloor = 0$ as (X, Δ) is klt. We set $\widetilde{\Delta} := \Delta + E$ and after possibly shrinking δ further, we may assume that the pair $(\widetilde{X}, \widetilde{\Delta})$ is klt.

As the exceptional locus of p does not dominate B, a general fiber of f has trivial canonical bundle and thus canonical singularities. The restriction of $K_{\widetilde{X}} + \widetilde{\Delta}$ to the general fiber of $f \circ p$ is semi-ample, so in particular the pair $(\widetilde{X}, \widetilde{\Delta}) \times_B U$ is a good minimal model (over U). By [HX13, Theorem 1.1], the pair $(\widetilde{X}, \widetilde{\Delta})$ has a good minimal model (X_m, Δ_m) over B. Let $\psi : \widetilde{X} \dashrightarrow X_m$ denote the corresponding birational map for which $\Delta_m = \psi_*\widetilde{\Delta}$. We may assume that ψ is an isomorphism over U, as can be seen e.g. from [Lai11, Proposition 2.5]. As in [Lai11, Theorem 4.4] and [GLR13, 6.3.3], one shows that (X_m, Δ_m) is actually a minimal model of $(\widetilde{X}, \widetilde{\Delta})$ (i.e. not only over B) and that Δ_m is semi-ample and induces the morphism $X_m \longrightarrow B$. □

Note that in the proof of [Lai11, Theorem 4.4], the morphism $p : \widetilde{X} \longrightarrow X$ was chosen to be a resolution of X and the linear system $|d\Delta|$. This is the only point where we slightly deviate from his argument, as we did not want to affect the fibration over U. As in our case, X has canonical singularities and $\widetilde{\Delta}$ is already semi-ample over U, the argument goes through nevertheless.

Combining Theorem 4.4 and Theorem 4.6, we immediately obtain an almost holomorphic version of Matsushita's theorem.

Theorem 4.7. *Let X be a projective primitive symplectic variety and $\mathfrak{F} \subset \mathscr{B}(X)$ be a closed subspace whose general point corresponds to an integral cycle. Let $f : X \dashrightarrow B$ be an almost holomorphic map whose general fibers are $\mathfrak{F}$-equivalence classes. If $\dim B \notin \{0, \dim X\}$, then f is a rational Lagrangian fibration.* □

5. Non-hyperbolicity of symplectic varieties

The purpose of this section is to prove our main result, Theorem 1.1, which is concerned with the non-hyperbolicity of symplectic varieties and vanishing of the Kobayashi pseudometric. We will indeed prove a slightly stronger version, see Theorem 5.3, and in order to formulate it, we recall the notion of the rational rank of a period. Let Λ be a lattice of signature $(3, n)$ and consider the period domain

$$\Omega_\Lambda := \{[x] \in \mathbb{P}(\Lambda \otimes \mathbb{C}) \mid (x, x) = 0, (x, \bar{x}) > 0\}. \tag{5.1}$$

It parametrizes Hodge structures of weight 2 of primitive symplectic varieties with $(H^2(X, \mathbb{Z}), q_X) \cong \Lambda$. For $p \in \Omega_\Lambda$ we will denote the corresponding Hodge decomposition by

$$\Lambda \otimes \mathbb{C} = H_p^{2,0} \oplus H_p^{1,1} \oplus H_p^{0,2}.$$

Definition 5.1. The *rational rank* of a period $p \in \Omega_\Lambda$ is defined as

$$\mathrm{rrk}(p) := \dim_{\mathbb{Q}} \left(\left(H_p^{2,0} \oplus H_p^{0,2} \right) \cap \Lambda \otimes \mathbb{Q} \right) \in \{0, 1, 2\}.$$

We define the rational rank of a primitive symplectic variety X, denoted $\mathrm{rrk}(X)$, to be the rational rank of its period $\mu_{\mathbb{C}}(H^{2,0}(X))$ after having chosen some *marking*, that is, an isometry $\mu : H^2(X, \mathbb{Z}) \longrightarrow \Lambda$. Note that the rational rank of X does not depend on the choice of marking.

