

Proficiency Algebra in Problems

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1. Numbers, variables, and operations

1.1. Order of operations

Problem 1. Indicate the order of operations and evaluate the following expressions:

a) $-1 + 2 \cdot 3$ b) $-4 \div (3 + 2 \cdot (1 - 2) + 1)$ c) $3 - 2 \cdot (4 + 3 \cdot (-1)) - 3 \div (-1)$

Solution. a) Multiplication ① is performed first, addition ② is performed after:

$$-1 + 2 \overset{\textcircled{1}}{\cdot} 3 = -1 + 6 = 5$$

b) We start with subtraction ① in the inner parentheses, followed by multiplication ②, addition ③ and addition ④. At last, we do division ⑤:

$$-4 \overset{\textcircled{5}}{\div} (3 + 2 \overset{\textcircled{2}}{\cdot} (1 - 2) \overset{\textcircled{1}}{+} 1) = -4 \div (3 + 2 \cdot (-1) + 1) = -4 \div (3 - 2 + 1) = -4 \div 2 = -2$$

c) In the parentheses, do multiplication ① followed by addition ②. Then perform multiplication ③ and division ④. At last, do subtractions ⑤ and ⑥:

$$3 \overset{\textcircled{5}}{-} 2 \overset{\textcircled{3}}{\cdot} (4 + 3 \overset{\textcircled{2}}{\cdot} (-1)) \overset{\textcircled{6}}{-} 3 \overset{\textcircled{4}}{\div} (-1) = 3 - 2 \cdot (4 - 3) - 3 \div (-1) = 3 - 2 \cdot 1 - 3 \div (-1) = 3 - 2 + 3 = 4$$

□

Problem 2. Calculate without calculator:

a) $4 - 2 \div (5 - (2 + 1))$ b) $2 \cdot (3 + 4 \cdot (5 + 6))$ c) $(6 - 3 \cdot (3 \div (-3))) \cdot (-2)$

Problem 3. Indicate the order of operations in the following algebraic expressions:

a) $x + 3 \cdot (2 - x)$ b) $(x - y) \cdot 2 - 2 \cdot (x + 3)$ c) $-x + 3 - (x + 1) \div (x + 3)$

Solution. a) $x + 3 \overset{\textcircled{3}}{\cdot} (2 \overset{\textcircled{2}}{-} x) \overset{\textcircled{1}}{+}$

□

Problem 4. Which of the following algebraic expressions make sense and which don't?

a) $2x$ b) $x2$ c) $2 \cdot x$ d) $x \cdot 2$ e) $-2x$ f) $x \cdot (-2)$ g) $x \cdot -2$ h) $(x) \cdot (-2)$

Solution. Correctly written expressions are a), c), d), e), f), h). Notice, that $2 \cdot x$ and $x \cdot 2$ are usually written as $2x$, and $x \cdot (-2)$ and $(x)(-2)$ are written as $-2x$.

Expression b) is poorly written, it should be rewritten as $x \cdot 2$ or $2x$. Expression g) is incorrect since two operations can't be right next to each other. If the goal is to multiply x by -2 , it should be written as $x \cdot (-2)$ or $-2x$. □

1.2. Parentheses make a difference!

Problem 5. Explain why $1 + 2 \cdot 3 \neq (1 + 2) \cdot 3$

Solution. $1 + 2 \cdot 3 = 1 + 6 = 7$, but $(1 + 2) \cdot 3 = 3 \cdot 3 = 9$. □

Problem 6. Explain why $x + 2 \cdot x \neq (x + 2) \cdot x$

Problem 7. Insert the parentheses according to the indicated order of operations and calculate:

a) $6 \overset{\textcircled{1}}{+} 1 \overset{\textcircled{3}}{-} 5 \overset{\textcircled{2}}{-} 3$ b) $-3 \overset{\textcircled{3}}{-} 2 \overset{\textcircled{1}}{-} 1 \overset{\textcircled{2}}{+} 4 \overset{\textcircled{4}}{-} 6$

Solution. a) $(6 \overset{\textcircled{1}}{+} 1) \overset{\textcircled{3}}{-} (5 \overset{\textcircled{2}}{-} 3) = 7 - 2 = 5$ □

Problem 8. Insert parentheses to get a true statement

a) $1 - 2 - 3 - 4 = 0$ b) $1 - 2 - 3 - 4 = 6$ c) $1 - 2 - 3 - 4 = -8$ d) $1 - 2 - 3 - 4 = -4$

Solution. a) $(1 - 2) - (3 - 4) = 0$ □

1.3. Distributive laws

Problem 9. Explain why $1 + (2 - 3) = 1 + 2 - 3$ but $1 - (2 - 3) \neq 1 - 2 - 3$.

Solution. Let us calculate $1 + (2 - 3)$ performing operations in their order:

$$1 + (2 - 3) = 1 + (-1) = 0.$$

Now calculate $1 + 2 - 3$. Since we have here only one type of operations, we perform them from left to right:

$$1 + 2 - 3 = (1 + 2) - 3 = 3 - 3 = 0.$$

These calculations show that $1 + (2 - 3) = 1 + 2 - 3$.

Let us show now that $1 - (2 - 3) \neq 1 - 2 - 3$. The left hand side is

$$1 - (2 - 3) = 1 - (-1) = 1 + 1 = 2,$$

the right hand side is

$$1 - 2 - 3 = (1 - 2) - 3 = -1 - 3 = -4.$$

Since $2 \neq -4$, then we conclude that $1 - (2 - 3) \neq 1 - 2 - 3$.

Notice that one can't omit parentheses if there is a "−" sign in front on them. □

Problem 10. Verify that

a) $2 \cdot (3 - 4) = 2 \cdot 3 - 2 \cdot 4$ b) $a(b - c) = ab - ac$ (distributive law for subtraction)

Solution. a) Calculate the left hand side: $2 \cdot (3 - 4) = 2 \cdot (-1) = -2$. Now calculate the right hand side: $2 \cdot 3 - 2 \cdot 4 = 6 - 8 = -2$.

b) $a(b - c) = a(b + (-c)) = ab + a \cdot (-c) = ab + (-ac) = ab - ac$. □

Problem 11. Clear parentheses:

a) $-(2 - 3)$ b) $-(x - 3)$ c) $-(3 - x)$ d) $-(a - b)$

Solution. a) $-(2 - 3) = (-1) \cdot (2 - 3) = (-1) \cdot 2 - (-1) \cdot 3 = -2 - (-3) = -2 + 3 = 1$.
Alternatively, $-(2 - 3) = -(-1) = 1$

b) $-(x - 3) = (-1)(x + (-3)) = (-1) \cdot x + (-1) \cdot (-3) = -x + 3$. So $-(x - 3) = -x + 3$.

d) $-(a - b) = (-1) \cdot (a + (-b)) = (-1) \cdot a + (-1) \cdot (-b) = -a + b$.

Remember:

$$-(a - b) = -a + b$$

□

Problem 12. Calculate the value of the following expressions in two different ways: first, by performing operations in their order, and then by clearing the parentheses:

a) $-2 - (3 - 4)$ b) $-2 \cdot (3 - 4) + 5 \cdot (-6 + 7)$ c) $(1 - 2) \cdot 3 - 2 \div (-1)$

Solution. a) Perform operations in their order: $-2 - (3 - 4) = -2 - (-1) = -2 + 1 = -1$.
Clear parentheses: $-2 - (3 - 4) = -2 - 3 + 4 = -5 + 4 = -1$. Notice that the answer is the same. □

Problem 13. Simplify the following expressions (clear parentheses and combine similar terms):

a) $2 + (x - 3)$ b) $2 - (x - 3)$ c) $1 + (2x - 3y - 4)$ d) $1 - (2x - 3y - 4)$

Problem 14. Simplify the following expressions (clear parentheses and combine similar terms):

a) $-2x + 1 - (3x - 4)$ b) $-2x + 1 - 2(3x - 4)$ c) $-2x + 1 - (3x - 4) \cdot 2$
d) $-2x + 1 + (-2) \cdot (3x - 4)$ e) $-2x + 1 - (-2) \cdot (3x - 4)$ f) $-2x + 1 - (3x - 4) \cdot (-2)$

Problem 15. Clear parentheses

a) $a + (b - c - d)$ b) $a - (b - c - d)$ c) $a - (-b - c - d)$

Problem 16. Calculate the value of the following expressions in two different ways: first, by performing operations in their order, and then by distribution:

a) $(1 + 2) \cdot (3 + 4)$ b) $(1 - 2) \cdot (3 + 4)$ c) $(1 - 2) \cdot (3 - 4)$

Solution. a) Perform operations in their order: $(1 + 2) \cdot (3 + 4) = 3 \cdot 7 = 21$. Use distribution: $(1 + 2) \cdot (3 + 4) = 1 \cdot (3 + 4) + 2 \cdot (3 + 4) = 1 \cdot 3 + 1 \cdot 4 + 2 \cdot 3 + 2 \cdot 4 = 3 + 4 + 6 + 8 = 21$.
Observe, the the answer is the same. □

Problem 17. Clear parentheses:

a) $(x + 1)(y - 2)$ b) $(a - 1)(b - c + d)$ c) $(2a - b + 3c)(d - e)$

1.4. Calculation tricks

Problem 18. Use properties of arithmetic operations to calculate in the most efficient way:

a) $35 + 49 + 11 + 15$ b) $5 \cdot 29 \cdot 4$ c) $125 \cdot 92 + 92 \cdot 75$ d) $73 + 28 - (128 + 83)$

Solution. a) $35 + 49 + 11 + 15 = (35 + 15) + (49 + 11) = 50 + 60 = 110$

b) $5 \cdot 29 \cdot 4 = 5 \cdot 4 \cdot 29 = (5 \cdot 2) \cdot (2 \cdot 29) = 10 \cdot 58 = 580$

c) $125 \cdot 92 + 92 \cdot 75 = 92 \cdot (125 + 75) = 92 \cdot 200 = 18,400$

d) $73 + 28 - (128 + 83) = 73 + 28 - 128 - 83 = (73 - 83) + (28 - 128) = -10 - 100 = -110$ □

Problem 19. Compute $1 + 2 + 3 + 4 + \cdots + 98 + 99 + 100$ without calculator.

Solution.

$$\begin{aligned} 1 + 2 + 3 + 4 + \cdots + 98 + 99 + 100 &= \\ (1 + 100) + (2 + 99) + (3 + 98) + \cdots + (49 + 52) + (50 + 51) &= \\ 101 + 101 + 101 + \cdots + 101 + 101 &= 101 \cdot 50 = 5050 \end{aligned}$$

□

Problem 20. Calculate in the most efficient way. Do not use calculator.

a) $34 \cdot 99 - 102 \cdot 34$ b) $103 \cdot 52$ c) $12 \cdot 75 - 63 \cdot 11$

1.5. Fractions, our friends

Problem 21. Reduce to the simplest form (bring to the lowest terms):

a) $\frac{96}{120}$ b) $\frac{26}{35}$ c) $\frac{42}{98}$ d) $\frac{31}{105}$ e) $\frac{84}{54}$ f) $\frac{93}{108}$

Solution. a) $\frac{96}{120} = \frac{\cancel{4} \cdot 24}{\cancel{4} \cdot 3 \cdot 10} = \frac{4 \cdot \cancel{3} \cdot \cancel{2}}{\cancel{3} \cdot \cancel{2} \cdot 5} = \frac{4}{5}$

b) $\frac{26}{35} = \frac{2 \cdot 13}{5 \cdot 7}$. The numerator and denominator have no factors in common, so the fraction is *irreducible*. □

Problem 22. Reduce to the simplest form:

a) $\frac{2x+6}{4x}$ b) $\frac{-3x}{9x-6}$ c) $\frac{x-1}{1-x}$ d) $\frac{6x-4}{2-3x}$

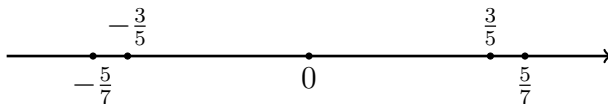
Solution. a) $\frac{2x+6}{4x} = \frac{2(x+3)}{4x} = \frac{\cancel{2}(x+3)}{\cancel{2} \cdot 2x} = \frac{x+3}{2x}$ □

Problem 23. Explain why the calculation $\frac{\cancel{2}}{1+\cancel{2}x} = \frac{1}{1+x}$ is **incorrect**.

Problem 24. Show the locations of the numbers $\frac{3}{5}$, $-\frac{3}{5}$, $\frac{5}{7}$, $-\frac{5}{7}$ on a number line.

Solution. Numbers $\frac{3}{5}$ and $\frac{5}{7}$ are positive, so they are located to the right from 0 on a number line. Numbers $-\frac{3}{5}$ and $-\frac{5}{7}$ are negative, so they are located to the left from 0. Let us compare the fractions $\frac{3}{5}$ and $\frac{5}{7}$ by bringing them to a common denominator:

$$\frac{3}{5} = \frac{3 \cdot 7}{5 \cdot 7} = \frac{21}{35} \quad \text{and} \quad \frac{5}{7} = \frac{5 \cdot 5}{7 \cdot 5} = \frac{25}{35}. \quad \text{Therefore, } \frac{3}{5} < \frac{5}{7} \quad \text{and} \quad -\frac{3}{5} > -\frac{5}{7}.$$



□

Problem 25. Compare (insert “<”, “>” or “=”) the following fractions. Justify your answer.

$$\text{a) } \frac{5}{9} \text{ — } \frac{8}{15} \quad \text{b) } -\frac{5}{8} \text{ — } -\frac{19}{28} \quad \text{c) } -\frac{9}{108} \text{ — } \frac{13}{109}$$

Problem 26. Explain why we do **not** add fractions like this: $\frac{a}{b} + \frac{c}{d} = \frac{a+c}{b+d}$.

