MAT544 HW2

Due 9/20 in class. Each problem is worth 10 points

Problem 1. Use Zorn's lemma to prove that any prime ideal P contains a minimal prime ideal.

Problem 2. Let A be a commutative ring with 1.

(a) Prove that A is local if and only if all the nonunits of A form an ideal.

(b) Let I be a maximal ideal such that $1 + x$ is a unit for every $x \in I$. Prove that A is a local ring.

Problem 3. If (A, m, k) is a local ring, M is a finite A-module, and $s_1, \cdots s_n \in M$, write $\bar{s}_1, \cdots \bar{s}_n \in \bar{M} = M/mM$ for their images under the quotient map. Prove that $\bar{s}_1, \cdots \bar{s}_n$ form a basis of $\bar{M}$ if and only if $s_1, \cdots s_n$ form a minimal generating set of M.

Problem 4. Let $0 \to L \xrightarrow{\alpha} M \xrightarrow{\beta} N \to 0$ be an exact sequence, and M_1, M_2 submodules of M. Decide whether the following implication is always true: $\beta(M_1) = \beta(M_2)$ and $\alpha^{-1}(M_1) = \alpha^{-1}(M_2)$ implies that $M_1 = M_2$.