Verbitsky [Ver15, Theorem 4.8], [Ver17, Theorem 2.5] classified the possible orbits under the action of any arithmetic group, see also [BL21, Proposition 3.11].

Theorem 5.2 (Verbitsky)**.** *Assume $\mathrm{rk}(\Lambda) \geq 5$. For $p \in \Omega_\Lambda$ there are three types of orbits of p under the action of $\Gamma := \mathrm{O}(\Lambda)$, depending on the rational rank:*

1. *If $\mathrm{rrk}(p) = 0$, then the orbit is dense, i.e., $\overline{\Gamma \cdot p} = \Omega_\Lambda$;*
2. *If $\mathrm{rrk}(p) = 1$, then $\overline{\Gamma \cdot p}$ is a (countable) union of totally real submanifolds of Ω_Λ of real dimension equal to $\dim_{\mathbb{C}} \Omega_\Lambda$;*
3. *If $\mathrm{rrk}(p) = 2$, then the orbit is closed, i.e., $\overline{\Gamma \cdot p}$ is countable.*

Clearly, a general period has rational rank 0. Periods of rational rank 2 are said to have *maximal Picard rank*.

Theorem 5.3. *Let X be a primitive symplectic variety. Suppose that every primitive symplectic variety which is a locally trivial deformation of X satisfies the rational SYZ conjecture. Then the following hold.*

(1) *If $b_2(X) \geq 5$, then X is non-hyperbolic.*
(2) *If $b_2(X) \geq 5 + \mathrm{rrk}(X)$, then $d_X \equiv 0$.*

The proof of this theorem will occupy the rest of the section. The main geometric tool will be Lagrangian fibrations whose transversality properties we will study next.

Definition 5.4. Let X be a primitive symplectic variety. Two rational Lagrangian fibrations $f_1 : X \dashrightarrow B_1$, $f_2 : X \dashrightarrow B_2$ are said to be *transversal* if the f_i are induced by movable line bundles L_i for $i = 1, 2$ that are non-proportional.

Note that any rational Lagrangian fibration that is induced by a line bundle (i.e. any Lagrangian fibration if X is $\mathbb{Q}$-factorial) is also induced by a movable line bundle, so we can always choose the line bundle to be movable. It is however important to require the *movable* line bundles to be non-proportional, as Example 1.3 shows. Indeed, if we suppose for simplicity that the classes of the fiber ℓ and the section σ generate the Picard group of the elliptic K3 surface, the positive cone is spanned by the two isotropic vectors ℓ and $\ell + \sigma$. The first one is movable, the second is not, and in fact both linear systems induce the same fibration.

Proposition 5.5. *Two rational Lagrangian fibrations $f_1 : X \dashrightarrow B_1$, $f_2 : X \dashrightarrow B_2$ on a primitive symplectic variety X, which are induced by line bundles L_1, L_2, are transversal if the rational map $(f_1, f_2) : X \dashrightarrow B_1 \times B_2$ is generically finite. The converse is true if we assume $b_2(X) \geq 5$.*

Proof. By subtracting the fixed part, we may assume that L_1, L_2 are movable. The fibrations f_1, f_2 being transversal is now equivalent to L_1 and L_2 not being proportional which is in turn equivalent to f_1, f_2 being distinct fibrations (on their domain of definition). Notice that in either case, X is projective. Indeed, if (f_1, f_2) is generically finite, X is Moishezon and hence projective by [Nam02, Theorem 1.6]. If conversely there are two non-proportional vectors on the boundary of the positive cone in $\mathrm{Pic}(X)_{\mathbb{R}}$, their sum must have positive square so that X is projective by the projectivity criterion [BL22, Theorem 6.9].