Solution. This “rule” gives contradictory results. Indeed, for $a = 1$, $b = 2$, $c = 1$ and $d = 2$ we get $\frac{a}{b} + \frac{c}{d} = \frac{1}{2} + \frac{1}{2} = 1$ and $\frac{a+c}{b+d} = \frac{1+1}{2+2} = \frac{2}{4} = \frac{1}{2}$. Since $1 \neq \frac{1}{2}$, $\frac{1}{2} + \frac{1}{2} \neq \frac{1+1}{2+2}$. □

Problem 27. Perform the following operations. Give your answer in the simplest form.

$$\text{a) } \frac{3}{8} + \frac{5}{8} \quad \text{b) } \frac{7}{12} - \frac{5}{6} \quad \text{c) } -\frac{4}{11} + \frac{8}{7} \quad \text{d) } \frac{4}{3} \cdot \frac{5}{6} \quad \text{e) } \frac{24}{55} \cdot \left(-\frac{33}{40}\right)$$

$$\text{f) } -\frac{7}{9} \div \frac{35}{24} \quad \text{g) } -\frac{10}{3} \div \left(-\frac{5}{18}\right) \quad \text{h) } \frac{\frac{3}{4} + \frac{1}{5}}{\frac{3}{2} \cdot \frac{1}{3}}$$

$$\text{Solution. a) } \frac{3}{8} + \frac{5}{8} = \frac{3+5}{8} = \frac{8}{8} = 1$$

$$\text{b) } \frac{7}{12} - \frac{5}{6} = \frac{7-5 \cdot 2}{12} = \frac{-3}{12} = -\frac{1}{4}$$

$$\text{c) } -\frac{4}{11} + \frac{8}{7} = \frac{-4 \cdot 7 + 8 \cdot 11}{11 \cdot 7} = \frac{-28 + 88}{77} = \frac{60}{77}$$

$$\text{d) } \frac{4}{3} \cdot \frac{5}{6} = \frac{\cancel{2} \cdot 2 \cdot 5}{3 \cdot \cancel{2} \cdot 3} = \frac{10}{9}$$

$$\text{e) } \frac{24}{55} \cdot \left(-\frac{33}{40}\right) = -\frac{\cancel{8} \cdot 3 \cdot \cancel{11} \cdot 3}{\cancel{11} \cdot 5 \cdot \cancel{8} \cdot 5} = -\frac{9}{25}$$

$$\text{f) } -\frac{7}{9} \div \frac{35}{24} = -\frac{7}{9} \cdot \frac{24}{35} = -\frac{7 \cdot 24}{9 \cdot 35} = -\frac{\cancel{7} \cdot 8 \cdot \cancel{3}}{\cancel{3} \cdot 3 \cdot \cancel{7} \cdot 5} = -\frac{8}{15}$$

$$\text{g) } -\frac{10}{3} \div \left(-\frac{5}{18}\right) = \frac{\cancel{10}^2}{\cancel{3}_1} \cdot \frac{\cancel{18}^6}{\cancel{5}_1} = \frac{12}{1} = 12$$

$$\text{h) } \frac{\frac{3}{4} + \frac{1}{5}}{\frac{3}{2} \cdot \frac{1}{3}} = \left(\frac{3}{4} + \frac{1}{5}\right) \div \left(\frac{3}{2} \cdot \frac{1}{3}\right) = \frac{3 \cdot 5 + 1 \cdot 4}{4 \cdot 5} \div \frac{3 \cdot 1}{2 \cdot 3} = \frac{19}{20} \cdot \frac{2}{1} = \frac{19}{10}$$

□

Problem 28. Perform the following operations. Give your answer in the simplest form.

$$\text{a) } \frac{1}{2} + \frac{2}{5} \quad \text{b) } 1 - \frac{5}{6} \quad \text{c) } \frac{6}{35} \cdot \left(-\frac{25}{24}\right) \quad \text{d) } \frac{3 - \frac{7}{2}}{\frac{2}{3} + \frac{3}{4}}$$

Problem 29. Perform the following operations. Give the answer in a fractional form:

$$\text{a) } x + \frac{x}{2} \quad \text{b) } \frac{x}{2} - \frac{x}{3} \quad \text{c) } 1 + \frac{1}{x} \quad \text{d) } -2 + \frac{x}{x+1} \quad \text{e) } -x - \frac{x-1}{4}$$

$$\text{Solution. a) } x + \frac{x}{2} = \frac{2x+x}{2} = \frac{3x}{2}$$

$$\text{b) } \frac{x}{2} - \frac{x}{3} = \frac{3x-2x}{6} = \frac{x}{6}$$

$$\text{c) } 1 + \frac{1}{x} = \frac{x+1}{x}$$

$$\text{d) } -2 + \frac{x}{x+1} = \frac{-2(x+1)+x}{x+1} = \frac{-2x-2+x}{x+1} = \frac{-x-2}{x+1}$$

$$\text{e) } -x - \frac{x-1}{4} = \frac{-x \cdot 4 - (x-1)}{4} = \frac{-4x-x+1}{4} = \frac{-5x+1}{4}$$

□

Problem 30. Perform the operations. Give the answer in a fractional form:

$$\text{a) } \frac{x+1}{x} \cdot \frac{x}{2} \quad \text{b) } \frac{x}{2} \div \frac{x}{3} \quad \text{c) } 1 - x \cdot \frac{1}{x} \quad \text{d) } \frac{2x+2}{x} \cdot \frac{x}{x+1} \quad \text{e) } -x \div \frac{x-1}{4}$$

Problem 31. Present the decimals as common fractions:

$$\text{a) } 1.3 \quad \text{b) } 0.4 \quad \text{c) } 3.25 \quad \text{d) } 0.5 \quad \text{e) } 0.075$$

$$\text{Solution. a) } 1.3 = 1 + \frac{3}{10} = \frac{10+3}{10} = \frac{13}{10}$$

□

Problem 32. Indicate the order of operations and evaluate the following expressions:

$$\text{a) } \frac{2}{3} - \frac{1}{9} \div \frac{3}{4} + 2 \cdot \left(1 - \frac{1}{4}\right) \quad \text{b) } \frac{3}{4} + \frac{1}{4} \cdot \left(-\frac{1}{3} + 1.3\right)$$

Problem 33. Perform the operations. Give the answer in a fractional form:

$$\text{a) } \frac{2x}{3} - \frac{1}{9} \div \frac{3}{4x} + 2 \left(x - \frac{1}{4}\right) \quad \text{b) } \frac{3}{4x} + \frac{1}{4} \cdot \left(-\frac{1}{3x} + 1.3\right)$$

1.6. Powers: positive integer exponents

Problem 34. Explain why $2^3 \neq 2 \cdot 3$

Solution. $2^3 = 2 \cdot 2 \cdot 2 = 8$, while $2 \cdot 3 = 2 + 2 + 2 = 6$. □

Problem 35. Rewrite the following expressions using power notation

a) $x \cdot x \cdot x \cdot x$ b) $x \cdot x \cdot x \cdot x \cdot x \cdot x$ c) $a \cdot a$ d) $\frac{1}{x} \cdot \frac{1}{x} \cdot \frac{1}{x}$ e) $\frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3}$

Problem 36. Explain why $(2x)^3 = 8x^3$.

Solution. $(2x)^3 = (2x) \cdot (2x) \cdot (2x) = (2 \cdot 2 \cdot 2)(x \cdot x \cdot x) = 8x^3$. □

Problem 37. Calculate without calculator:

a) 2^5 b) $(-2)^5$ c) $\left(\frac{1}{2}\right)^5$ d) 0^5 e) 1^5 f) $(-1)^5$ g) $\left(\frac{2}{3}\right)^5$ h) $\left(-\frac{2}{3}\right)^5$

Solution. a) $2^5 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 32$

b) $(-2)^5 = (-2) \cdot (-2) \cdot (-2) \cdot (-2) \cdot (-2) = -32$

c) $\left(\frac{1}{2}\right)^5 = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{32}$

d) $0^5 = 0 \cdot 0 \cdot 0 \cdot 0 \cdot 0 = 0$

e) $1^5 = 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 = 1$

f) $(-1)^5 = (-1) \cdot (-1) \cdot (-1) \cdot (-1) \cdot (-1) = -1$

g) $\left(\frac{2}{3}\right)^5 = \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} = \frac{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}{3 \cdot 3 \cdot 3 \cdot 3 \cdot 3} = \frac{32}{243}$

h) $\left(-\frac{2}{3}\right)^5 = \left(-\frac{2}{3}\right) \cdot \left(-\frac{2}{3}\right) \cdot \left(-\frac{2}{3}\right) \cdot \left(-\frac{2}{3}\right) \cdot \left(-\frac{2}{3}\right) = -\frac{32}{243}$ □

Problem 38. Find the value of the expression x^3 if

a) $x = 1$ b) $x = -2$ c) $x = \frac{1}{2}$

Problem 39. Explain why $(-2)^4 \neq -2^4$

Solution. $(-2)^4 = (-2) \cdot (-2) \cdot (-2) \cdot (-2) = 16$, while $-2^4 = -(2 \cdot 2 \cdot 2 \cdot 2) = -16$. □

Problem 40. Calculate without calculator:

a) 1^{2015} b) $(-1)^{2015}$ c) $(-1)^{2016}$ d) -1^{2016} e) -1^{2015} f) $(-1)^0$ g) -1^0

Problem 41. a) Let $x = 2$. Find $(-x)^2$ and $-x^2$.

b) Let $x = -2$. Find $(-x)^2$ and $-x^2$.

c) Let $x = 0$. Find $(-x)^2$ and $-x^2$.

d) Explain why $(-x)^2 \neq -x^2$ for an arbitrary value of x .

Problem 42. Explain why $(-x)^3 = -x^3$ for any value of x .

Problem 43. Prove that $(-x)^n = x^n$ if n is even, and $(-x)^n = -x^n$ if n is odd.

1.7. Powers: negative integer exponents

Problem 44. Explain why $2^{-3} \neq -8$

Solution. $2^{-3} = \frac{1}{2^3} = \frac{1}{8} \neq -8.$ □

Problem 45. Explain why $2x^{-3} \neq \frac{1}{2x^3}.$

Problem 46. a) Is it true that $(-x)^{-5} = -\frac{1}{x^5}$? Justify your answer.

b) Is it true that $(-x)^{-4} = -\frac{1}{x^4}$? Justify your answer.

Problem 47. Calculate without calculator:

a) 2^{-5} b) $(-2)^{-5}$ c) $\left(\frac{1}{2}\right)^{-5}$ d) $\left(-\frac{1}{2}\right)^{-5}$ e) 1^{-5} f) $(-1)^{-5}$

Solution. a) $2^{-5} = \frac{1}{2^5} = \frac{1}{32}$

b) $(-2)^{-5} = \frac{1}{(-2)^5} = \frac{1}{-32} = -\frac{1}{32}$

c) $\left(\frac{1}{2}\right)^{-5} = \frac{1}{(1/2)^5} = \frac{1}{1/32} = 32$

d) $\left(-\frac{1}{2}\right)^{-5} = \frac{1}{(-1/2)^5} = \frac{1}{-1/32} = -32$

e) $1^{-5} = \frac{1}{1^5} = \frac{1}{1} = 1$

f) $(-1)^{-5} = \frac{1}{(-1)^5} = \frac{1}{-1} = -1$ □

Problem 48. Calculate without calculator:

a) $\left(\frac{2}{3}\right)^{-4}$ b) $\left(-\frac{2}{3}\right)^{-4}$ c) $\left(\frac{2}{3}\right)^{-5}$ d) $\left(-\frac{2}{3}\right)^{-5}$ e) $\left(\frac{2}{3}\right)^0$ f) $\left(-\frac{2}{3}\right)^0$

Problem 49. Calculate without calculator:

a) $3^{-2} - 2^{-3}$ b) $(-3)^{-3} \cdot (-2)^{-2}$ c) $\left(\frac{3}{2}\right)^{-1} - \left(\frac{1}{4}\right)^{-2} + \left(\frac{1}{2}\right)^{-3}$ d) $2^{(2^3)}$ e) $(2^2)^3$

1.8. Exponential rules

Problem 50. Explain why $2^2 \cdot 2^3 \neq 2^6$.

Solution. $2^2 \cdot 2^3 = (2 \cdot 2) \cdot (2 \cdot 2 \cdot 2) = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^5 \neq 2^6$. □

Problem 51. Explain why $2^2 + 2^3 \neq 2^5$.

Solution. $2^2 + 2^3 = (2 \cdot 2) + (2 \cdot 2 \cdot 2) = 4 + 8 = 12 \neq 2^5 = 32$. □

Problem 52. Calculate using exponential rules:

a) $2^4 \cdot 2^5$ b) $2^4 \cdot 2^{-5}$ c) $2^{-4} \cdot 2^{-5} \cdot 2^3$ d) $(-2)^4 \cdot 2^{-5} \cdot (-2)^3$ e) $\left(\frac{1}{2}\right)^4 \cdot 2^{-5}$

Solution. a) $2^4 \cdot 2^5 = 2^{4+5} = 2^9 = 512$

d) $(-2)^4 \cdot 2^{-5} \cdot (-2)^3 = 2^4 \cdot 2^{-5} \cdot (-2)^3 = -2^{4-5+3} = -2^2 = -4$ □

Problem 53. Simplify and leave your answer in power form:

a) $(2^3)^5$ b) $(2^{-3})^5$ c) $(2^3 \cdot 3^2)^5$ d) $(2^{-3} \cdot 3^{-2})^{-5}$ e) $\left(\left(\frac{3}{5}\right)^5 \cdot 3^{-4}\right)^{-3}$

Problem 54. Simplify

a) $x^3 \cdot x^4$ b) $x^{-3} \cdot x^4$ c) $x^{-3} \cdot x^{-4}$ d) $(-x)^3 \cdot x^4$ e) $(-x)^3 \cdot (-x)^4$

Problem 55. Simplify

a) $(2ab)^2$ b) $(-2ab)^2$ c) $(-2ab)^3$ d) $(2a^3b)^2$ e) $(-2a^3b)^2$ f) $(-2a^3b)^3$

Problem 56. a) $(2ab)^2 = 2^2 a^2 b^2 = 4a^2 b^2$

f) $(-2a^3b)^3 = (-2)^3 (a^3)^3 b^3 = -8a^9 b^3$

Problem 57. Simplify and leave your answer in power form:

a) $(2x^3)^4 \cdot (4x^3)^{-2}$ b) $\left(\frac{1}{3}x^{-2}\right)^3$ c) $\left(\frac{1}{3}x^{-2}\right)^{-3}$ d) $(x^2y^3)^4$ e) $(x^{-2}y^3)^{-4}$

Problem 58. Simplify and leave your answer in power form:

a) $10^3 \cdot 15^4$ b) $24^2 \cdot 27^5$ c) $(-30)^7 \cdot 8^{10}$

Solution. a) $10^3 \cdot 15^4 = (2 \cdot 5)^3 \cdot (3 \cdot 5)^4 = 2^3 \cdot 5^3 \cdot 3^4 \cdot 5^4 = 2^3 \cdot 3^4 \cdot 5^{3+4} = 2^3 \cdot 3^4 \cdot 5^7$ □

Problem 59. Explain why $2^6 \div 2^3 \neq 2^2$

Solution. $2^6 \div 2^3 = \frac{2^6}{2^3} = \frac{\cancel{2} \cdot \cancel{2} \cdot \cancel{2} \cdot 2 \cdot 2 \cdot 2}{\cancel{2} \cdot \cancel{2} \cdot \cancel{2}} = \frac{2 \cdot 2 \cdot 2}{1} = 2^3 \neq 2^2$. □

Problem 60. Calculate using exponential rules:

a) $2^{100} \div 2^{95}$ b) $2^{-2} \div 2^{-3}$ c) $2^2 \div 2^{-3}$ d) $\left(\frac{2}{3}\right)^{67} \div \left(\frac{2}{3}\right)^{64}$ e) $\left(\frac{2}{3}\right)^{-67} \div \left(\frac{2}{3}\right)^{-64}$

Solution. a) $2^{100} \div 2^{95} = \frac{2^{100}}{2^{95}} = 2^{100-95} = 2^5 = 32$ □

Problem 61. Simplify

a) $\frac{x^4}{x^7}$ b) $\frac{x^{-4}}{x^7}$ c) $\frac{x^{-4}}{x^{-7}}$ d) $\frac{x^4}{(-x)^7}$ e) $\left(\frac{x^4}{x^7}\right)^3$ f) $\left(\frac{x^4}{x^7}\right)^{-3}$

Solution. a) $\frac{x^4}{x^7} = x^{4-7} = x^{-3}$. Alternatively, $\frac{x^4}{x^7} = \frac{1}{x^{7-4}} = \frac{1}{x^3}$. □

Problem 62. Let $n^5 = 243$. Find 5^n .

Problem 63. Which number is larger, 10^{-3} or 2^{-10} ?

Problem 64. Using no calculator, put the following numbers in increasing order:

a) 2^{24} , 3^{18} , 4^{15} , 5^6 b) 2^{55} , 3^{44} , 5^{33} , 6^{22}

Solution. a) $2^{24} = (2^4)^6 = 16^6$, $3^{18} = (3^3)^6 = 27^6$, $4^{15} = (2^2)^{15} = 2^{30} = (2^5)^6 = 32^6$.

Since $5^6 < 16^6 < 27^6 < 32^6$, then $5^6 < 2^{24} < 3^{18} < 4^{15}$. □

Problem 65. Let $\frac{2^{2015}}{2^n} = \frac{1}{32}$. Find n .

