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COMPLEX STRUCTURES ON RULED SURFACES

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We study the twistor space Z of $\mathbf{D} \times \mathbb{CP}^1$. If the scalar curvature of $\mathbf{D} \times \mathbb{CP}^1$ is zero, then it is known that Z is a complex manifold, so every almost complex structure of $\mathbf{D} \times \mathbb{CP}^1$, compatible with the metric is integrable. Our main result is that the set of all integrable structures of $\mathbf{D} \times \mathbb{CP}^1$ is a real quadric, which we describe explicitly. As a corollary we get the same result for ruled surfaces of genus $g \geq 2$ and of an even degree.

1. Preliminaries. Here we introduce the twistor space of an oriented even dimensional Riemannian manifold M, following the notations of [2], [3] and [5].

Let (M, g) be an oriented connected Riemannian manifold of real dimension $2n$. Let P be the $SO(2n)$-principle bundle of oriented g-orthonormal frames on M. Denote by $\pi : P \to M$ the canonical projection. Then $SO(2n)$ acts on the right on P.

Consider local coordinates $\{x_1, \ldots, x_{2n}\}$ in a nieghborhood U of $x \in M$, and let $\{\theta_1, \ldots, \theta_{2n}\}$ be a local oriented g-orthonormal frame.

The fibre $\pi^{-1}(x)$ is diffeomorphic to $SO(2n)$. Let us denote by $i : \pi^{-1}(x) \to P$ the fibre's inclusion, then

$$\tilde{V_a} = i_{*|a}(T_a \pi^{-1}(x))$$

is called the vertical tangent space at the point a.

Let $\{\theta_1^*, \ldots, \theta_{2n}^*\}$ be the local coframe dual to $\{\theta_1, \ldots, \theta_{2n}\}$, then the covariant derivative ∇ on M defined by the Levi-Civita connection of g is locally expressed by: $\nabla\theta_j = \Gamma_{ij}^k \theta_i^* \otimes \theta_k$, and the Christoffel's symbols satisfy $\Gamma_{ij}^k = -\Gamma_{ik}^j$, so the matrix $(\Gamma_{i\cdot}^{\cdot}) \in so(2n)$. Hence (according to [4]), the Riemannian connection on P at the point $a = (x; \tilde{X}) = (x; (X_j^i))$, induced by g, can be expressed by:

$$\omega_j^i(x; \tilde{X}) = \frac{1}{2}(X_i^r dX_j^r - X_j^r dX_i^r) + X_i^r \Gamma_{ml}^r(x) X_j^l \theta_m^*(x) \tag{1}$$

The connection $\omega = (\omega_j^i)$ on P induces a splitting of the tangent bundle in horizontal and vertical subbundles, at the point a:

$$T_a P = \tilde{H_a} \oplus \tilde{V_a},$$

where

$$\tilde{H_a} := \{Y \in T_a P \mid \omega(Y) = 0\}$$

is the horizontal tangent space at the point a.

$SO(2n)$ acts transitively on $\dfrac{SO(2n)}{U(n)}$. Then we have an action of $SO(2n)$ on $P \times \dfrac{SO(2n)}{U(n)}$, defined by:

$$SO(2n) \times P \times \frac{SO(2n)}{U(n)} \to P \times \frac{SO(2n)}{U(n)}, \qquad (2)$$
$$(A, a, XU(n)) \to (aA, A^{-1}XU(n)).$$

Definition 1. *The twistor space of (M, g) is the associated bundle to P defined as the quotient of $P \times \dfrac{SO(2n)}{U(n)}$ with respect to the action (2). It will be denoted by $Z(M, g)$, or simply by Z, when the manifold (M, g) is understood.*

Z is a bundle over M with fibre $\dfrac{SO(2n)}{U(n)}$ and structure group $SO(2n)$. Denote by $\Pi : P \to Z$ and by $r : Z \to M$ the bundle projections, and by $Z_x := r^{-1}(x)$ the fibre of Z at the point $x \in M$. The geometric meaning of Z is clear from the following:

Theorem 1.1. *Z_x parametrizes the complex structures on T_xM compatible with the metric g and the orientation.*

The splitting $T_aP = \tilde{H_a} \oplus \tilde{V_a}$, via Π induces a corresponding splitting of T_pZ for $p \in Z$. Let $a \in \Pi^{-1}(p)$. Then

$$T_pZ = (\Pi)_{*|a}(\tilde{H_a}) \oplus (\Pi)_{*|a}(\tilde{V_a}) := H_p \oplus V_p.$$

We have immediately that $r_{*|p}(H_p) = T_{r(p)}M$. Then for every $p \in Z$ and for every tangent vector $X \in T_pZ$, we have a decomposition of X in horizontal and vertical component:

$$X = X_h + X_v, X_h \in H_p, X_v \in V_p.$$

Let us define an almost complex structure J on Z: for $X = X_h + X_v \in T_pZ$, we set:

$$J(X) = r^{-1}_{*|r(p)} \circ p \circ r_{*|p}(X_h) + J_V(X_v),$$

where p acts on $T_{r(p)}M$ according to Theorem 1.1, and J_V is the almost complex structure of the symmetric space $Z_{r(p)} \cong \dfrac{SO(2n)}{U(n)}$.