Let us assume that the f_i are transversal. By replacing X by a bimeromorphically equivalent primitive symplectic variety thanks to Lemma 2.12, we may assume that one of the fibrations is holomorphic. Let $f_1 : X \longrightarrow B_1$ be this fibration. We will show that (f_1, f_2) is generically finite if f_1 and f_2 are distinct. Let A be a general fiber of f_1. As f_1 and f_2 are distinct and of the same relative dimension, the restriction of f_2 to A is nontrivial. As A is abelian, $f_2|_A$ extends to a holomorphic map $f_2 : A \longrightarrow B$ given by (possibly a subsystem of) the linear system of $L_2|_A$. It suffices to show that this linear system is ample. This however follows from the properties of the BBF form. Note that $q_X(L_1 + L_2) > 0$ as both L_i are movable and the positive cone of a hyperbolic lattice does not contain isotropic planes. This property is moreover equivalent by the Fujiki relations to

$$0 < L_1^n.L_2^n = c \cdot A.L_2^n = c \cdot (L_2|_A)^n$$

for some positive constant c. An effective divisor on an abelian variety with positive self-intersection is necessarily ample. In particular, it cannot contract any curve.

The other direction is immediate. □

The main geometric ingredient for the proof of Theorem 5.3 is the following result.

Theorem 5.6. *Let X be a projective primitive symplectic variety with $b_2(X) \geq 5$ and let $f : X \dashrightarrow B$ be a rational Lagrangian fibration induced by a line bundle L on X. If X satisfies the rational SYZ conjecture, then $d_{X'} \equiv 0$ for every compact variety X' birational to X.*

Proof. We will argue by induction on the Picard rank of X. For the inductive argument, we need X to be $\mathbb{Q}$-factorial which we may assume replacing it by a $\mathbb{Q}$-factorialization, see [BCHM10, Corollary 1.4.3]. This may increase the Picard number once, but during the inductive process, we will always remain $\mathbb{Q}$-factorial. By subtracting the fixed part, we may assume that L is movable. The class $[L] \in \operatorname{Pic}(X)$ is isotropic, so taking a rational plane containing it and passing through the interior of the positive cone, one finds a non-proportional isotropic class $[L'] \in \operatorname{Pic}(X)$ whose sign we choose in such a way that $[L']$ lies on the boundary of the positive cone. Note that X has Picard rank at least 2. As X satisfies the rational SYZ conjecture, L' gives rise to a rational Lagrangian fibration $f' : X \dashrightarrow B'$.

The first step in the proof is to show that either X admits a rational Lagrangian fibration that is not almost holomorphic or that X admits two transverse rational Lagrangian fibrations. We may therefore assume that both f and f' are almost holomorphic. If L' is movable, then f and f' are transversal by definition. If $[L'] \notin \overline{\operatorname{Mov}}(X)$, we can write

$$|mL'| = |D| + F$$

as in (2.2), i.e. D is movable and F is the divisorial stable base locus. We infer from Lemma 2.17 that the BBF-intersection matrix of F is negative definite. Let us choose $\Delta = D + F \in |mL'|$ and run a $K_X + \varepsilon\Delta = \varepsilon\Delta$ log-MMP where $\varepsilon > 0$ is chosen such that $(X, \varepsilon\Delta)$ is klt. This leads to a sequence

$$X = X_0 \overset{\phi_0}{\dashrightarrow} X_1 \overset{\phi_1}{\dashrightarrow} X_2 \overset{\phi_2}{\dashrightarrow} \ldots, \tag{5.2}$$

possibly infinite, where the ϕ_i are either flips or divisorial contractions. Denoting $\psi_i := \phi_i \circ \ldots \circ \phi_0$, we consider the rational Lagrangian fibration

$$f_i := f \circ \psi_i^{-1} : X_i \dashrightarrow B.$$

We may suppose that all f_i are almost holomorphic. By the argument of [Dru11, Théorème 3.3], the divisor F has to be contracted at some point in (5.2). If say $X_{j-1} \longrightarrow X_j$ is a divisorial contraction, the composition $f_j : X_j \dashrightarrow B$ is still almost holomorphic. Hence, the $\mathbb{Q}$-line bundle $L_j := (\psi_j)_* f^* L$ still has BBF-square zero (it cannot be positive as it intersects trivially a general curve in the fibers of f_i). step, we used the $\mathbb{Q}$-factoriality of X (which implies that all X_i are $\mathbb{Q}$-factorial as this property is preserved by the MMP). Now the Picard rank of X_j is strictly smaller than that of X and we proceed inductively. Note that if $\operatorname{rk}\operatorname{Pic}(X) = 2$, the $\mathbb{Q}$-line bundle L_j becomes ample on X_j, hence the fibration $f_j : X_j \longrightarrow B$ cannot be almost holomorphic.