2. Algebraic expressions

2.1. What is an algebraic expression?

Problem 66. Which of the following are algebraic expressions:

a) $3x + 1$ b) $2x + 1 = 4$ c) $\frac{x^2 - 1}{2x + 3}$ d) $xy - y^2$ e) $x(2(x + 1) - 3(x^2 + 1))$

f) $-x^2 + 3x \geq 1$ g) $(x + 2x)$ h) $(x + 2) - 1 \longrightarrow x + 1$ i) $x + 1 \neq x$ j) $\frac{2}{\frac{3}{4}}$

Problem 67. Find the value of the expression $1 - 2x$ when x is equal to

a) 0 b) 1 c) -1 d) $\frac{3}{5}$ e) a f) $a + 1$ g) $1 - a$ h) a^2 i) $\frac{a + 1}{a - 1}$

Solution. a) $(1 - 2x) \Big|_{x=0} = 1 - 2 \cdot 0 = 1$

c) $(1 - 2x) \Big|_{x=-1} = 1 - 2 \cdot (-1) = 1 + 2 = 3$

f) $(1 - 2x) \Big|_{x=a+1} = 1 - 2 \cdot (a + 1) = 1 - 2a - 2 = -1 - 2a$

i) $(1 - 2x) \Big|_{x=\frac{a+1}{a-1}} = 1 - 2 \cdot \frac{a+1}{a-1} = \frac{(a-1) - 2(a+1)}{a-1} = \frac{a-1-2a-2}{a-1} = \frac{-3-a}{a-1} = \frac{3+a}{1-a}$ \square

Problem 68. Find the value of the expression $\frac{-x^2 + 2}{2x + 1}$ when x is equal to

a) 0 b) 1 c) -1 d) $\frac{1}{2}$ e) a f) $a + 1$ g) $1 - a$ h) a^2 i) $\frac{3}{a}$

Problem 69. Find the value of the expression $\frac{x + y^2}{x^2 y}$ if $x = -\frac{1}{2}$ and $y = 2$.

2.2. Composing algebraic expressions from word descriptions

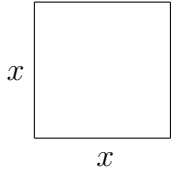
Problem 70. The length of a side of a square is x length units.

a) Compose an algebraic expression (in terms of x) for the perimeter of the square.

b) Find the value of the expression in a) for $x = 3$ cm.

c) Compose an algebraic expression (in terms of x) for the area of the square.

d) Find the value of the expression in c) for $x = 4$ cm.

Solution.

a) The perimeter is the sum of the lengths of all sides. If a square has a side of length x , then the perimeter of the square is $x + x + x + x = 4x$. Answer: the perimeter is $4x$ length units.

b) Plug in $x = 3$ cm in the expression for the perimeter:
 $4x \Big|_{x=3} = 4 \cdot 3 = 12x$ (cm). Answer: the perimeter is 12 cm.

c) The area of the square with a side of length x is x^2 (square unit lengths). Answer: the area is x^2 (square unit lengths).

d) Plug in $x = 4$ cm in the expression for the area: $x^2 \Big|_{x=4} = 4^2 = 16$ (cm²). Answer: the area is 16 cm². □

Problem 71. One side of a rectangle is x cm, the other side is 4 cm longer.

a) Compose an algebraic expression showing the perimeter of the rectangle. Find the value of this expression if $x = 3$.

b) Compose an algebraic expression showing the area of the rectangle. Find the value of this expression if $x = 3$.

Problem 72. One side of a rectangle is x cm, the other side is 4 times longer.

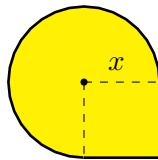
a) Compose an algebraic expression showing the perimeter of the rectangle. Find the value of this expression if $x = 3$.

b) Compose an algebraic expression showing the area of the rectangle. Find the value of this expression if $x = 3$.

Problem 73. In a triangle ABC , the side AB has length of x inches. The side BC is twice as long as AB . The side AC is 2 inches longer than BC . Compose an algebraic expression showing the perimeter of the triangle. Find the value of this expression if $x = 5$.

Problem 74. The width of a rectangular aquarium is x inches, the length is twice as long as the width, and the height is 3 inches more than the width. Compose an algebraic expression showing the volume of the aquarium.

Problem 75. Find an algebraic expression for the area of the shape below:



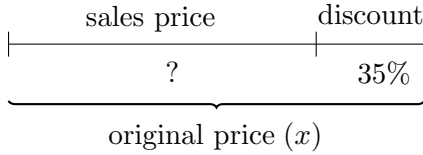
Problem 76. Bob earns x dollars per hour; Ann earns twice as much.

a) Compose an algebraic expression (in terms of x) showing how much Ann earns per hour.

b) Compose an algebraic expression (in terms of x) showing how much more Ann will earn than Bob, if she will work 3 hours and he will work 2 hours.

Problem 77. A pair of jeans costs x dollars. Compose an algebraic expression showing the price of this pair on a 35% sale.

Solution.



The sale price is the original price less the discount, that is $x - 0.35x = 0.75x$.

Answer: the sales price is $0.75x$ dollars.

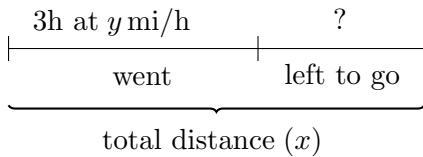
□

Problem 78. On Monday, Jim spent x minutes playing his favorite video game. On Tuesday, he played 10 minutes less than on Monday. On Wednesday, he played three times longer than on Monday and Tuesday combined. How much time did Jim spend playing video games on those three days?

Problem 79. Two identical boxes contain x chocolate bars. How many chocolate bars will be in five such boxes?

Problem 80. Cam has to cover the distance of x miles. He went 3 hours with a constant speed of y miles per hour. How many miles are left for Cam to go?

Solution.



Cam spent 3 hours at a speed of y miles per hour; so he went $3y$ miles. Since the total distance to cover is x miles, then there are $x - 3y$ miles left to go. Answer: Cam has $x - 3y$ miles left to go.

□

Problem 81. The mass of a melon is x lb, the mass of a watermelon is y lb more than the mass of the melon. How many times heavier is the watermelon than the melon?

Problem 82. Sam reads m pages per hour, his little sister reads 8 pages per hour less than Sam. How much faster will Sam read a book containing n pages?

Problem 83. A car spent 2 hours with a constant speed of x mi/h and then spent 3 hours in a jam moving 20 mi/h slower than before. During the next 2 hours, the car moved 10 mi/h faster than during the first two hours of its journey. Compose an algebraic expression showing the distance travelled by the car. Find the value of this expression if $x = 50$.

2.3. Operations with algebraic expressions

Problem 84. Simplify the following expressions (open parenthesis and combine similar terms):

a) $3x^2 - x(2 - 4x)$ b) $(x^2 - 3x + 1)(x - 2)$ c) $(2x^2 + x - 2)(-x^2 - 3x + 1)$

Problem 85. Perform the operations and give the answer as a simple fraction

a) $\frac{x-1}{x} - \frac{x}{x-1}$ b) $\frac{x^2+2x}{2x+4} \div \frac{x^3+2x^2}{2}$ c) $\frac{xy^2-xy}{xy} \cdot \frac{y^3}{y-1}$ d) $x + \frac{1}{3x} - \frac{1}{2x^2}$

Solution. a) $\frac{x-1}{x} - \frac{x}{x-1} = \frac{(x-1)(x-1) - x \cdot x}{x(x-1)} = \frac{x^2 - 2x + 1 - x^2}{x(x-1)} = \frac{1-2x}{x(x-1)}$

$$\text{b) } \frac{x^2 + 2x}{2x + 4} \div \frac{x^3 + 2x^2}{2} = \frac{x(x + 2)}{2(x + 2)} \div \frac{x^2(x + 2)}{2} = \frac{x}{2} \cdot \frac{2}{x^2(x + 2)} = \frac{1}{x(x + 2)}$$

$$\text{c) } \frac{xy^2 - xy}{xy} \cdot \frac{y^3}{y - 1} = \frac{xy(y - 1)}{xy} \cdot \frac{y^3}{y - 1} = y^3$$

$$\text{d) } x + \frac{1}{3x} - \frac{1}{2x^2} = \frac{x \cdot (6x^2) + 2x - 3}{6x^2} = \frac{6x^3 + 2x - 3}{6x^2} \quad \square$$

Problem 86. Perform the operations and write the answer as a simple fraction

$$\text{a) } \frac{x}{x + 1} - \frac{x - 1}{x} \quad \text{b) } \frac{x^2 - x}{x + 1} \div \frac{2x - 2}{3x^2 + 3x} \quad \text{c) } (x + y) \left(\frac{1}{x} + \frac{1}{y} \right) \quad \text{d) } \frac{x}{2} - \frac{3}{x} - x$$

Problem 87. Simplify the following expressions:

$$\text{a) } \frac{(x^2y^3)^2}{(2x)^3} \quad \text{b) } \frac{(-3xy^3)^3}{(x^2y)^{-2}} \quad \text{c) } \frac{(-2x^3y^2)^{-2}}{(xy^2)^{-3}} \quad \text{d) } \frac{(-xy^4)^{-1}}{(2xy^2)^3} \quad \text{e) } \frac{(-2x^3y^2)^2}{(x^{-2}y^3)^{-2}}$$

Solution. a) $\frac{(x^2y^3)^2}{(2x)^3} = \frac{x^4y^6}{8x^3} = \frac{xy^6}{8}$

b) $\frac{(-3xy^3)^3}{(x^2y)^{-2}} = (-3xy^3)^3(x^2y)^2 = (-3)^3x^3y^9x^4y^2 = -27x^7y^{11}$

c) $\frac{(-2x^3y^2)^{-2}}{(xy^2)^{-3}} = \frac{(xy^2)^3}{(-2x^3y^2)^2} = \frac{x^3y^6}{(-2)^2x^6y^4} = \frac{y^2}{4x^3} \quad \square$

2.4. Short multiplication

Problem 88. Prove that for any x and y the *short multiplication* formulas hold true

$$\begin{aligned} (x + y)^2 &= x^2 + 2xy + y^2 \\ (x - y)^2 &= x^2 - 2xy + y^2 \end{aligned}$$

Solution. $(x + y)^2 = (x + y)(x + y) = x^2 + xy + yx + y^2 = x^2 + 2xy + y^2$.

To prove the second identity, we plug in $-y$ in the first one:

$$(x - y)^2 = (x + (-y))^2 = x^2 + 2x(-y) + (-y)^2 = x^2 - 2xy + y^2. \quad \square$$

Problem 89. For which values of x and y , $(x + y)^2 \neq x^2 + y^2$?

Solution. By the short multiplication, $(x + y)^2 = x^2 + 2xy + y^2$. So

$$(x + y)^2 \neq x^2 + y^2 \iff x^2 + 2xy + y^2 \neq x^2 + y^2 \iff xy \neq 0.$$

That is, if neither x nor y is zero, then $(x + y)^2 \neq x^2 + y^2$. \square

Problem 90. Use the short multiplication formula to expand the following expressions:

a) $(x + 1)^2$ b) $(x - 2)^2$ c) $(3 + y)^2$ d) $(a + b)^2$ e) $(3x - 1)^2$ f) $(4 - x)^2$
 g) $(-3a - 5)^2$ h) $(ab + c)^2$ i) $(x + \pi)^2$ j) $\left(\frac{x}{3} - 2\right)^2$ k) $(3 + \sqrt{2})^2$

Solution. a) $(x + 1)^2 = x^2 + 2 \cdot x \cdot 1 + 1^2 = x^2 + 2x + 1$

b) $(x - 2)^2 = x^2 - 2 \cdot x \cdot 2 + 2^2 = x^2 - 4x + 4$

g) $(-3a - 5)^2 = (-1 \cdot (3a + 5))^2 = (-1)^2 \cdot (3a + 5)^2 = (3a + 5)^2 = (3a)^2 + 2 \cdot (3a) \cdot 5 + 5^2 = 9a^2 + 30a + 25$

k) $(3 + \sqrt{2})^2 = 3^2 + 2 \cdot 3 \cdot \sqrt{2} + (\sqrt{2})^2 = 9 + 6\sqrt{2} + 2 = 11 + 6\sqrt{2}$ □

Problem 91. Factor the following expressions using the *perfect trinomial formulas*

$$x^2 \pm 2xy + y^2 = (x \pm y)^2$$

a) $x^2 + 6x + 9$ b) $4x^2 - 4x + 1$ c) $16a^2 + 16ab + 4b^2$ d) $25 - 20x + 4x^4$

Solution. a) The expression $x^2 + 6x + 9$ is a perfect square trinomial. Indeed,

$$x^2 + 6x + 9 = x^2 + 2 \cdot x \cdot 3 + 3^2 = (x + 3)^2.$$

□

Problem 92. Prove that $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$ for any x and y .

Problem 93. Expand and simplify:

a) $(x + y)^2 - (x - y)^2$ b) $(x + y)^2 + (x - y)^2$

2.5. Difference of squares

Problem 94. Prove that for any x and y the *difference of squares* formula holds true

$$x^2 - y^2 = (x - y)(x + y)$$

Solution. Let us expand the right hand side:

$$(x - y)(x + y) = x^2 + xy - yx - y^2 = x^2 - y^2.$$

□

Problem 95. Factor the following expressions:

a) $x^2 - 1$ b) $x^2 - 9$ c) $16 - x^2$ d) $1 - 4x^2$ e) $\frac{x^2}{25} - 64$ f) $x^2 - 5$ g) $3x^2 - 1$

Solution. a) $x^2 - 1 = x^2 - 1^2 = (x - 1)(x + 1)$

c) $16 - x^2 = 4^2 - x^2 = (4 - x)(4 + x)$

$$\mathbf{f)} \quad x^2 - 5 = x^2 - (\sqrt{5})^2 = (x - \sqrt{5})(x + \sqrt{5})$$

□

Problem 96. Simplify the following expressions

$$\mathbf{a)} \quad \frac{x^2 - 1}{x + 1} \quad \mathbf{b)} \quad \frac{x - 2}{4 - x^2} \quad \mathbf{c)} \quad \frac{x^2 + 2x + 1}{x + 1} \quad \mathbf{d)} \quad \frac{x^2 - 3x}{x^2 - 6x + 9} \quad \mathbf{e)} \quad \frac{x^2 + 2x + 1}{x + 1}$$

Solution. $\mathbf{b)} \quad \frac{x - 2}{4 - x^2} = \frac{x - 2}{(2 - x)(2 + x)} = \frac{-(2 - x)}{(2 - x)(2 + x)} = -\frac{1}{2 + x}$

$$\mathbf{d)} \quad \frac{x^2 - 3x}{x^2 - 6x + 9} = \frac{x(x - 3)}{(x - 3)^2} = \frac{x}{x - 3}$$

□

3. Linear equations

3.1. What is a linear equation and its solution

Problem 97. Which of the following are **linear expressions**?

- a) $3x - 4$ b) $x^2 + 2$ c) $\frac{x}{4}$ d) $\frac{4}{x}$ e) $\frac{3x - 5}{2}$ f) $\frac{1}{2}$ g) $x + 1 = 2$

Solution. a) $3x - 4$ is linear.

b) $x^2 + 2$ is not linear, because $x^2 = x \cdot x$ is the product of two variables, and we are not allowed to multiply variables together in linear expressions.

c) $\frac{x}{4}$ is linear: $\frac{x}{4} = \frac{1}{4} \cdot x$.

d) $\frac{4}{x}$ is not linear: we are not allowed to divide by a variable.

e) $\frac{3x - 5}{2}$ is linear: $\frac{3x - 5}{2} = \frac{1}{2} \cdot (3x - 5)$.

f) $\frac{1}{2}$ is linear. Any constant comprises a linear expression.

g) $x + 1 = 2$ is a linear **equation**, but **not** an expression. □

Problem 98. Which of the following are **linear equations**?