2. Main results. We explore the complex inclusions of $\mathbf{D} \times \mathbb{CP}^1$ in its twistor space and in consequence we get the explicit record of its complex structures.

We use the notations of the previous section. Let $\mathbf{D}$ be the one-dimensional disc equipped with Poincare metric with scalar curvature $-1-\varepsilon$, $(\varepsilon > -1)$, and $\mathbb{CP}^1$ denotes the projective 1-dimensional space with Fubini-Study metric with scalar curvature $+1$. We denote the real local coordinates on $\mathbf{D}$ and $\mathbb{CP}^1$ with x_1, y_1 and x_2, y_2, respectively. According to [8] their metrics are:

$$g_{\mathbf{D}} = \frac{dx_1 \otimes dx_1 + dy_1 \otimes dy_1}{1 - \frac{1+\varepsilon}{4}(x_1^2 + y_1^2)}, \quad g_{\mathbb{CP}^1} = \frac{dx_2 \otimes dx_2 + dy_2 \otimes dy_2}{1 + \frac{1}{4}(x_2^2 + y_2^2)}.$$

We set:

$$A := \frac{1}{1 - \frac{1+\varepsilon}{4}(x_1^2 + y_1^2)}, \quad B := \frac{1}{1 + \frac{1}{4}(x_2^2 + y_2^2)}.$$

We choose $x=(x_1,y_1,x_2,y_2)\in \mathbf{D}\times\mathbb{CP}^1$. An orthonormal frame for $T_x(\mathbf{D}\times\mathbb{CP}^1)$ is

$$\left\{\theta_1=\frac{1}{A}\frac{\partial}{\partial x_1},\quad \theta_2=\frac{1}{A}\frac{\partial}{\partial y_1},\quad \theta_3=\frac{1}{B}\frac{\partial}{\partial x_2},\quad \theta_4=\frac{1}{B}\frac{\partial}{\partial y_2}\right\}$$

and an orthonormal frame for $T^*_x(\mathbf{D}\times\mathbb{CP}^1)$ is

$$\{\theta^*_1=Adx_1,\quad \theta^*_2=Ady_1,\quad \theta^*_3=Bdx_2,\quad \theta^*_4=Bdy_2\}.$$

From the condition $\nabla\theta_j=\Gamma^k_{ij}\theta^*_i\otimes\theta_k$ for the metric of $\mathbf{D}\times\mathbb{CP}^1$, we compute its Christoffel's symbols and therefore we get the following:

Lemma 2.1. *The curvature components of the metric of* $\mathbf{D}\times\mathbb{CP}^1$ *defined above are:*

$$R^1_{212}=\varepsilon A^2,\quad R^3_{434}=\varepsilon B^2,$$

$$R^1_{313}=R^1_{414}=R^2_{323}=R^2_{424}=\varepsilon AB,$$

$R^i_{jkl}=0$, *else, where* $i<j,k<l$.

In order to parametrize the fibre in more convenient way, we use the following:

Lemma 2.2.

$$\frac{SO(4)}{U(2)}\cong\mathbb{CP}^1.$$

Proof. Let $\mathbb{H}\cong\mathbb{R}^4$ be the space of quaternions with a basis $\{\mathbf{1},\mathbf{i},\mathbf{j},\mathbf{k}\}$ over $\mathbb{R}$ with the relations:

$$\mathbf{i}^2=\mathbf{j}^2=\mathbf{k}^2=-\mathbf{1},\ \mathbf{ij}=\mathbf{k}=-\mathbf{ji},\ \mathbf{jk}=\mathbf{i}=-\mathbf{kj},\ \mathbf{ki}=\mathbf{j}=-\mathbf{ik}.$$

