MAT 536 SPRING 2020 HOMEWORK 1

Problem 1. (a) Let $\{z_n\}$ be a sequence of complex numbers such that
$$|z_m - z_n| \leq \frac{1}{1+|m-n|}$$
for all m and n. What can you say about this sequence in terms of convergence?

(b) Let $\{z_n\}$ be a sequence of complex numbers such that $\lim_{n\to\infty} z_n = 0$ and let $\{w_n\}$ be a bounded sequence. Show that $\lim_{n\to\infty} w_n z_n = 0$.

(c) Let $\{z_n\}$ be a sequence of complex numbers such that $\lim_{n\to\infty} z_n = A$. Prove that
$$\lim_{n\to\infty} \frac{z_1 + \cdots + z_n}{n} = A.$$

Problem 2. Verify the Cauchy-Riemann equations for the function $f(z) = z^3$ by splitting f into real and imaginary parts.

Problem 3. Let $x = r\cos\theta$ and $y = r\sin\theta$. Show that the Cauchy-Riemann equations for $f = u + iv$ in polar coordinates take form
$$r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta} \quad \text{and} \quad r\frac{\partial v}{\partial r} = -\frac{\partial u}{\partial \theta}.$$

Problem 4. Determine the radius of convergence of the following infinite series:
$$\sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad \sum_{n=1}^{\infty} \frac{z^n}{n}, \quad \sum_{n=0}^{\infty} n! z^n.$$

Problem 5. Does there exist a holomorphic function f on $\mathbb{C}$, whose real part is

(a) $u(x,y) = e^x$

(b) $u(x,y) = e^x(x\cos y - y\sin y)$.

If the answer is ‘yes’, exhibit the function, if the answer is ‘no’, prove it.