- a) $3(x - 2) - x = x + 1$ b) $5(x - 2) - 4x = x + 1$ c) $x(x - 2) = 3$ d) $5x - 7$
e) $\frac{-4x + 7}{3} + 2 = \frac{x}{6}$ f) $x - 2(x + 1) = -2 - x$ g) $\pi x - 3x = x + \sqrt{2}$

Solution. a) is a linear equation, since both sides of it are linear expressions.

b) is a linear equation by the same reason as a) is.

c) is not linear, because two variables are being multiplied together.

d) $5x - 7$ is not an equation, it is just an expression (linear though).

e) may look complicated, but it is linear.

f) is linear.

g) looks unusual due to irrational coefficients, but it is linear. □

Problem 99. Explain why the equation $x + 1 = x + 2$ has no solutions.

Solution. For any value of x , the left hand side is less than the right hand side by one. They can never be equal. Therefore, there are no values of x which can make $x + 1 = x + 2$ true. □

Problem 100. Explain why all real numbers are solutions to the equation $x + 2 = x + 2$.

Solution. For any value of x , the left hand side and the right hand side have the same value. They are always equal. Therefore, any x solves the equation. \square

3.2. How to solve a linear equation

Problem 101. Solve the following equations and check your answer by substitution:

a) $x - 3(3 - 2x) = 5$ b) $\frac{x}{2} - 4 = 2x - 2$ c) $2(x + \pi) = x + 3$

Solution.

a) $x - 3(3 - 2x) = 5$ (Distribute -3)
 $x - 9 + 6x = 5$ (Combine x -terms)
 $7x - 9 = 5$ (Move -9 to RHS)
 $7x = 5 + 9$
 $7x = 14$ (Divide both sides by 7)
 $x = 2$

Check:
 $2 - 3(3 - 2(2)) \stackrel{?}{=} 5$
 $2 - 3(3 - 4) \stackrel{?}{=} 5$
 $2 - 3(-1) \stackrel{?}{=} 5$
 $2 + 3 \stackrel{?}{=} 5$
 $5 = 5 \checkmark$

b) $\frac{x}{2} - 4 = 2x - 2$ (Multiply both sides by 2)
 $2 \cdot \left(\frac{x}{2} - 4\right) = 2 \cdot (2x - 2)$ (Distribute 2)
 $x - 8 = 4x - 4$ (Move x -terms to RHS)
 $-8 = 4x - 4 - x$ (Combine x -terms)
 $-8 = 3x - 4$ (Move constant term -4 to LHS)
 $-8 + 4 = 3x$
 $-4 = 3x$ (Divide both sides by 3)
 $-\frac{4}{3} = x$
 $x = -\frac{4}{3}$

Check:
 $\frac{-\frac{4}{3}}{2} - 4 \stackrel{?}{=} 2 \left(-\frac{4}{3}\right) - 2$
 $-\frac{4}{6} - 4 \stackrel{?}{=} -\frac{8}{3} - 2$
 $-\frac{2}{3} \stackrel{?}{=} -\frac{8}{3} + 2$
 $-\frac{2}{3} + \frac{8}{3} \stackrel{?}{=} 2$
 $\frac{6}{3} \stackrel{?}{=} 2$
 $2 = 2 \checkmark$

c) $2(x + \pi) = x + 3$ (Distribute 2)
 $2x + 2\pi = x + 3$ (Move x to LHS and 2π to RHS)
 $2x - x = 3 - 2\pi$
 $x = 3 - 2\pi$

Check:
 $2(3 - 2\pi + \pi) \stackrel{?}{=} 3 - 2\pi + 3$
 $2(3 - \pi) \stackrel{?}{=} 3 - 2\pi$
 $6 - 2\pi = 6 - 2\pi \checkmark$

\square

Problem 102. Solve the following equations and check your answer by substitution:

a) $2 - (x + 3) = 2x - 4$ b) $2 + \frac{x}{2} = 3(x - 1)$ c) $1 + 3(x - 1) = x - 1$

$$\text{d) } 2 - 2(1 - 2x) = -4 - 2(x - 2) \quad \text{e) } \frac{x-1}{3} + 2 = 3x \quad \text{f) } -2(x + 1) - 3x = 4(x - 3)$$

Problem 103. Solve the following linear equations. Can you check the solution?

$$\text{a) } 4x - 2(5 + 2x) = 6 \quad \text{b) } 4x - 2(5 + 2x) = -10$$

Solution.

$$\text{a) } 4x - 2(5 + 2x) = 6 \text{ (Distribute } -2) \quad \text{b) } 4x - 2(5 + 2x) = -10 \text{ (Distribute } -2)$$

$$4x - 10 - 4x = 6$$

$$-10 = 6 \text{ Contradiction!}$$

This equation has **no** solutions

(so there is nothing to check).

$$4x - 10 - 4x = -10$$

$$-10 = -10 \text{ Always true!}$$

All real numbers are solutions to this equation.

It is impossible to check the solution. \square

Problem 104. Solve the following linear equations. Can you check the solution?

$$\text{a) } 5(x - 1) - 4x = x + 1 \quad \text{b) } \frac{x}{3} + 1 = 2(1 - x) + \frac{7x}{3} \quad \text{c) } 2(1 - 3x) = -3(2x + 1)$$

$$\text{d) } 3 - (2x - 1) = 2(2 - x) \quad \text{e) } 1 - \frac{x+1}{2} = -\frac{1}{2}(x - 1) \quad \text{f) } 4(x + 2) - 10 = 2(x - 1) + 2x$$

3.3. Linear equations in science

Problem 105. For temperature conversion from Fahrenheit to Celsius, the following formula is used:

$$C = (F - 32) \cdot \frac{5}{9},$$

where C is the temperature in Celsius and F is the temperature in Fahrenheit.

a) Using this formula, get a conversion formula from Celsius to Fahrenheit.

b) Convert 20°C to Fahrenheit.

c) Convert -10°F to Celsius.

d) Is there a temperature at which a Fahrenheit thermometer gives the same numerical reading as a Celsius thermometer? If so, find it.

Solution. a) We have to solve the equation $C = (F - 32) \cdot \frac{5}{9}$ for F . This is a linear equation in two variables C and F . To solve the equation for F , we have to isolate F on one side of the equation:

$$C = (F - 32) \cdot \frac{5}{9} \text{ (Multiply both sides by } \frac{9}{5})$$

$$\frac{9}{5}C = F - 32 \text{ (Move } -32 \text{ to LHS)}$$

$$\frac{9}{5}C + 32 = F \text{ (Done! Just rewrite starting with } F)$$

$$\boxed{F = \frac{9}{5}C + 32}$$

b) To convert $20^\circ C$ to Fahrenheit, we plug in $C = 20^\circ$ in the equation $F = \frac{9}{5}C + 32$:

$$F = \frac{9}{5} \cdot 20 + 32 = 9 \cdot 4 + 32 = 68.$$

So $20^\circ C = 68^\circ F$.

c) To convert $-10^\circ F$ to Celsius, we plug in $F = -10^\circ$ in the equation $C = (F - 32) \cdot \frac{5}{9}$:

$$C = (-10 - 32) \cdot \frac{5}{9} = -42 \cdot \frac{5}{9} = -14 \cdot \frac{5}{3} = -\frac{70}{3} = -23\frac{1}{3}$$

So $-10^\circ F = -23\frac{1}{3}^\circ C$.

d) Assume that F° is a temperature such that $F^\circ = C^\circ$. Then

$$F = (F - 32) \cdot \frac{5}{9} \iff 9F = 5(F - 32) \iff 9F = 5F - 160 \iff 4F = -160 \iff F = -40.$$

Therefore, $-40^\circ F = -40^\circ C$. □

Problem 106. According to the **ideal gas law**,

$$PV = nRT,$$

where P is the pressure of the gas, V is the volume of the gas, n is the number of moles of the gas, R is the universal gas constant, and T is the temperature of the gas.

a) Find P in terms of V, n, R , and T .

b) Find n in terms of P, V, R , and T .

c) Find T in terms of P, V, n , and R .

Solution. a) We have to solve the equation $PV = nRT$ for P , that is, to isolate P :

$$PV = nRT \quad (\text{Divide both sides by } V)$$

$$P = \frac{nRT}{V} \quad (\text{Done!})$$

b) We have to solve the equation $PV = nRT$ for n , that is, to isolate n :

$$PV = nRT \quad (\text{Divide both sides by } RT)$$

$$\frac{PV}{RT} = n \quad (\text{Done! Just rewrite starting from } n)$$

$$n = \frac{PV}{RT}$$

□

Problem 107. The area A of a trapezoid with bases a, b and height h is calculated by the the formula

$$A = \frac{a+b}{2} h.$$

a) Using this formula, find a in terms of A, b, h .

b) Using this formula, find h in terms of A, a, b .

Solution. a) We have to solve the equation for a , that is, to isolate a :

$$A = \frac{a+b}{2}h \quad (\text{Divide both sides by } h)$$

$$\frac{A}{h} = \frac{a+b}{2} \quad (\text{Multiply both sides by } 2)$$

$$\frac{2A}{h} = a+b \quad (\text{Move } b \text{ to LHS})$$

$$\frac{2A}{h} - b = a \quad (\text{Done! Just rewrite starting from } a)$$

$$a = \frac{2A}{h} - b$$

□

Problem 108. The average molecular kinetic energy E is proportional to the gas temperature T :

$$E = \frac{3}{2}RT,$$

where R is the universal gas constant. Find T in terms of E and R .

Problem 109. The velocity v of a particle moving with a constant acceleration a is given by the formula

$$v = at + v_0,$$

where t is the time interval and v_0 is initial velocity. Find t in terms of v, a, v_0 .

Problem 110. According to Newton's law of universal gravitation,

$$F = G \frac{m_1 m_2}{R^2},$$

where F is the force between the masses of m_1 and m_2 , G is the gravitational constant, and R is the distance between the centers of the masses. Use this equation to find m_1 in terms of F, G, m_2 and R .

Problem 111. The volume V of a cone of radius r and height h is given by the formula

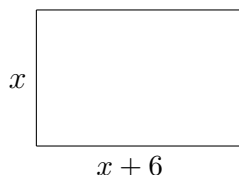
$$V = \frac{1}{3}\pi hr^2.$$

Find h in terms of V and r .

3.4. Word problems leading to linear equations

Problem 112. One side of a rectangle is 6 cm longer than another. The perimeter of the rectangle is 52 cm. What are the lengths of the sides of the rectangle?

Solution.



Let x cm be length the shorter side of the rectangle. Then the length of the longer side is given by $x + 6$ cm. The perimeter of the rectangle (the sum of the lengths of all sides) is given by $x + x + (x + 6) + (x + 6)$. Since the latter expression is simplified to $4x + 12$, the perimeter is $4x + 12$ cm.

It is given that the perimeter of the rectangle is 52 cm. So $4x + 12 = 52$. We have to solve this

equation to find x :

$$4x + 12 = 52 \iff 4x = 52 - 12 \iff 4x = 40 \iff x = 10.$$

Therefore, the shorter side of the rectangle is $x = 10$ cm, the longer side is $x + 6 = 10 + 6 = 16$ cm. Let us check that the perimeter of 10×16 rectangle is indeed 52 cm:

$$10 + 10 + 16 + 16 = 52.$$

Answer: The lengths of the sides of the rectangle are 10 cm and 16cm. \square

Problem 113. A bathtub contains 40 gallons of water. You start to pouring in additional water at a rate of 8 gallons per minute. How long will it take to fill the bathtub up to 96 gallons?

Solution. Let x be the number of minutes needed to fill the bathtub up to 96 gallons. Since the rate of filling is 8 gallons per minute, $8x$ gallons of water will be added to the bathtub after x minutes. After filling, the total amount of water will be $40 + 8x$ gallons. It is given that after filling the bathtub should contain 96 gallons. Therefore, $40 + 8x = 96$. Let us solve this equation to find x :

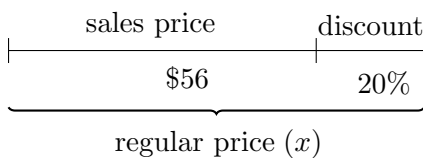
$$40 + 8x = 96 \iff 8x = 96 - 40 \iff 8x = 56 \iff x = 7.$$

Hence, it takes 7 minutes to fill the bathtub. Let us check our solution. If the bathtub is filled during 7 minutes at the rate of 8 gallons per minute, then 56 gallons of water will be added to 40 gallons of water, which was in the bathtub. In total, we will have $40 + 56 = 96$ gallons, as it should be.

Answer: It will take 7 minutes. \square

Problem 114. A gadget costs \$56 while on a 20% sale. What was the regular price of the gadget?

Solution.



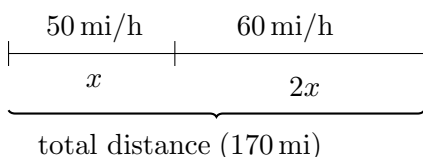
Let x dollars be the regular price for the gadget. Then the 20% discount of x is $0.2x$. The regular price is the sales price plus the discount, therefore $56 + 0.2x = x$. To find x , we solve this equation.

$$56 + 0.2x = x \iff 56 = x - 0.2x \iff 56 = 0.8x \iff \frac{56}{0.8} = x \iff x = 70.$$

Answer: the regular price was \$70. \square

Problem 115. Ann drove a car for several hours at a constant speed of 50 mi/h. Then she increased the speed to 60 mi/h and drove for twice as much time as before. Find the time spent while driving on both intervals, if Ann drove 170 mi in total.

Solution.



Let x hours be the time of driving on the first interval. Then the time of driving on the second interval is $2x$ hours. Under a uniform motion, distance = time \times speed.

So the distance covered on the first interval is $x \cdot 50 = 50x$ miles, and the distance covered on the second interval is (time \times speed) $2x \cdot 60 = 120x$ miles. It is given that the total distance is 170 miles. Therefore, $50x + 120x = 170$. Let us solve this equation:

$$50x + 120x = 170 \iff 170x = 170 \iff x = 1.$$

Answer: Ann spent 1 hour on the first interval and 2 hours on the second interval. \square

Problem 116. In triangle ABC , angle A is 20° greater than angle B , and angle C is twice the size of angle A . Find the angles of the triangle.

Problem 117. Elspark, an electric utility company, charges a base rate of \$20 per month plus a power supply charge of 9 cents per 1 kWh. What was the electricity consumption last month if that month's bill shows \$104.60 in charges?

Problem 118. Winnie the Pooh ate 16 jars of honey, which comprise 40% of all his honey supply. How many jars of honey did Winnie the Pooh have?

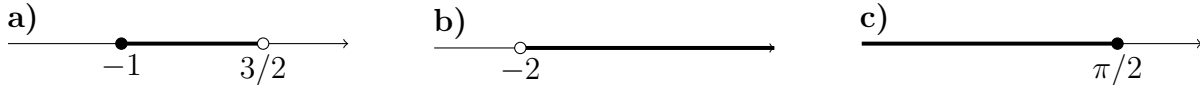
Problem 119. Ken drove 69 mi/h when he was stopped by police. The police officer claimed that Ken exceeded the speed limit by 15%. What was the speed limit?

Problem 120. A car went 2 hours with a constant speed and then spent 3 hours in a jam moving 20 mi/h slower than before. During the next 2 hours, the car moved 10 mi/h faster than during the first two hours of its journey. Find the speed of the car during the first two hours of its journey, if it is known that the car has travelled 380 miles.

