MAT 536 SPRING 2020 HOMEWORK 2

Problem 1. Assume that the function $f : \mathbb{C} \to \mathbb{C}$ is holomorphic. Let $f = u + iv$ be its decomposition into real and imaginary parts. Show that if $u = v^2$ everywhere, then f is constant.

Problem 2. Let $\{y_n\}$ be an increasing sequence of real numbers such that $y_n \to \infty$. Prove that (Stolz theorem)

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{x_n - x_{n-1}}{y_n - y_{n-1}}$$

if the limit in the right-hand side exists (or equals $\pm\infty$). Show that Problem 1(c) in HW 1 (due to Cauchy), immediately follows from Stolz's theorem.

Problem 3. Let $\{a_n\}$ and $\{b_n\}$ be positive sequences.

(a) Show that

$$\overline{\lim}\, a_n b_n \leq \overline{\lim}\, a_n \, \overline{\lim}\, b_n,$$

provided the right-hand side is not of the indeterminate form $0 \times \infty$. Give an example when strict inequality holds.

(b) If $\lim_{n\to\infty} a_n$ exists, show that

$$\overline{\lim}\, a_n b_n = \lim_{n\to\infty} a_n \, \overline{\lim}\, b_n,$$

if the right-hand side is not of the indeterminate form.

Problem 4. Let $\{a_n\}$ be a positive sequence such that $\lim_{n\to\infty} a_{n+1}/a_n$ exists. Show that $\lim_{n\to\infty} \sqrt[n]{a_n}$ also exists and

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

Problem 5. Give an example of a power series whose radius of convergence is 1, and such that the corresponding holomorphic function is continuous on the closed unit disk $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$.