In the second step, we will show that if $f : X \dashrightarrow B$ is a Lagrangian fibration which is not almost holomorphic, then X is chain connected by the fibers of f. This essentially is a direct consequence of Campana's theorem. The family $\mathfrak{F}$ of analytic cycles determined by the fibers of f, see Proposition 4.1, gives rise to an almost holomorphic map $g : X \dashrightarrow B$ by Theorem 4.4. Clearly, if f were almost holomorphic, then g and f would coincide up to birational isomorphism. But as f is not almost holomorphic, the fibers of g have dimension $> \dim X/2$. Hence, by Theorem 4.7, the base B must be a point. In particular, X is chain connected by cycles in $\mathfrak{F}$.

As a final step, we deduce the birational vanishing statement for the Kobayashi pseudometric. Let $X' \longrightarrow X$ be a birational map from a compact variety X'. Note that we may always replace X' by a blow up. First suppose that X is birational to a primitive symplectic variety having two transverse rational Lagrangian fibrations. Each of those can be made holomorphic on a birational primitive symplectic variety by Lemma 2.12. By composition and replacing X' by a blow up, we obtain holomorphic $f_i : X' \longrightarrow B_i$ for $i = 1, 2$ with the following properties:

(1) The general fiber of f_1 resp. f_2 are birational to abelian varieties, see Theorem 2.8.
(2) The morphism $(f_1, f_2) : X' \longrightarrow B_1 \times B_2$ is generically finite by Proposition 5.5.

Note that a birational image of an abelian variety always has vanishing Kobayashi distance, see e.g. Lemma 3.4 (5). So by (2) above, each general pair of points on X' can be joined by varieties with vanishing Kobayashi pseudometric. We conclude that $d_{X'} \equiv 0$.

Finally, we treat the case where X is birational to a variety with a not almost holomorphic Lagrangian fibration. As above, we may assume that X itself has such a fibration, say $f : X \dashrightarrow B$. After replacing X' by a suitable blowup, we obtain a resolution of indeterminacy

$$\begin{tikzcd} & X' \arrow[dl, "\pi"'] \arrow[dr, "f'"] & \\ X \arrow[rr, dashed, "f"] & & B \end{tikzcd}$$

where the fibers of f' have vanishing Kobayashi metric and any two general points of X can be joined by a chain of fibers of f. Hence, by [HM07, Corollary 1.5], any two general points in X' can be joined by a chain of varieties which are fibers of f' or rationally chain connected varieties. Notice that whenever $f : X \dashrightarrow B$ is a fibration whose general fibers have vanishing Kobayashi metric, the d_X restricted to *all* fibers of f vanishes. We conclude that $d_{X'} \equiv 0$. $\square$

Now that we have established the vanishing of the Kobayashi pseudometric for primitive symplectic varieties admitting Lagrangian fibrations, we use an ergodicity argument to transport this property to all varieties in the same component of the moduli space. For this, we need the following preliminary consideration. Let X and X' be primitive symplectic varieties, which are equivalent by locally trivial deformations. We choose a marking μ on X and endow X' with a marking μ' that is obtained from the one of X by parallel transport. Let us denote $p := \mu(H^{2,0}(X))$, $p' := \mu'(H^{2,0}(X')) \in \Omega_\Lambda$ the periods of X, X' thus obtained where Λ is a lattice isometric to $(H^2(X,\mathbb{Z}), q_X)$.

Definition 5.7. We say that X *is in the* Mon-*orbit closure of* X' if $p \in \overline{\mathrm{Mon}.p'}$.