The standart complex structure J_2 of $\mathbb{R}^4$ is identified with the left multiplication with $\mathbf{i}$. Let S^3 be the unit sphere in $\mathbb{R}^4$, i.e. the space of unit quaternions. Let $q\in S^3$, $q=a+\mathbf{i}\,b+\mathbf{j}c+\mathbf{k}\,d=z_1+z_2\mathbf{j}$, where $z_1=a+\mathbf{i}b$, $z_2=c+\mathbf{i}d$. We define the following maps:

$$c_1:\mathbb{CP}^1\to S^3,\quad [z_1:z_2]\mapsto q;\quad c_2:S^3\to\mathbb{CP}^1,\quad q\mapsto[z_1:z_2].$$

For every $q\in S^3$ we define a matrix $A_q\in SO(4)$, which corresponds to a left multiplication with the conjugate of q, in other words, $A_qx=\bar{q}\cdot x$, where $x\in\mathbb{H}=\mathbb{R}^4$ and "$\cdot$" is the quaternionic multiplication. Reversly, for every matrix $A\in SO(4)$, we define a unit quaternion q with the same equation. We obtain the following maps:

$$s_1:S^3\to SO(4),\quad q\mapsto A_q;\quad s_2:SO(4)\to S^3,\quad A_q\mapsto q.$$

Note that for $q_1,q_2\in S^3$, $A_{q_1}=A_{q_2}$ if and only if $\bar{q}_1^{-1}\bar{q}_2\in U(2)$

Now we can define the maps:

$$p_1:\mathbb{CP}^1\to\frac{SO(4)}{U(2)},\quad [z_1:z_2]\mapsto A_qU(2), p_1=pr\circ s_1\circ c_1$$

and

$$p_2:\frac{SO(4)}{U(2)}\to\mathbb{CP}^1,\quad A_qU(2)\mapsto[z_1:z_2],\quad p_2=c_2\circ s_2\circ i,$$

where $pr:SO(4)\to\dfrac{SO(4)}{U(2)}$ and $i:\dfrac{SO(4)}{U(2)}\to SO(4)$.

It is easily seen that the maps p_1 and p_2 are correctly defined and they are isomorphisms between $\mathbb{CP}^1$ and $\dfrac{SO(4)}{U(2)}$. $\square$

Remark 1. We associate to A_q the almost complex structure $J_q = A_q^{-1} J_2 A_q$, comming from Theorem 1.1. After some computations we get:

$$J_q = \begin{pmatrix} 0 & -S & 2(ad+bc) & 2(bd-ac) \\ S & 0 & 2(bd-ac) & -2(ad+bc) \\ -2(ad+bc) & 2(ac-bd) & 0 & -S \\ 2(ac-bd) & 2(ad+bc) & S & 0 \end{pmatrix},$$

where $S = a^2 + b^2 - c^2 - d^2$.

In order to compute a local frame for $\tilde{H}_a, a \in \pi^{-1}(U)$, it suffices to compute horizontal lifts $\tilde{\theta}_j$ of θ_j for $1 \leq j \leq 4$. In the notations of the previous paragraph, $\tilde{\theta}_j$ is uniquely determined by: $\pi_{*|a}(\tilde{\theta}_j) = \theta_j(\pi(a))$ and $\omega(a)(\tilde{\theta}_j) = 0$ for all $a \in \pi^{-1}(U)$, where ω is defined by (1). The local frames $\hat{\theta}_j, 1 \leq j \leq 4$, for $H_p, p \in Z$ are the projections of $\tilde{\theta}_j$:

$$\hat{\theta}_j = \Pi_{*|a}(\tilde{\theta}_j).$$

We consider local coordinates $\{u_1, u_2\}$ on the fibre $\mathbb{CP}^1$, where $u := \dfrac{z_2}{z_1}, u = u_1 + \mathbf{i}u_2$. So the local coordinates on the twistor space Z are $\{x_1, y_1, x_2, y_2, u_1, u_2\}$. If we choose local coordinates $\{v_1, v_2\}$ on the fibre $\mathbb{CP}^1$, where $v := \dfrac{z_1}{z_2}, v = v_1 + \mathbf{i}v_2$, then the computations will be analogous. Let $p = (x_1, y_1, x_2, y_2, u_1, u_2) \in Z$. According to Remark 1, we can compute J in the chosen local coordinates, as well as $J\hat{\theta}_j, 1 \leq j \leq 4$.

Now we can compute explicitely all terms of the Njienhuis tensor

$$N(J)(p)(X,Y) = [JX, JY] - J[X, JY] - J[JX, Y] - [X, Y].$$

For convenience we will write $N(X,Y)$ instead of $N(J)(p)(X,Y)$.

