MAT544 HW6

Due 11/15 in class. Each problem is worth 10 points

Problem 1. Let $\rho : G \to GL(V)$ be any representation of the finite group G on an n-dimensional complex vector space V, and suppose that for any $g \in G$ the determinant of $\rho(g)$ is 1. Show that the spaces $\Lambda^k V$ and $\Lambda^{n-k} V^*$ are isomorphic as representations of G. [Hint: the bilinear map $\Lambda^k V \otimes \Lambda^{n-k} V \to \Lambda^n V = \mathbb{C}$ is a perfect pairing.]

Problem 2. Show that if $\dim V = 2$, there are isomorphisms

$$\mathrm{Sym}^p(\mathrm{Sym}^q V) \cong \mathrm{Sym}^q(\mathrm{Sym}^p V)$$

of $GL(V)$-representations for all p and q. Note: you could do it for $p = 2, q = 3$ instead of the general case.

Problem 3. For $\mathrm{Sym}^2 V$, verify that

$$\chi_{\mathrm{Sym}^2 V}(g) = \frac{1}{2}[\chi_V(g)^2 + \chi_V(g^2)].$$

Note that this is compatible with the decomposition $V \otimes V = \mathrm{Sym}^2 V \oplus \Lambda^2 V$.

Problem 4. (Fixed-point formula). If V is the permutation representation associated to the action of a group G on a finite set X, show that $\chi_V(g)$ is the number of elements of X fixed by g.