Note that this definition does not depend on the choice of μ as long as μ' is chosen as explained above. The following is the analog of [KLV14, Theorem 2.1] in the smooth case. The idea of proof is essentially the same, but for convenience, we spell out the details.

Proposition 5.8. *Let X be a projective primitive symplectic variety with a rational Lagrangian fibration induced by a line bundle. Assume that $b_2(X) \geq 5$ and that the rational SYZ conjecture holds. Then every primitive symplectic variety X' locally trivially deformation equivalent to X such that X is in the* Mon-*orbit closure of X' satisfies $d_{X'} \equiv 0$.*

Proof. Let $\mathscr{X} \longrightarrow \mathrm{Def}^{\mathrm{lt}}(X) =: S$ be the universal deformation of X and let $\pi : \mathscr{Y} \longrightarrow \mathscr{X}$ be a simultaneous resolution of singularities, which exists by [BGL22, Corollary 2.27]. Let us denote by $\pi_0 : Y := \mathscr{Y}_0 \longrightarrow X$ the central fiber and consider the diameter function

$$\mathrm{diam} : S \to \mathbb{R}_{\geq 0}, \quad s \mapsto \mathrm{diam}(\mathscr{Y}_s)$$

for the Kobayashi pseudometric. It was shown in [KLV14, Theorem 2.1] that diam is upper semi-continuous for families of smooth varieties. Hence, for all $\varepsilon > 0$ the sets

$$U_\varepsilon := \{s \in S \mid \mathrm{diam}(\mathscr{Y}_s) < \varepsilon\}$$

are open (and nonempty, as $0 \in U_\varepsilon$ by Theorem 5.6). By the local Torelli theorem [BL22, Proposition 5.5], we can identify S with a small open set in the period domain Ω_Λ where $\Lambda \cong (H^2(X,\mathbb{Z}), q_X)$. Let us consider the action of the monodromy group $\mathrm{Mon} \subset \Gamma$ of X on Ω_Λ. Let X' be as in the statement of the proposition, and let us adopt the notation of Definition 5.7. Then the Mon-orbit of the period $p' \in \Omega_\Lambda$ of X' has the period $p \in \Omega_\Lambda \cap S$ of X in its closure.

The sets U_ε are saturated for the Mon-action in the sense that $(\mathrm{Mon}.U_\varepsilon) \cap S = U_\varepsilon$. In particular, the set $\mathrm{Mon}.p' \cap S$ is contained in U_ε for every $\varepsilon > 0$. It follows that $\mathrm{diam}(\mathscr{Y}_s) = 0$ for all $s \in \mathrm{Mon}.p' \cap S$. For each such s, the Global Torelli theorem

[BL22, Theorem 1.1] implies that X' and $\mathscr{Y}_s$ are bimeromorphic. Let us chose a bimeromorphism $\mathscr{Y}_s \dashrightarrow X'$ and a resolution of indeterminacies, i.e., a diagram

$$\begin{tikzcd} & W \arrow[dl, bend right, "q"'] \arrow[dr, bend left, "p"] & \\ \mathscr{Y}_s \arrow[rr, dashed] & & X \end{tikzcd}$$

where p, q are bimeromorphic morphisms and W is a smooth and compact variety. By item (5) of Lemma 3.4, also W has Kobayashi diameter 0. But p is distance decreasing and surjective, so the same holds for X and the claim follows. □

Proof of Theorem 5.3. We start by proving (2). In view of Proposition 5.8, we need to find a small locally trivial deformation Y of X which is projective, admits a rational Lagrangian fibration, and is contained in the Mon-orbit closure of X. Note that the subgroup $\mathrm{Mon} \subset \Gamma$ has finite index by [BL22, Theorem 1.1], so the analogous trichotomy to Theorem 5.2 holds for orbit closures of Mon.