Lemma 2.3.

$$N(\hat{\theta}_1, \hat{\theta}_2) = N(\hat{\theta}_3, \hat{\theta}_4) = \frac{8\varepsilon(u_2\frac{\partial}{\partial u_1} - u_1\frac{\partial}{\partial u_2})((A^2+B^2)(u_1^2+u_2^2) - 2AB)}{(1+u_1^2+u_2^2)^2},$$

$$N(\hat{\theta}_1, \hat{\theta}_3) = -N(\hat{\theta}_2, \hat{\theta}_4) = -\frac{4\varepsilon(1+u_1^2-u_2^2)((A^2+B^2)(u_1^2+u_2^2)-2AB)}{(1+u_1^2+u_2^2)^2}\frac{\partial}{\partial u_1} - \\ -\frac{8\varepsilon u_1 u_2((A^2+B^2)(u_1^2+u_2^2)-2AB)}{(1+u_1^2+u_2^2)^2}\frac{\partial}{\partial u_2},$$

$$N(\hat{\theta}_1, \hat{\theta}_4) = N(\hat{\theta}_2, \hat{\theta}_3) = -\frac{8\varepsilon u_1 u_2((A^2+B^2)(u_1^2+u_2^2)-2AB)}{(1+u_1^2+u_2^2)^2}\frac{\partial}{\partial u_1} - \\ -\frac{4\varepsilon(1-u_1^2+u_2^2)((A^2+B^2)(u_1^2+u_2^2)-2AB)}{(1+u_1^2+u_2^2)^2}\frac{\partial}{\partial u_2}.$$

According to the Newlander-Nirenberg Theorem, an almost complex structure is integrable if and only if its Nijenhuis tensor vanishes. We apply it for the horizontal subbundle of Z and get the following result:

Theorem 2.1. *If $\varepsilon = 0$, then Z is a complex manifold, else $\mathbf{D} \times \mathbb{CP}^1$ embeds in Z*

as a complex submanifold with the help of the section

$$\phi : \mathbf{D} \times \mathbb{CP}^1 \to Z,$$

$$(x_1, y_1, x_2, y_2) \mapsto (x_1, y_1, x_2, y_2, u_1, u_2)$$

if and only if

$$u_1^2 + u_2^2 = \frac{2AB}{A^2 + B^2},$$

where A *and* B *are defined above.*

Corollary 2.1.1. *If* $\varepsilon = 0$*, then all almost complex structures of* $\mathbf{D} \times \mathbb{CP}^1$*, compatible with the metric, are integrable, else the set of integrable complex structures is a quadric.*

Remark 2. In the case $\varepsilon \neq 0$, all integrable almost complex structures, compatible with the metric are in the equivalent class with representative of the form: $J = \begin{pmatrix} J_H & 0 \\ 0 & J_V \end{pmatrix}$, where $J_V = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$,

$$J_H = \frac{1}{1 + u_1^2 + u_2^2} \begin{pmatrix} 0 & 1 - u_1^2 - u_2^2 & 2u_2 & -2u_1 \\ u_1^2 + u_2^2 - 1 & 0 & -2u_1 & 2u_2 \\ -2u_2 & 2u_1 & 0 & 1 - u_1^2 - u_2^2 \\ 2u_1 & 2u_2 & u_1^2 + u_2^2 - 1 & 0 \end{pmatrix},$$

and moreover $u_1^2 + u_2^2 = \dfrac{2AB}{A^2 + B^2}$.

Remark 3. It is interesting that the above properties we got do not depend on the exact value of ε, but only on the condition if it is equel or not to zero.

Remark 4. Although the submanifolds $\phi(\mathbf{D} \times \mathbb{CP}^1)$ of Z are the same as smooth manifolds for different values of ε, their induced metrics are different.

Remark 5. Since $\mathbf{D}^1$ is the universal cover of Riemmanian surfaces of genus $g \geq 2$, then Theorem 2.1 is true for every ruled surface of genus $g \geq 2$ and of an even degree (see [6]).

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КОМПЛЕКСНИ СТРУКТУРИ ВЪРХУ ЛИНИРАНИ ПОВЪРХНИНИ

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Изучаваме твисторното пространство Z на $\mathbf{D} \times \mathbb{CP}^1$. Ако скаларната кривина на $\mathbf{D} \times \mathbb{CP}^1$ се анулира, добре известно е, че Z е комплексно многообразие, откъдето всяка почти комплексна структура на $\mathbf{D} \times \mathbb{CP}^1$, съвместима с метриката е интегруема. Основният ни резултат е, че множеството от всички интегруеми структури на $\mathbf{D} \times \mathbb{CP}^1$ е реална квадрика, която описваме в явен вид. Като следствие получаваме същият резултат за линирани повърхнини от род $g \geq 2$ и от четна степен.