4. Life on a number line

4.1. Intervals and their notations

Problem 121. Give the interval notation for the following intervals:



Problem 122. Draw the following intervals on a number line:

a) $(3, 5]$ b) $(-\infty, 0]$ c) $(-\sqrt{2}, \infty)$ d) $[-\pi, \frac{2}{3})$ e) $(-2, \frac{\sqrt{3}}{4}]$

Problem 123. Explain what is wrong with the following interval notations:

a) $[3, -5)$ b) $[2, -\infty)$ c) $[0, \infty]$ d) $\{-1, 3\}$

Solution. a) The left end point of the interval should be less than the right end point. Since $3 > -5$, the notation $[3, -5)$ makes **no** sense. It could be corrected to $(-5, 3]$.

b) $2 > -\infty$, so the notation $[2, -\infty)$ is incorrect. Correction: $(-\infty, 2]$.

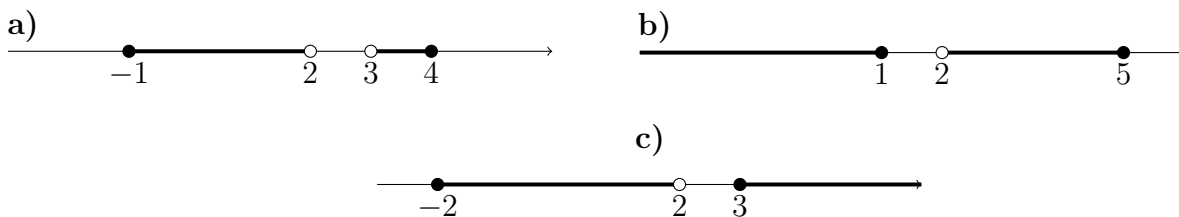
c) ∞ is **not** a number, it should be **never** included into an interval. Correction: $[0, \infty)$.

d) Notation $\{-1, 3\}$ means a set consisting of two numbers, -1 and 3 . It's not an interval. \square

Problem 124. Find the following intersections and show them on a number line:

a) $(-\infty, 3] \cap (1, 5)$ b) $[-2, 4) \cap (-3, 5]$ c) $(2, 3) \cap (-\infty, 4]$ d) $(-\infty, 1] \cap (2, \infty)$

Problem 125. Give the interval notation for the following unions of intervals:



4.2. Absolute value

Problem 126. Calculate:

a) $|1 + |-2| - 3|$ b) $|-1 + |-2| + 3|$ c) $|-1 - |-2| - 3|$

Solution.

a) $|1 + |-2| - 3| = |1 + 2 - 3| = |0| = 0$

$$\text{b) } \left| -1 + |-2| + 3 \right| = \left| -1 + 2 + 3 \right| = |4| = 4$$

$$\text{c) } \left| -1 - |-2| - 3 \right| = \left| -1 - 2 - 3 \right| = |-6| = 6 \quad \square$$

Problem 127. Calculate:

$$\text{a) } -| - | -2| | \quad \text{b) } | - | -2| -3 | \quad \text{c) } | - | -2| - |3-4| |$$

Problem 128. Calculate the value of $|-x|$, if

$$\text{a) } x = 2 \quad \text{b) } x = -2 \quad \text{c) } x = 0$$

Problem 129. Is it true that $|-x| = |x|$ for arbitrary real number x ? Explain!

Solution. Yes, it is true. The expression $|x|$ means geometrically the distance from the point x to the origin. The expression $|-x|$ stands for the distance from the point $-x$ to the origin. Since the points x and $-x$ are symmetrically located on a number line, the distances from them to the origin are equal. \square

Problem 130. Which number is larger, x or $-x$? Explain!

Problem 131. Is it true that $|x + y| = |x| + |y|$ for arbitrary numbers x and y ? Explain!

Solution. No, it is not true. Here is a counterexample: let $x = 1$ and $y = -1$. Then the left hand side will be $|x + y| = |1 + (-1)| = |0| = 0$. The right hand side will be $|x| + |y| = |1| + |-1| = 1 + 1 = 2$. Since $0 \neq 2$, the statement is not true. \square

4.3. Linear equations involving absolute value

Problem 132. Solve the equations

$$\text{a) } |x| = 3 \quad \text{b) } |x| = 0 \quad \text{c) } |x| = -3 \quad \text{d) } |-x| = 3 \quad \text{e) } |-x| = -3.$$

Solution. a) To solve the equation $|x| = 3$, we have to find **all** real numbers located at the distance of 3 from the origin. There are two such numbers, 3 and -3 . So the solutions are $x = 3$ and $x = -3$. We may write them down in an abbreviated way as $x = \pm 3$.

b) $|x| = 0$ if and only if $x = 0$.

c) The absolute value is always **non-negative**. Therefore, there are no numbers having the absolute value of -3 . So the equation $|x| = -3$ has **no** solutions.

d) Since $|-x| = |x|$, the equation $|-x| = 3$ is equivalent to $|x| = 3$, which has the solutions $x = \pm 3$.

e) The absolute value can't be negative. Therefore, the equation $|-x| = -3$ has no solutions. \square

Problem 133. Solve the equations and check your solution by substitution (if possible)

$$\text{a) } |3x + 1| = 7 \quad \text{b) } |3x + 1| = 0 \quad \text{c) } |3x + 1| = -7$$

Solution. a) The equation $|3x + 1| = 7$ says the the absolute value of the unknown number $3x + 1$ is equal to 7. It may happen only in two cases: when $3x + 1 = 7$ or $3x + 1 = -7$. We have to solve these two equations separately:

$$\begin{array}{c}
 |3x + 1| = 7 \\
 \swarrow \quad \searrow \\
 3x + 1 = 7 \quad \text{or} \quad 3x + 1 = -7 \\
 3x = 7 - 1 \quad 3x = -7 - 1 \\
 3x = 6 \quad 3x = -8 \\
 x = 2 \quad x = -\frac{8}{3}
 \end{array}$$

So $|3x + 1| = 7 \iff x = 2$ or $x = -\frac{8}{3}$. Therefore, there are two solutions: $x = 2$ and $x = -\frac{8}{3}$.

Let us check **both** solutions.

Substitute $x = 2$:

$$\begin{array}{l}
 |3 \cdot 2 + 1| \stackrel{?}{=} 7 \\
 |7| \stackrel{?}{=} 7 \\
 7 = 7 \quad \checkmark
 \end{array}$$

Substitute $x = -\frac{8}{3}$:

$$\begin{array}{l}
 \left| 3 \cdot \left(-\frac{8}{3}\right) + 1 \right| \stackrel{?}{=} 7 \\
 |-8 + 1| \stackrel{?}{=} 7 \\
 |-7| \stackrel{?}{=} 7 \\
 7 = 7 \quad \checkmark
 \end{array}$$

b) The equation says that the absolute value of the unknown number $3x + 1$ is equal to 0. It may happen only if this number is equal to 0:

$$|3x + 1| = 0 \iff 3x + 1 = 0 \iff 3x = -1 \iff x = -\frac{1}{3}.$$

Check: $\left| 3 \cdot \left(-\frac{1}{3}\right) + 1 \right| \stackrel{?}{=} 0 \iff |-1 + 1| \stackrel{?}{=} 0 \iff |0| \stackrel{?}{=} 0 \iff 0 = 0 \quad \checkmark$

c) The equation $|3x + 1| = -7$ has no solutions since the absolute value can't be negative. \square

Problem 134. Solve the equations and check your solution by substitution (if possible)

a) $|2 - 3x| = 4$ b) $|2 - 3x| = 0$ c) $|2 - 3x| = -4$

Problem 135. Solve the equations

a) $-2 + |3x + 1| = 4$ b) $-2 + 2 \cdot |3x + 1| = 4$ c) $-2 + \left| \frac{3x}{2} + 1 \right| = 4$

Solution. c) $-2 + \left| \frac{3x}{2} + 1 \right| = 4 \iff \left| \frac{3x}{2} + 1 \right| = 4 + 2 \iff \left| \frac{3x}{2} + 1 \right| = 6$

$$\left| \frac{3x}{2} + 1 \right| = 6$$

$$\swarrow \quad \searrow$$

$$\frac{3x}{2} + 1 = 6 \quad \text{or} \quad \frac{3x}{2} + 1 = -6$$

$$\frac{3x}{2} = 6 - 1 \quad \frac{3x}{2} = -6 - 1$$

$$\frac{3x}{2} = 5 \quad \frac{3x}{2} = -7$$

$$x = \frac{10}{3} \quad x = -\frac{14}{3}$$

So $-2 + |3x + 1| = 4 \iff x = \frac{10}{3}$ or $x = -\frac{14}{3}$.

Answer: The solutions are $x = \frac{10}{3}$ and $x = -\frac{14}{3}$. □

4.4. Linear inequalities

Problem 136. Solve each of the following inequalities, show the solution on a number line and give the answer in interval notation

a) $2 + 3x \leq -3$ b) $2 - 3x > -3$ c) $-2 - 3x \geq -3$

Solution.

a) $2 + 3x \leq -3$

$$3x \leq -5$$

$$x \leq -\frac{5}{3}$$

$$\left(-\infty, -\frac{5}{3}\right]$$

b) $2 - 3x > -3$

$$-3x > -5$$

$$5 > 3x$$

$$\frac{5}{3} > x$$

$$x < \frac{5}{3}$$

$$\left(-\infty, \frac{5}{3}\right)$$

c) $-2 - 3x \geq -3$

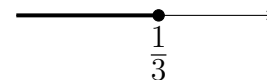
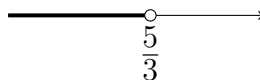
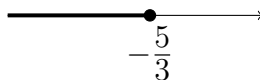
$$-3x \geq -1$$

$$1 \geq 3x$$

$$\frac{1}{3} \geq x$$

$$x \leq \frac{1}{3}$$

$$\left(-\infty, \frac{1}{3}\right]$$



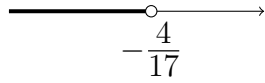
□

Problem 137. Solve each of the following inequalities, show the solution on a number line and give the answer in interval notation

a) $1 - 3x > \frac{5}{3} - \frac{x}{6}$ b) $3 - 2\left(x + \frac{2}{3}\right) \leq 4$ c) $3(x - 1) - 3(x + 1) \geq 1$

Solution.

$$\begin{array}{ll}
 \text{a) } 1 - 3x > \frac{5}{3} - \frac{x}{6} & \text{Multiply both sides by 6} \\
 6 \cdot (1 - 3x) > 6 \cdot \left(\frac{5}{3} - \frac{x}{6} \right) & \text{Distribute} \\
 6 - 18x > 10 - x & \text{Move } -18x \text{ to RHS and 10 to LHS} \\
 6 - 10 > -x + 18x & \text{Combine similar terms} \\
 -4 > 17x & \text{Divide both sides by 17} \\
 -\frac{4}{17} > x & \text{Rewrite starting with } x \\
 x < -\frac{4}{17} &
 \end{array}$$



Answer: $\left(-\infty, -\frac{4}{17}\right)$ □

Problem 138. Solve the systems of inequalities:

$$\text{a) } \begin{cases} 3 - 2x \leq 4x + 5 \\ 1 + 6x > 0 \end{cases} \quad \text{b) } \begin{cases} 2x - 4 \geq 5x + 2 \\ 3 - 2x \leq 3(x + 1) \end{cases} \quad \text{c) } \begin{cases} -3x - 1 < 3x + 1 \\ 2x + 2 \leq 4 - x \end{cases}$$

Solution. a)

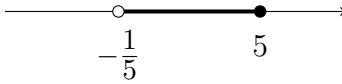
$$\begin{aligned}
 \begin{cases} 3 - 2x \leq 4x + 5 \\ 1 + 6x > 0 \end{cases} & \iff \begin{cases} 3 - 5 \leq 4x + 2x \\ 6x > -1 \end{cases} & \iff \begin{cases} -2 \leq 6x \\ x > -\frac{1}{6} \end{cases} & \iff \begin{cases} -\frac{1}{3} \leq x \\ x > -\frac{1}{6} \end{cases} & \iff \\
 \begin{cases} x \geq -\frac{1}{3} \\ x > -\frac{1}{6} \end{cases} & \iff x > -\frac{1}{6} & \iff x \in \left(-\frac{1}{6}, \infty\right) & \iff & \square
 \end{aligned}$$

Problem 139. Solve the following inequalities and give the answers in interval notations.

$$\text{a) } -3x + 1 < 2x + 2 \leq x + 7 \quad \text{b) } 0 \leq \frac{4x + 1}{3} \leq 3x - 4 \quad \text{c) } 2x - 2 < -x + 1 < 2(x - 1) - 3$$

Solution. a) The double inequality $-3x + 1 < 2x + 2 \leq x + 7$ is equivalent to the system

$$\begin{aligned}
 \begin{cases} -3x + 1 < 2x + 2 \\ 2x + 2 \leq x + 7 \end{cases} & \iff \begin{cases} 1 - 2 < 2x + 3x \\ 2x - x \leq 7 - 2 \end{cases} & \iff \begin{cases} -1 < 5x \\ x \leq 5 \end{cases} & \iff \begin{cases} -\frac{1}{5} < x \\ x \leq 5 \end{cases} & \iff \\
 \begin{cases} x > -\frac{1}{5} \\ x \leq 5 \end{cases} & \iff -\frac{1}{5} < x \leq 5 & \iff x \in \left(-\frac{1}{5}, 5\right] & \iff & \square
 \end{aligned}$$



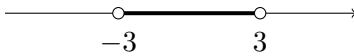
Answer: $\left[-\frac{1}{5}, 5\right]$ □

Problem 140. Solve the inequalities, show the solution on a number line and give the answer in interval notations

- a) $|x| < 3$ b) $|x| \geq 3$ c) $|x| \leq -3$ d) $|x| > -3$ e) $|x| \leq 0$ f) $|x| > 0$

Solution. a) The inequality $|x| < 3$ says that the distance from a number x to the origin is less than 3. Where are **all** such numbers x located? Obviously, in the interval $(-3, 3)$:

$$|x| < 3 \iff -3 < x < 3 \iff x \in (-3, 3).$$



Answer: $(-3, 3)$

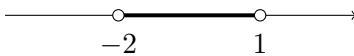
c) Absolute value of any number is non-negative, so there are no numbers whose absolute value is -3 or less. Therefore, the inequality $|x| \leq -3$ has **no** solution. □

Problem 141. Solve the inequalities, show the solution on a number line and give the answer in interval notations

- a) $|2x + 1| < 3$ b) $|2x + 1| \geq 3$ c) $|2x + 1| \leq -3$ d) $|2x + 1| > -3$ e) $|2x + 1| \leq 0$

Solution. a) The inequality $|2x + 1| < 3$ says that the number $2x + 1$ is located at the distance less than 3 from the origin, so

$$\begin{aligned} |2x + 1| < 3 &\iff -3 < 2x + 1 < 3 \iff -3 - 1 < 2x < 3 - 1 \iff -4 < 2x < 2 \iff \\ &\iff -2 < x < 1 \iff x \in (-2, 1). \end{aligned}$$



Answer: $(-2, 1)$. □

5. Line on a plane

5.1. Linear equations in two variables and their graphs

Problem 142. Which of the following equations are linear equations in two variables?

- a) $2x - 3y = 5$ b) $x = \sqrt{3} + \frac{y}{2}$ c) $y = \frac{2}{x}$ d) $-x^2 + y = 0$ e) $x + 2y + 3z = 4$
f) $x = 1$ g) $x = 0$ h) $y = -1$ i) $y = 0$ j) $0 = 1$ k) $xy = 1$ l) $\frac{x}{-2} + \frac{y}{3} = 1$

Solution. A linear equation in two variables may be written in the *standard* form

$$Ax + By = C,$$

where A , B , and C are constants. Therefore, equations **a)**, **b)**, and equations **f) – j)** are linear. Indeed, **a)** is a linear equation in the standard form with $A = 2$, $B = -3$ and $C = 5$.