Let us chose a marking μ on X and let us fix a lattice Λ that is isometric to $(H^2(X,\mathbb{Z}), q_X)$. The assumption $b_2(X) \geq 5 + \mathrm{rrk}(X)$ together Meyer's theorem shows that there is an isotropic class

$$\alpha \in \Lambda \cap \mu\left(H^2(X,\mathbb{Q}) \cap (H^{2,0}(X) \oplus H^{0,2}(X))\right)^{\perp}.$$

If $\mathrm{rrk}(X) = 2$, the class α is of type $(1,1)$ on X itself and the rational SYZ conjecture allows to conclude. If $\mathrm{rrk}(X) = 0$, we first choose a period in $\alpha^\perp$ of rational rank ≤ 1 and hence find a primitive symplectic variety Y in the same component of the marked moduli space realizing that period by[BL22, Theorem 1.1]. We may assume Y to be projective by [BL22, Corollary 6.10] and the assumption on $b_2(X)$. It remains to treat the case where $\mathrm{rrk}(X) = 1$. Let λ be a generator of $H^2(X,\mathbb{Q}) \cap (H^{2,0}(X) \oplus H^{0,2}(X))$. As $\lambda \in \alpha^\perp$, we just need to choose any other $\mu \in \Lambda_\mathbb{R} \cap \alpha^\perp$ for which $\langle \lambda, \mu \rangle$ is a positive 2-space and not rational. This 2-space defines a period of rational rank 1 that is in the orbit closure of X. The sought-for variety Y is again obtained from [BL22, Theorem 1.1] and (2) follows.

Finally, (1) follows from (2) as being non-hyperbolic is closed in families, see Theorem 3.2. □

While primitive symplectic varieties form a large class of singular symplectic varieties, one may wonder whether assuming primitivity is really necessary. The following observation shows that it is indeed superfluous. By a *symplectic variety*, we mean a variety with rational singularities having a symplectic form on the regular part.

Proposition 5.9. *If the Kobayashi pseudometric vanishes for every irreducible symplectic variety, then the same holds true for any compact Kähler symplectic variety.*

Proof. Let X be a compact Kähler symplectic variety. By the decomposition theorem [BGL22, Theorem A], we know that up to a finite quasi-étale cover X is a product of irreducible symplectic varieties and complex tori of even dimension. The finite cover is distance decreasing, so we are reduced to showing the claim separately for tori and irreducible symplectic varieties, see Lemma 3.4. For tori, the claim is obvious and for irreducible symplectic factors the claim holds by assumption. □

6. Applications and Examples

Here we discuss some examples of (orbifold) primitive symplectic varieties with small second Betti numbers b_2. Still, it is possible to show the vanishing of their Kobayashi pseudometrics as they are quotients of primitive symplectic varieties with $b_2 \geq 7$ so that our result applies to the covering variety (if the covering variety has $b_2 \geq 13$, one can of course also use [KLV14]). We also discuss some crepant partial resolutions of these quotients.

Example 6.1. Fu and Menet [FM21, Example 5.2] construct the following quotients based on Mongardi's PhD thesis work [Mon13, Section 4.5]: $M_{11}^i = X_i/\sigma_i$, where X_i are $K3^{[2]}$-type manifolds endowed with special symplectic automorphisms σ_i of order 11, where $i = 1, 2$. Both primitive symplectic orbifolds M_{11}^i have second Betti number $b_2(M_{11}^i) = 3$. Since the Kobayashi pseudometric of X_i vanishes by [KLV14, Remark 1.2, Theorem 1.3], the Kobayashi pseudometric of the quotients M_{11}^i also vanishes by Lemma 3.4.

Similarly, based on Mongardi's PhD thesis work [Mon13, Section 4.4] Fu and Menet [FM21, Example 5.3] construct a quotient $M_7 = X/\sigma$, where X is a $K3^{[2]}$-type manifold endowed with a symplectic automorphism σ of order 7. In this case, $b_2(M_7) = 5$. As above, one concludes that the Kobayashi pseudometric of M_7 vanishes.