Equation **b)** is linear since it may be rewritten as $1 \cdot x - \frac{1}{2}y = \sqrt{3}$, so $A = 1$, $B = -\frac{1}{2}$ and $C = \sqrt{3}$.

Equation **f)** may be rewritten as $1 \cdot x - 0 \cdot y = 1$, so $A = 1$, $B = 0$ and $C = 1$. Similarly, we rewrite equations **g)**, **h)**, **j)**.

Equation **j)** is curious, but it may be rewritten as $0 \cdot x + 0 \cdot y = 1$, so $A = 0$, $B = 0$ and $C = 1$. \square

Problem 143. Which of the following equations are linear equations written in the standard form?

- a) $y = 2x + 3$ b) $\pi x - y = 2015$ c) $2(x + 1) + 4y = 3$ d) $x = 0$ e) $y = -1$

Problem 144. Determine if the points given by their coordinates belong to the line $-2x + 3y = 6$

- a) $(3, 0)$ b) $(0, 2)$ c) $(3, 2)$ d) $(-3, -2)$ e) $\left(\frac{1}{3}, \frac{20}{9}\right)$ f) $\left(t, \frac{2}{3}t + 2\right)$, where t is an arbitrary number.

Solution. The given point belongs a given line if and only if the coordinates of the point satisfy the equation of the line.

a) We have to check if $x = 3$, $y = 0$ is a solution of $-2x + 3y = 6$. To do this, we plug in $x = 3$ and $y = 0$ into the equation $-2x + 3y = 6$:

$$-2 \cdot 3 + 3 \cdot 0 = 6 \iff -6 = 6,$$

which is false. Therefore, $x = 3$, $y = 0$ is **not** a solution of $-2x + 3y = 6$ and the point $(3, 0)$ doesn't belong to the line.

b) Plug in $x = 0$ and $y = 2$ into the equation $-2x + 3y = 6$:

$$-2 \cdot 0 + 3 \cdot 2 = 6 \iff 6 = 6,$$

which is true. Therefore, the equation is satisfied and the point $(3, 0)$ belongs to the line.

f) Plug in $x = t$ and $y = \frac{2}{3}t + 2$ into the equation $-2x + 3y = 6$:

$$-2 \cdot t + 3 \cdot \left(\frac{2}{3}t + 2\right) = 6 \iff -2t + 2t + 6 = 6 \iff 6 = 6,$$

which is true. Therefore, the equation is satisfied and the point $\left(t, \frac{2}{3}t + 2\right)$ belongs to the line for all t . \square

Problem 145. Is it true that the graph of any linear equation is a line?

Solution. No. For example, for the linear equation $0 \cdot x + 0 \cdot y = 1$, there are no points (x, y) satisfying it. So the graph is *empty set*.

For the linear equation $0 \cdot x + 0 \cdot y = 0$, any point (x, y) on a plane satisfies it. So the graph is the whole plane. \square

5.2. Slope and intercepts of a line

Problem 146. Determine the slopes of the following lines

a) $2x + 3y = 5$ b) $y = -4x + 2$ c) $x = \frac{2}{3}y + 1$ d) $2x = 5$ e) $3y = -2$

Solution. a) Let us rewrite the equation $2x + 3y = 5$ in the *slope-intercept* form $y = mx + b$:

$$2x + 3y = 5 \iff 3y = -2x + 5 \iff y = -\frac{2}{3}x + \frac{5}{3}.$$

The slope is the coefficient m in the slope-intercept form $y = mx + b$. In our case, it is $-\frac{2}{3}$. \square

Problem 147. Determine the slope of the line passing through the points

a) $(-1, 2)$ and $(3, -4)$ b) $(-1, 2)$ and $(3, 2)$ c) $(-1, 2)$ and $(-1, 1)$

Solution. a) The slope m of the line passing through the points (x_1, y_1) and (x_2, y_2) with $x_1 \neq x_2$ is given by the formula

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Notice that $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$.

For $(x_1, y_1) = (-1, 2)$ and $(x_2, y_2) = (3, -4)$, the slope formula gives

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-4 - 2}{3 - (-1)} = \frac{-6}{4} = -\frac{3}{2}.$$

c) For $(x_1, y_1) = (-1, 2)$ and $(x_2, y_2) = (-1, 1)$, the slope formula gives

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1 - 2}{-1 - (-1)} = \frac{-1}{0},$$

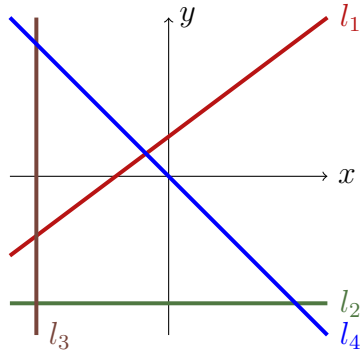
which makes **no** sense since division by zero is prohibited. Let us come back to the formula and read it carefully. The formula $m = \frac{y_2 - y_1}{x_2 - x_1}$ is applicable **only if** $x_1 \neq x_2$.

In our case, $x_1 = x_2$, that is the points $(-1, 2)$ and $(-1, 1)$ have the same x -coordinate of -1 . Geometrically it means that line is vertical and the slope is **not** defined. \square

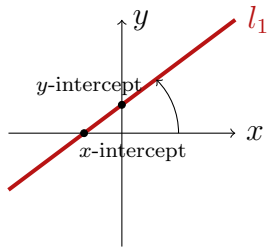
Problem 148. a) What is the slope of a horizontal line?

b) What is the slope of a vertical line?

Problem 149. For each of the given lines, determine, if possible, the **sign** of the slope, the **sign** of the x -intercept and the **sign** of the y -intercept.



Solution.



For line l_1 , the angle, measured from the positive direction of the x -axis counterclockwise to the line, is **acute**. Therefore the slope is positive.

Alternatively, we may reason as follows. Let us stand on the leftmost point of the line and move along the line. Do we move up or down? We move **up** along the line. So the slope is positive.

Moving from left to right along the line is moving in the direction of increasing of the x -coordinate. If y -coordinate is also increasing along this move, then the slope is positive.

This line intersects the x -axis to the left from origin, therefore, the x -intercept is negative. The line intersects the y -axis from above the origin, therefore, the y -intercept is positive. \square

Problem 150. a) Draw a line with a negative slope and a positive y -intercept.

b) Draw a line with a negative slope and a negative x -intercept.

c) Draw a line with a positive slope, a positive x -intercept and a negative y -intercept.

d) Can you draw a line with a positive slope, a negative x -intercept and a negative y -intercept?

Problem 151. Prove that if the equation of a line is written in the form $\frac{x}{a} + \frac{y}{b} = 1$, where a and b are non-zero numbers, then a is the x -intercept and b is the y -intercept.

Solution. To find x -intercept, we plug $y = 0$ into the equation of the line and find x :

$$\frac{x}{a} + \frac{0}{b} = 1 \implies \frac{x}{a} = 1 \implies x = a.$$

So the x -intercept is a .

To find y -intercept, we plug $x = 0$ into the equation of the line and find y :

$$\frac{0}{a} + \frac{y}{b} = 1 \implies \frac{y}{b} = 1 \implies y = b.$$

So the y -intercept is b . □

Problem 152. Determine the x -intercept and the y -intercept of the following lines

a) $2x + 4y = 3$ b) $y = \sqrt{2}x - \frac{2}{3}$ c) $x = -2$ d) $y = 0$ e) $x = 2y$

Solution. a) Let us rewrite the equation of the line in the *two intercept* form:

$$2x + 4y = 3 \iff \frac{2}{3}x + \frac{4}{3}y = 1 \iff \frac{x}{3/2} + \frac{y}{3/4} = 1$$

The coefficients $\frac{3}{2}$ and $\frac{3}{4}$ are x - and y -intercepts respectively.

c) The line $x = -2$ is a vertical line crossing the x -axis at point $x = -2$. So the x -intercept of this line is -2 , and y -intercept doesn't exist because the line doesn't cross the y -axis. □

Problem 153. a) How many x -intercepts may a line have?

b) How many y -intercepts may a line have?

c) Can the x -intercept coincide with the y -intercept?

5.3. Various forms of linear equations

Problem 154. Rewrite, if possible, the following line equations in the *slope-intercept* form. Determine the slope and the y -intercept.

a) $3x + 2y = 6$ b) $\frac{x + 2y}{3} = 1$ c) $x = y$ d) $\frac{x}{2} + \frac{y}{-5} = 1$ e) $y = \sqrt{2}$

Solution. The *slope-intercept* form of the equation of a line is

$$y = mx + b$$

Here, m is the slope and b is the y -intercept.

a) To rewrite the equation $3x + 2y = 6$ in the slope-intercept form, we have to solve out y in terms of x :

$$3x + 2y = 6 \iff 2y = -3x + 6 \iff y = -\frac{3}{2}x + 3.$$

The slope of this line is $-\frac{3}{2}$ and the y -intercept is 3. □

Problem 155. Rewrite, if possible, the following line equations in the *two intercept* form. Determine the x -intercept and the y -intercept. Draw the graph of each line.

a) $-2x + 3y = 5$ b) $2x + 3y = -5$ c) $y = 2x + \frac{1}{3}$ d) $x = \pi y - 3$ e) $x = y$

Solution. The *two intercept* form of the equation of a line is

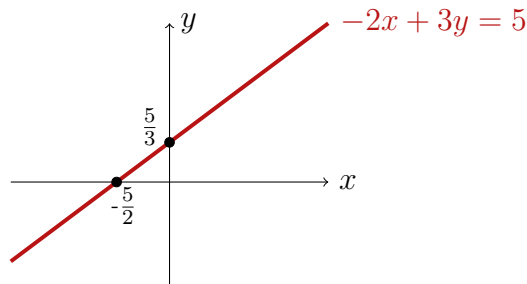
$$\frac{x}{a} + \frac{y}{b} = 1$$

Here, a is the x -intercept and b is the y -intercept.

a) to rewrite the equation $-2x + 3y = 5$ in the two slope form, we divide both sides by 5 to get 1 on RHS and then leave only x and y in the numerators:

$$-2x + 3y = 5 \iff -\frac{2x}{5} + \frac{3y}{5} = 1 \iff \frac{x}{-5/2} + \frac{y}{5/3} = 1.$$

So the x -intercept is $-\frac{5}{2}$ and y -intercept is $\frac{5}{3}$. For drawing, we place the intercepts on the axes and draw a line through them:



□

Problem 156. A line passes through the points $(2, -3)$ and $(1, -1)$. Find the equation of the line in

a) the standard form a) the slope-intercept form c) the two intercept form

Problem 157. Show that the equation of a line that passes through the point (x_0, y_0) and has the slope m can be written in the *point-slope* form

$$y - y_0 = m(x - x_0)$$

Is it true that the equation of **any** line can be written in the point-slope form?

Solution. We start with the slope-intercept form $y = mx + b$. The slope m is given, but b is unknown, which we find by plugging in $x = x_0$ and $y = y_0$ into the equation:

$$y_0 = mx_0 + b \implies b = y_0 - mx_0.$$

Hence,

$$y = mx + b \iff y = mx + (y_0 - mx_0) \iff y - y_0 = mx - mx_0 \iff y - y_0 = m(x - x_0).$$

It is **not** true that any line can be written in the point-slope form. All lines with undefined slope (namely, vertical lines) can't be written in this form. \square

Problem 158. Find the point-slope equation the line that passes through the point $(-1, 2)$ and has the slope of -3 . Draw the line.

Problem 159. Show that the equation of a line that passes through the points (x_1, y_1) and (x_2, y_2) can be written in the *two point* form $\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}$. Is it true that the equation of **any** line can be written in the two point form?

Solution. The slope of a line passing through (x_1, y_1) and (x_2, y_2) is $m = \frac{y_2 - y_1}{x_2 - x_1}$. The equation of the line is

$$\begin{aligned} y - y_1 = m(x - x_1) &\iff y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \iff \frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1} \iff \\ &\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}. \end{aligned}$$

\square

Problem 160. A line has the equation $\frac{x + 1}{2} = \frac{y - 3}{4}$. Find the slope of the line.

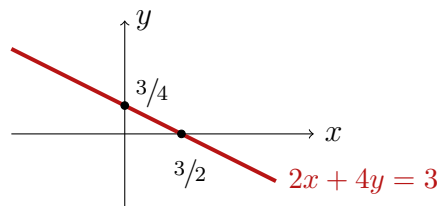
Problem 161. On the coordinate system, draw the following lines:

a) $2x + 4y = 3$ b) $y = -2x - 5$ c) $-100x + 3y = -1$ d) $\frac{2x}{3} + \frac{5y}{6} = -2$

Solution. a) Each line is determined by any two of its points. So let us pick up any two points satisfying the equation, and draw a line through them. Which points to choose? Any two! The easiest choice is the intercepts (if both of them exist, of course). To get the intercepts, we rewrite the equation in the two intercept form:

$$2x + 4y = 3 \iff \frac{2x}{3} + \frac{4y}{3} = 1 \iff \frac{x}{3/2} + \frac{y}{3/4} = 1.$$

We see that the x -intercept is $3/2$ and the y -intercept is $3/4$. Let us place those points on the plane and draw a line through them:



\square

Problem 162. Which of the following are **linear** functions?

a) $y = \sqrt{2}x$ b) $y = -2x + \frac{3}{4}$ c) $y = 1$ d) $y = \frac{2x+3}{5}$ e) $y = 3x^{-1} + 2$

Solution. A linear function can be written in the form $y = mx + b$, where m and b are given constants. Functions **a) - f)** are linear, **e)** is not linear. \square

Problem 163. Draw the graphs of the following linear functions

a) $y = x$ b) $y = x + 1$ c) $y = x - 1$ d) $y = -x$ e) $y = -x + 1$ f) $y = -x - 1$
g) $y = 2x$ h) $y = \frac{1}{2}x$ i) $y = \pi x$ j) $y = -\sqrt{2}x + 3$ k) $y = 0$ l) $y = -\pi$

Problem 164. Is it true that **any** line is the graph of a linear function?

Solution. No. Explain why! \square

Problem 165. Find a linear function $y = y(x)$ satisfying the following conditions: $y(1) = 3$ and $y(-2) = 2$. How many such functions exist?

Problem 166. The velocity of an object moving with a constant acceleration a is given by

$$v(t) = at + v_0,$$

where $v(t)$ is the velocity at time moment t and v_0 is the initial velocity. Sketch the graph of the function $y = v(t)$ assuming that **a)** $a > 0$ **b)** $a < 0$, **c)** $a = 0$.

Problem 167. According to the **ideal gas law**,

$$PV = nRT,$$

where P is the pressure, V is the volume, T is the temperature, n is the amount of the gas, and R is the universal gas constant. Assume that the amount and the volume of the gas are held constant. Show that the pressure is a linear function on the temperature. Find this function and sketch its graph.

5.5. Parallel and perpendicular lines

Any two vertical lines are *parallel*. Non-vertical lines are *parallel* if and only if they have the same slope.

Two lines with slopes m_1 and m_2 are *perpendicular* if and only if their slopes are negative reciprocals of each other, that is $m_2 = -\frac{1}{m_1}$ (or, equivalently, $m_1m_2 = -1$). Any horizontal line (slope 0) is *perpendicular* to any vertical line (no slope).

Problem 168. Prove algebraically that the lines $2x - 3y = 1$ and $-\frac{1}{3}x + \frac{1}{2}y = 2$ are parallel.

Solution. Let us calculate the slopes of both lines. It could be done in various ways. For example, we may bring the equations in the slope-intercept form:

$$2x - 3y = 1 \iff 3y = 2x - 1 \iff y = \frac{2}{3}x - \frac{1}{3}; \text{ the slope is } \frac{2}{3}$$

$$-\frac{1}{3}x + \frac{1}{2}y = 2 \iff -2x + 3y = 12 \iff 3y = 2x + 12 \iff y = \frac{2}{3}x + 4; \text{ the slope is } \frac{2}{3}.$$

We see that both lines have the same slope of $\frac{2}{3}$. Therefore, the lines are parallel. \square

Problem 169. Prove algebraically that the lines $x - 10y = 4$ and $-\frac{1}{2}x + 5y = 0$ are parallel.

Problem 170. Is it true that the lines $-x + 2y = 3$ and $2x - 4y = -6$ are parallel?

Problem 171. Let l_1 be the line that passes through the points $(0, 3)$ and $(2, 2)$, and l_2 be the line that passes through the points $(4, 0)$ and $(2, 1)$. Prove algebraically that the lines l_1 and l_2 are parallel.

Problem 172. Find an equation of a line that passes through the point $(1, -2)$ and is parallel to the line $x + 5y = 2015$.

Problem 173. Prove algebraically that the lines $2x - 3y = 1$ and $\frac{1}{3}x + \frac{2}{9}y = 2$ are perpendicular.

Solution. Let us calculate the slopes of both lines.

$$2x - 3y = 1 \iff 3y = 2x - 1 \iff y = \frac{2}{3}x - \frac{1}{3}, \text{ the slope is } \frac{2}{3}$$

$$\frac{1}{3}x + \frac{2}{9}y = 2 \iff 3x + 2y = 18 \iff 2y = -3x + 18 \iff y = -\frac{3}{2}x + 9, \text{ the slope is } -\frac{3}{2}$$

The product of the slopes is $\frac{2}{3} \cdot \left(-\frac{3}{2}\right) = -1$, so the lines are perpendicular. \square

Problem 174. Prove algebraically that the lines $3x + \frac{1}{2}y = 1$ and $-\frac{1}{2}x + 3y = 3$ are perpendicular.

Problem 175. Is it true that the lines $x = 3$ and $y = -6$ are perpendicular?

Problem 176. Let l_1 be the line that passes through the points $(0, 3)$ and $(2, 2)$, and l_2 be the line that passes through the points $(0, 4)$ and $(-2, 0)$. Prove algebraically that the lines l_1 and l_2 are perpendicular.

Problem 177. Find an equation of a line that passes through the point $(1, -2)$ and is perpendicular to the line $x + 5y = 2015$.

6. Linear systems

6.1. What is a solution

Problem 178. a) Show that $x = 2$, $y = 1$ is a solution of the system below, but $x = -4$, $y = -3$ is not a solution:

$$\begin{cases} 2x - 3y = 1 \\ 3x - 2y = 4 \end{cases}$$

b) On a coordinate system, draw the lines $2x - 3y = 1$ and $3x - 2y = 4$. Show that the point $(2, 1)$ belongs to both of the lines. How is the point $(-4, -3)$ located with respect to the lines?

c) Explain why the system above has no solutions besides $(x, y) = (2, 1)$.

Solution. a) We plug in $x = 2$, $y = 1$ in **both** equations of the system:

$$\begin{cases} 2 \cdot 2 - 3 \cdot 1 = 1 \\ 3 \cdot 2 - 2 \cdot 1 = 4 \end{cases} \iff \begin{cases} 1 = 1 \checkmark \\ 4 = 4 \checkmark \end{cases}$$

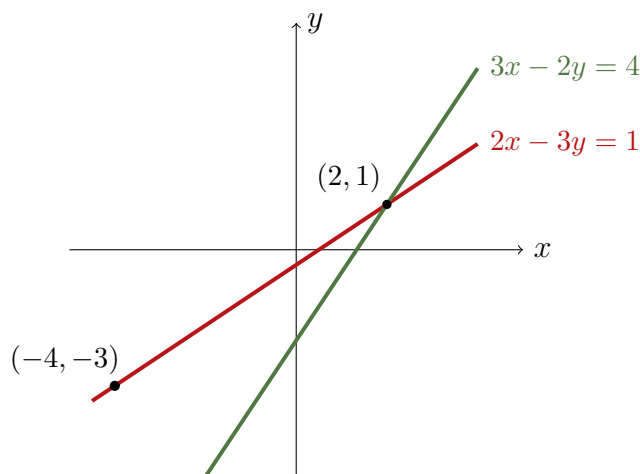
and see that the equations are satisfied, so $x = 2$, $y = 1$ is a solution indeed.

Now plug in $x = -4$, $y = -3$:

$$\begin{cases} 2 \cdot (-4) - 3 \cdot (-3) = 1 \\ 3 \cdot (-4) - 2 \cdot (-3) = 4 \end{cases} \iff \begin{cases} 1 = 1 \checkmark \\ -6 = 4 \times \end{cases}$$

The second equation is not satisfied, therefore, $x = -4$, $y = -3$ is **not** a solution for the system.

b) The point $(-4, -3)$ belongs to one line and doesn't belong to the other.



c) Geometrically, the system represents two intersecting lines. The solution of the system is a point belonging to both lines, that is the intersection point $(2, 1)$. Since two distinct lines can't intersect in more than one point, the only solution is $(x, y) = (2, 1)$.

□

Problem 179. a) Prove algebraically that the lines $-x + 2y = 0$ and $3x - 6y = 1$ are parallel and use this fact to prove that the system

$$\begin{cases} -x + 2y = 0 \\ 3x - 6y = 1 \end{cases}$$

has no solutions.

b) Draw the lines and give geometric interpretation of the system.

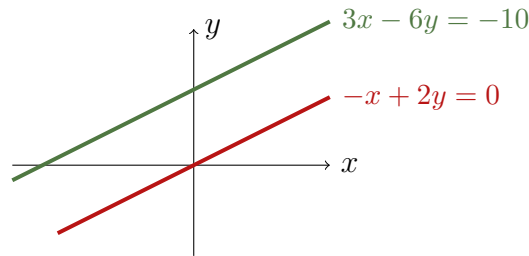
Solution. a) Let us find the slopes of the lines:

$$-x + 2y = 0 \iff 2y = x \iff y = \frac{1}{2}x, \text{ so the slope is } \frac{1}{2}$$

$$3x - 6y = 1 \iff 6y = 3x - 1 \iff y = \frac{1}{2}x - \frac{1}{6}, \text{ so the slope is } \frac{1}{2}.$$

Since the slopes are the equal and y -intercepts are distinct, then the lines are parallel. It means that there is no intersection between them and the system has **no** solutions.

b) Geometrically, the system represents two parallel lines:



□

Problem 180. a) Prove that $x = 2, y = 3$ is a solution of the system

$$\begin{cases} 2x - y = 1 \\ -4x + 2y = -2 \end{cases}$$

b) Prove that $x = t, y = 2t - 1$ is a solution of the same system for any value of real number t .

c) Prove that any solution of the system above may be written in the form $x = t, y = 2t - 1$, where t is a real number.

d) Draw the lines $2x - y = 1$ and $-4x + 2y = -2$. What is their mutual position? What is a geometric meaning of the system and its solution?

Solution. a) Plug in $x = 2, y = 3$ in the system

$$\begin{cases} 2 \cdot 2 - 3 = 1 \\ -4 \cdot 2 + 2 \cdot 3 = -2 \end{cases} \iff \begin{cases} 1 = 1 \checkmark \\ -2 = 2 \checkmark \end{cases}$$

and see that the equations are satisfied, so $x = 2, y = 1$ is a solution.

b) Plug in $x = t, y = 2t - 1$ in the system

$$\begin{cases} 2 \cdot t - (2t - 1) = 1 \\ -4 \cdot t + 2 \cdot (2t - 1) = -2 \end{cases} \iff \begin{cases} 2 \cdot t - 2t + 1 = 1 \\ -4 \cdot t + 4t - 2 = -2 \end{cases} \iff \begin{cases} 1 = 1 \checkmark \\ -2 = 2 \checkmark \end{cases}$$

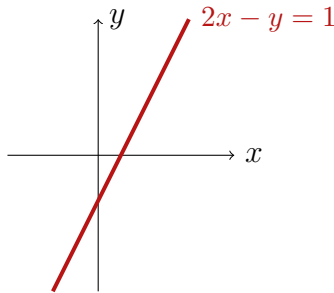
and see that the equations are satisfied, so $x = t, y = 2t - 1$ is a solution for any t .

c) Let us solve the system

$$\begin{cases} 2x - y = 1 \\ -4x + 2y = -2 \end{cases} \iff \begin{cases} y = 2x - 1 \\ 2y = 4x - 2 \end{cases} \iff \begin{cases} y = 2x - 1 \\ y = 2x - 1 \end{cases} \iff y = 2x - 1$$

The system is reduced to just one equation. It means that the solution of the system is the solution of this equation. What is the solution of the equation $y = 2x - 1$? It is any pair (x, y) of numbers satisfying the equation $y = 2x - 1$. If we let $x = t$, where t is an arbitrary real number, then $y = 2t - 1$.

d) The second equation of the system is obtained from the first one by multiplication by -2 . Since the equations are equivalent, they represent geometrically the **same** line:



One can say that the lines $2x - y = 1$ and $-4x + 2y = -2$ coincide. The solution of the system, as the intersection between these lines, is the line itself: any point (x, y) on this line solves the system. Since there are infinitely many points on the line, then there are infinitely many solutions for the system. \square

6.2. How to solve a system

Problem 181. Solve the following systems and check the answers by substitution (if possible). Give geometric interpretation of each system.

$$\text{a) } \begin{cases} 3x - 2y = -5 \\ -2x + 3y = 5 \end{cases} \quad \text{b) } \begin{cases} x + 2y = 1 \\ -2x - 4y = 3 \end{cases} \quad \text{c) } \begin{cases} 3x - 2y = -2 \\ -6x + 4y = -4 \end{cases}$$

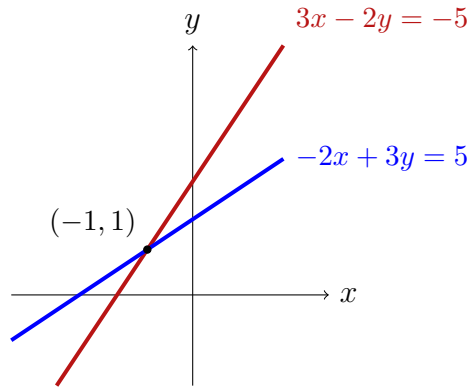
Solution.

$$\begin{aligned} \text{a) } \begin{cases} 3x - 2y = -5 \\ -2x + 3y = 5 \end{cases} &\iff \begin{cases} 2(3x - 2y) = 2 \cdot (-5) \\ 3(-2x + 3y) = 3 \cdot 5 \end{cases} \iff \begin{cases} 6x - 4y = -10 \\ -6x + 9y = 15 \end{cases} \iff \\ \begin{cases} 6x - 4y = -10 \\ 5y = 5 \end{cases} &\iff \begin{cases} 3x - 2y = -5 \\ y = 1 \end{cases} \iff \begin{cases} 3x - 2 \cdot 1 = -5 \\ y = 1 \end{cases} \iff \begin{cases} 3x = -3 \\ y = 1 \end{cases} \iff \\ \begin{cases} x = -1 \\ y = 1 \end{cases} & \end{aligned}$$

Now check the solution $x = -1, y = 1$ by substitution into **both** equations of the system:

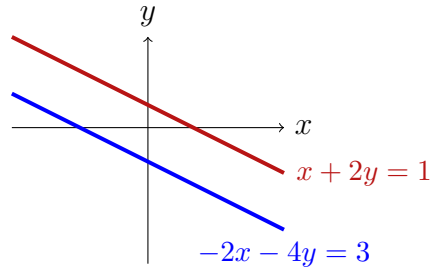
$$\begin{cases} 3 \cdot (-1) - 2 \cdot 1 \stackrel{?}{=} -5 \\ -2 \cdot (-1) + 3 \cdot 1 \stackrel{?}{=} 5 \end{cases} \iff \begin{cases} -5 \stackrel{\checkmark}{=} -5 \\ 5 \stackrel{\checkmark}{=} 5. \end{cases} \quad \text{So the system is solved correctly.}$$

For a geometric interpretation, we have to draw both lines representing the equations of the system and show that they intersect each other at the point $(-1, 1)$.



$$\text{b) } \begin{cases} x + 2y = 1 \\ -2x - 4y = 3 \end{cases} \iff \begin{cases} 2x + 4y = 2 \\ -2x - 4y = 3 \end{cases} \iff \begin{cases} 2x + 4y = 2 \\ 0 = 5 \end{cases}$$

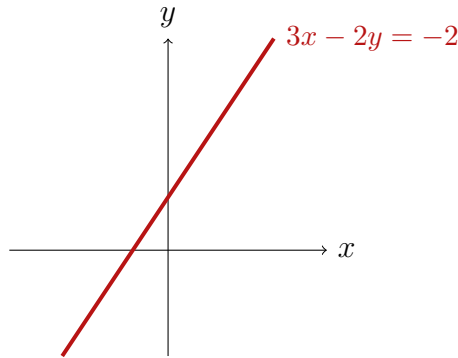
The contradiction $0 = 5$ implies that the system has **no** solution, and we may expect to see **parallel** lines on the picture:



$$\text{c) } \begin{cases} 3x - 2y = -2 \\ -6x + 4y = -4 \end{cases} \iff \begin{cases} 6x - 4y = -4 \\ -6x + 4y = -4 \end{cases} \iff \begin{cases} 6x - 4y = -4 \\ 0 = 0 \end{cases} \iff \begin{cases} 3x - 2y = -2 \\ 0 = 0 \end{cases}$$

$$\iff 3x - 2y = -2 \iff y = \frac{3}{2}x + 1 \iff \begin{cases} x = t \\ y = \frac{3}{2}t + 1, \end{cases} \quad \text{where } t \text{ is a real number.}$$

Notice that the second equation of the system is obtained from the first one by multiplication by -2 . It means that our system is equivalent to just one equation in two variables. Therefore, the system has infinitely many solutions, namely $x = t$, $y = \frac{3}{2}t + 1$, where t is an arbitrary real number. All the solutions comprise the line $3x - 2y = -2$:



Let us check our solutions $x = t$, $y = \frac{3}{2}t + 1$ by substitution into the first equation (the second equation is equivalent to the first one, so we may omit it):

$3t - 2\left(\frac{3}{2}t + 1\right) \stackrel{?}{=} -2 \iff 3t - 3t - 2 \stackrel{?}{=} -2 \iff -2 \stackrel{\checkmark}{=} -2$. So the system is solved correctly. \square

Problem 182. Solve the following systems and check the answers by substitution (if possible). Give geometric interpretation of each system.

$$\begin{array}{llll} \text{a)} \begin{cases} 2x + 3y = 5 \\ 7x + 2y = 9 \end{cases} & \text{b)} \begin{cases} 2x + y = 5 \\ x - y = 4 \end{cases} & \text{c)} \begin{cases} 4x + 3y = 0 \\ 7x + 4y = 5 \end{cases} & \text{d)} \begin{cases} 5x + 4y = 2 \\ 3x - 2y = -12 \end{cases} \\ \text{e)} \begin{cases} 2x + 3y = -5 \\ 4x + 6y = 10 \end{cases} & \text{f)} \begin{cases} \frac{1}{2}x - 3y = 0 \\ -x + 6y = 4 \end{cases} & \text{g)} \begin{cases} x + 5y = -2 \\ 0.2x + y = 1 \end{cases} & \text{h)} \begin{cases} -2x + 4y = 9 \\ x + 2y = -12 \end{cases} \\ \text{i)} \begin{cases} 2x + y = 1 \\ 6x + 3y = 3 \end{cases} & \text{j)} \begin{cases} 2x - y = -5 \\ -x + \frac{y}{2} = 2.5 \end{cases} & \text{k)} \begin{cases} 4x + 3y = 0 \\ -4x - 3y = 0 \end{cases} & \text{l)} \begin{cases} -x + 0.4y = 2 \\ \frac{5}{2}x - y = -5 \end{cases} \end{array}$$

6.3. Word problems leading to linear systems

Problem 183. A planet is populated by creatures of two types: threellies and fourlies. A threelly has three legs and a fourly has four legs. In a dancing hall, there are 13 creatures dancing with all their 47 legs. How many threellies and how many fourlies are dancing? (Solve the problem by composing a linear system).