Example 6.2. We learned this example from Giovanni Mongardi. Let S be Fermat's quartic $K3$ surface, and let us consider the symmetries coming from the symmetries of its defining equation. The automorphism group of S as a projective variety in $\mathbb{P}^3$ can be written down as $\mathrm{Aut}(S) = (\mathbb{Z}/4\mathbb{Z})^3 \rtimes S_4$, where S_4 is the symmetric group. Not all of these automorphisms preserve the symplectic form of S. The group of symplectic automorphisms of S is the kernel of the natural homomorphism $\mathrm{Aut}(S) = (\mathbb{Z}/4\mathbb{Z})^3 \rtimes S_4 \to \mathbb{C}^*$, which is denoted by F_{384}, and it is a subgroup of order 384 of the Mathieu group M_{24}, see [Muk88]. Let $n \geq 2$, let G be the induced group of symplectic automorphisms of $S^{[n]}$ preserving the degree four polarization, and let $X = S^{[n]}/G$. Then $b_2(X) = 4$ by [Has12], see in particular section 10.3 there (F_{384} is the group number 80 in the list). Since the Kobayashi pseudometric of $S^{[n]}$ vanishes by [KLV14, Remark 1.2, Theorem 1.3], the Kobayashi pseudometric of the quotient X also vanishes by Lemma 3.4.

Example 6.3. Let T be a complex 2-torus equipped with a symplectic automorphism σ_4 of order 4 as constructed by Fu and Menet [FM21, Example 5.4]. Let $K_2(T)$ be the generalized Kummer variety associated to T, and let $\sigma_4^{[2]}$ be the automorphism extending σ_4 on $K_2(T)$. Fu and Menet construct a proper birational map $K_4' \longrightarrow K_2(T)/\sigma_4^{[2]}$ where K_4' is a crepant resolution in codimension 2. The primitive symplectic orbifold K_4' has second Betti number $b_2(K_4') = 6$, by construction is dominated by a blow up of $K_2(T)$ so that its Kobayashi pseudometric vanishes by Lemma 3.4. We use that the Kobayashi pseudometric of $K_2(T)$ vanishes by Theorem 5.3.

The question remains whether the Kobayashi pseudometric also vanishes on all locally trivial deformations of K_4' as the generic such deformation will no longer be birational to a quotient of a generalized Kummer variety. We do not know whether the SYZ conjecture holds for deformations of K_4'. However, instead of Lagrangian fibered varieties, we can use quotients as an input and then argue as in Proposition 5.8. With this modification, the argument of our main result Theorem 5.3 implies that all deformations of K_4' are non-hyperbolic and that all of them except for maybe those with maximal Picard rank have vanishing Kobayashi pseudometric.

Example 6.4. Let X be a projective fourfold of $K3^{[2]}$-type, admitting a symplectic involution ι. The moduli space of such pairs of objects (X, ι) is described in [CGKK21, Section 2 and 3]. The fixed loci of symplectic involutions of $K3^{[2]}$-type manifolds are classified in [Mon12, Theorem 4.1], and more generally, the fixed loci of symplectic involutions of $K3^{[n]}$-type manifolds are classified in [KMO22, Theorem 1.1]. The irreducible symplectic orbifolds $Y \to X/\iota$ obtained as a partial resolution of X/ι for certain fourfolds X of $K3^{[2]}$-type, and for certain symplectic involutions ι are called *Nikulin orbifolds*, see e.g. [CGKK21, Definition 3.1]. Menet [Men15, Theorem 2.5] has computed their integral second cohomology, and $b_2(Y) = 16$. Since the Kobayashi pseudometric of X vanishes by [KLV14, Remark 1.2, Theorem 1.3], the Kobayashi pseudometric of the quotients X/ι and the partial resolutions Y also vanishes by Lemma 3.4.

Note that a general deformation of Y is no longer a partial resolution of the quotient X/ι, so [KLV14] no longer applies. Our main result Theorem 5.3 would guarantee the vanishing of the Kobayashi pseudometric if the rational SYZ conjecture were satisfied. Verifying the (rational) SYZ conjecture for this class of examples seems to be a interesting and valuable task.

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