Solution. Let x and y be the numbers of dancing threellies and fourlies respectively. Since there are 13 creatures in total, then $x + y = 13$. How many legs do they all have? It's $3x + 4y$. Since there are 47 legs in total, $3x + 4y = 47$. Therefore, we get the following system

$$\begin{cases} x + y = 13 \\ 3x + 4y = 47. \end{cases}$$

We multiply the first equation by 3 and subtract it from the second equation:

$$\begin{cases} x + y = 13 \\ 3x + 4y = 47 \end{cases} \iff \begin{cases} x + y = 13 \\ y = 47 - 39 \end{cases} \iff \begin{cases} x + y = 13 \\ y = 8 \end{cases} \iff \begin{cases} x = 5 \\ y = 8. \end{cases}$$

Let us check the solution: the total number of creatures is $5 + 8 = 13$, and the total number of their legs if $3 \cdot 5 + 4 \cdot 8 = 47$. \square

Problem 184. It took 3 hours for Wilma, the witch, to fly 75 km on her broomstick from her house to the Halloween Party with a tailwind. The way back, against the wind, took 5 hours. Find the speed of Wilma's broomstick and the speed of the wind, assuming that the motion is uniform. (Solve the problem by composing a system of linear equations.)

Solution. Let x km/h be the the speed of Wilma's broomstick, and y km/h be the speed of the wind. When Wilma flies with a tailwind, her speed is $(x + y)$ km/h. With a tailwind, it took her 3 hours to cover 75 km. Since for a uniform motion $\text{time} \times \text{speed} = \text{distance}$, then $3(x + y) = 75$.

When she flies against the wind, her speed is $(x - y)$ km/h. The way back (also 75km) took 5 hours. Therefore, $5(x - y) = 75$. Here is our system:

$$\begin{cases} 3(x + y) = 75 \\ 5(x - y) = 75. \end{cases}$$

We divide the first equation by 3 and divide the second equation by 5:

$$\begin{cases} x + y = 25 \\ x - y = 15 \end{cases} \iff \begin{cases} x + y = 25 \\ 2x = 40 \end{cases} \iff \begin{cases} x + y = 25 \\ y = 20 \end{cases} \iff \begin{cases} x = 5 \\ y = 20. \end{cases}$$

\square

Problem 185. Find an equation of a line (in the slope-intercept form) passing through the points $(-1, 2)$ and $(3, 4)$. Solve the problem by composing a linear system.

Problem 186. A family of two adults and five kids pays \$61 for tickets in a movie theater. A family of three adults and two kids pays \$53. Find the prices for an adult ticket and for a kid ticket. (Solve the problem by composing a linear system).

Problem 187. It takes three hours for a plane to fly 2,400 mi from A to B with a tailwind and 3 hours 20 min from B to A against the wind. Find the speed of the plane and the speed of the wind, assuming that the motion is uniform. (Solve the problem by composing a system of linear equations.)

Problem 188. A car rental company rents compact cars at \$55 per day per car, and SUVs at \$70 per day per car. One day, 25 cars were rented for a total cost of \$1,600. How many compact cars and how may SUVs were rented that day? (Solve the problem by composing a system of linear equations.)

Problem 189. On a farm, there are sheep and chickens. All together, they have 44 feet and 17 heads. How many sheep and how many chickens are on the farm? (Solve the problem by composing a system of equations.)

Problem 190. Ben worked for 30 hours in a library and 50 hours in a cafeteria and earned \$820 in total. The cafeteria pays \$2 more per hour than the library. Find the pay rates at the library and the cafeteria. (Solve the problem by composing a system of equations.)

Problem 191. A winemaker has in his cellar 1620 liters of wine aging in three small and five large barrels. Find the volumes of the barrels if the large barrel contains 20 liters more than the small one. (Solve the problem by composing a system of linear equations.)

7. Radicals

7.1. What is a radical?

Problem 192. What is $\sqrt{49}$ equal to and why?

Solution. $\sqrt{49} = 7$. Why? Because $\sqrt{49}$ (the radical of 49) is, by definition, a **non-negative** real number whose square is equal to 49. Since $7 \geq 0$ and $7^2 = 49$, then $\sqrt{49} = 7$. \square

Problem 193. Between which integers is $\sqrt{47}$ located?

Solution. We are looking for integers whose squares are closest to 47. Since $7^2 = 49$ and $6^2 = 36$, we conclude that the number, whose square is 47, is between 6 and 7. So $6 < \sqrt{47} < 7$. \square

Problem 194. What is $\sqrt{-9}$?

Solution. Nothing is, because the square of any real number is non-negative. \square

Problem 195. Find **all** real numbers which square is equal to

a) 16 b) 0 c) -16

Problem 196. Explain why the following identities are **incorrect**:

a) $\sqrt{2} = 4$ b) $\sqrt{4} = -2$ c) $\sqrt{4} = \pm 2$ d) $\sqrt{-4} = 2$ e) $\sqrt{-4} = -2$

Solution. a) $\sqrt{2} \neq 4$ since $4^2 = 16 \neq 2$.

b) $\sqrt{4} \neq -2$ since a radical is non-negative.

c) $\sqrt{4} \neq \pm 2$ since a radical is a **single** number.

d) $\sqrt{-4} \neq 2$ since $2^2 \neq -4$ (note that -4 is a negative number and $\sqrt{-4}$ is **not** defined).

e) $\sqrt{-4} \neq -2$ by two reasons. First, $\sqrt{-4}$ is **not** defined since -4 is a negative number, and second, a radical is always non-negative and couldn't be equal to -2 . \square

Problem 197. For which values of x are the following expressions defined

a) \sqrt{x} b) $\sqrt{x-1}$ c) $\sqrt{x+2}$ d) $\frac{1}{\sqrt{x}}$ e) $\frac{2}{\sqrt{x+1}}$ f) $\frac{\sqrt{x}}{\sqrt{x-2}}$

Solution. a) \sqrt{x} is defined for all $x \geq 0$.

b) $\sqrt{x-1}$ is defined when $x-1 \geq 0$, that is for $x \geq 1$.

e) $\frac{\sqrt{x}}{\sqrt{x-2}}$ is defined when $x \geq 0$ and $x-2 > 0$ (since $x \neq 2$), that is for $x > 2$. \square

Problem 198. Let $a > 0$. Prove that if $x^2 = a$ for some x , then $(-x)^2 = a$.

Solution. $(-x)^2 = ((-1) \cdot x)^2 = (-1)^2 \cdot x^2 = x^2 = a$. \square

Problem 199. Solve the equations

a) $x^2 = 5^2$ b) $x^2 = (-5)^2$ c) $x^2 = a^2$, where a is an arbitrary real number.

Solution. a) To solve the equation $x^2 = 5^2$, that is, to find **all** the solutions, we observe that $x = 5$ is a solution. Then $x = -5$ is also a solution, see previous problem.

Are there other solutions besides $x = \pm 5$? To answer this question, let us rewrite the equation in the form $x^2 - 5^2 = 0$. By factoring, we get $(x - 5)(x + 5) = 0$. The product of two quantities is equal to 0 if and only if one of the quantities is 0. So

$$(x - 5)(x + 5) = 0 \iff x - 5 = 0 \text{ or } x + 5 = 0 \iff x = 5 \text{ or } x = -5.$$

Hence the equation has no solutions besides $x = \pm 5$.

b) The equation $x^2 = (-5)^2$ is equivalent to $x^2 = 5^2$ since $(-5)^2 = 5^2$, and has the same solution.

c) $x^2 = a^2 \iff x^2 - a^2 = 0 \iff (x - a)(x + a) = 0 \iff x = a \text{ or } x = -a$. Notice that there are two solutions if $a \neq 0$ and there is only one solution if $a = 0$. \square

Problem 200. Evaluate the following radicals. Do not use calculator.

a) $\sqrt{1}$ b) $\sqrt{4}$ c) $\sqrt{9}$ d) $\sqrt{16}$ e) $\sqrt{25}$ f) $\sqrt{36}$ g) $\sqrt{49}$ h) $\sqrt{64}$
 i) $\sqrt{81}$ j) $\sqrt{100}$ k) $\sqrt{121}$ l) $\sqrt{144}$ m) $\sqrt{169}$ n) $\sqrt{196}$ o) $\sqrt{225}$

Problem 201. Evaluate the following radicals. Do not use calculator.

a) $\sqrt{0.01}$ b) $\sqrt{0.0001}$ c) $\sqrt{\frac{1}{4}}$ d) $\sqrt{\frac{9}{16}}$ e) $\sqrt{\frac{25}{64}}$ f) $\sqrt{\frac{100}{81}}$

Problem 202. Calculate without calculator

a) $(\sqrt{2})^2$ b) $(\sqrt{3})^2$ c) $(\sqrt{4})^2$ d) $(\sqrt{5})^2$ e) $(\sqrt{2015})^2$

Problem 203. Is it true that $(\sqrt{x})^2 = x$ for every real x ?

Solution. No. Explain why! \square

Problem 204. Calculate without calculator

a) $\sqrt{2^2}$ b) $\sqrt{(-2)^2}$ c) $\sqrt{3^2}$ d) $\sqrt{(-3)^2}$ e) $\sqrt{2015^2}$ f) $\sqrt{(-2015)^2}$

Problem 205. Is it true that $\sqrt{x^2} = x$ for every real number?

Solution. No. Explain why! \square

Problem 206. Prove that $\sqrt{x^2} = |x|$ for any real x .

Problem 207. Simplify the following expressions

a) $(\sqrt{3})^4$ b) $(\sqrt{3})^5$ c) $(\sqrt{3})^{2015}$ d) $(\sqrt{3})^{2016}$ e) $(-\sqrt{3})^{2017}$

Solution. a) $(\sqrt{3})^4 = ((\sqrt{3})^2)^2 = 3^2 = 9$. Alternatively, $(\sqrt{3})^4 = (\sqrt{3} \cdot \sqrt{3}) \cdot (\sqrt{3} \cdot \sqrt{3}) = 3 \cdot 3 = 9$.

b) $(\sqrt{3})^5 = (\sqrt{3})^{4+1} = (\sqrt{3})^4 \cdot \sqrt{3} = 9\sqrt{3}$

c) $(\sqrt{3})^{2015} = (\sqrt{3})^{2014+1} = (\sqrt{3})^{2 \cdot 1007+1} = ((\sqrt{3})^2)^{1007} \cdot \sqrt{3} = 3^{1007} \cdot \sqrt{3}$ □

Problem 208. Let x and y be nonnegative numbers. Explain why

a) $\sqrt{xy} = \sqrt{x} \cdot \sqrt{y}$ b) $\sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}}$, if $y \neq 0$

Solution. The square of the left hand side of the identity is $(\sqrt{xy})^2 = xy$. The square of the right hand side of the identity is $(\sqrt{x} \cdot \sqrt{y})^2 = (\sqrt{x})^2 \cdot (\sqrt{y})^2 = xy$. Therefore, $(\sqrt{xy})^2 = (\sqrt{x} \cdot \sqrt{y})^2$ and, since the expressions in the parentheses are nonnegative, $\sqrt{xy} = \sqrt{x} \cdot \sqrt{y}$. □

Problem 209. Simplify the following expressions

a) $\sqrt{3^4}$ b) $\sqrt{3^5}$ c) $\sqrt{3^{2015}}$ d) $\sqrt{3^{2016}}$

Solution. a) $\sqrt{3^4} = \sqrt{(3^2)^2} = 3^2 = 9$.

b) $\sqrt{3^5} = \sqrt{(3^2)^2 \cdot 3} = \sqrt{(3^2)^2} \cdot \sqrt{3} = 9\sqrt{3}$. □

Problem 210. Calculate without calculator

a) $\sqrt{3} \cdot \sqrt{12}$ b) $\frac{\sqrt{3}}{\sqrt{12}}$ c) $\sqrt{2} \cdot \sqrt{7} \cdot \sqrt{14}$ d) $\frac{\sqrt{2}}{\sqrt{7} \cdot \sqrt{14}}$

Solution. a) $\sqrt{3} \cdot \sqrt{12} = \sqrt{3 \cdot 12} = \sqrt{36} = 6$. Alternatively, $\sqrt{3} \cdot \sqrt{12} = \sqrt{3} \cdot \sqrt{4 \cdot 3} = \sqrt{3} \cdot \sqrt{4} \cdot \sqrt{3} = (\sqrt{3})^2 \cdot 2 = 3 \cdot 2 = 6$. □

Problem 211. Bring to simplest radical form

a) $\sqrt{2^6 \cdot 3^3}$ b) $\sqrt{3^5 \cdot 5^3}$ c) $\sqrt{72}$ d) $\sqrt{124}$ e) $\sqrt{200}$

Solution. a) $\sqrt{2^6 \cdot 3^3} = \sqrt{2^6} \cdot \sqrt{3^3} = \sqrt{(2^3)^2} \cdot \sqrt{3^2 \cdot 3} = 2^3 \cdot 3\sqrt{3} = 24\sqrt{3}$.

c) $\sqrt{72} = \sqrt{36 \cdot 2} = 6\sqrt{2}$. □

Problem 212. Bring to simplest radical form

a) $\frac{1}{\sqrt{2}}$ b) $\frac{5}{\sqrt{7}}$ c) $\frac{2}{\sqrt{12}}$ d) $\frac{\sqrt{6}}{\sqrt{15}}$ e) $\frac{1}{1 + \sqrt{2}}$ f) $\frac{2}{1 - 3\sqrt{2}}$

Solution. a) $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2} \cdot \sqrt{2}} = \frac{\sqrt{2}}{2}$

e) $\frac{1}{1 + \sqrt{2}} = \frac{1 - \sqrt{2}}{(1 + \sqrt{2})(1 - \sqrt{2})} = \frac{1 - \sqrt{2}}{1^2 - (\sqrt{2})^2} = \frac{1 - \sqrt{2}}{1 - 2} = \sqrt{2} - 1$ □

Problem 213. Which number is greater?

a) $2\sqrt{10}$ or $3\sqrt{5}$ b) $\frac{1}{3}\sqrt{27}$ or $\frac{1}{5}\sqrt{50}$ c) $\frac{1}{3-\sqrt{2}}$ or 1?

Solution. a) Let us square both numbers: $(2\sqrt{10})^2 = 4 \cdot 10 = 40$ and $(3\sqrt{5})^2 = 9 \cdot 5 = 45$. Since $40 < 45$, then $2\sqrt{10} < 3\sqrt{5}$.

c) Bring the first number to simplest radical form:

$$\frac{1}{3-\sqrt{2}} = \frac{3+\sqrt{2}}{(3-\sqrt{2})(3+\sqrt{2})} = \frac{3+\sqrt{2}}{3^2 - (\sqrt{2})^2} = \frac{3+\sqrt{2}}{9-2} = \frac{3+\sqrt{2}}{7}.$$

Since $\sqrt{2} < 4$ then $3 + \sqrt{2} < 3 + 4 = 7$. Therefore $\frac{3+\sqrt{2}}{7} < \frac{7}{7} = 1$, so $\frac{1}{3-\sqrt{2}} < 1$.

□

Problem 214. Simplify the following expressions assuming that x and y are nonnegative

a) $\sqrt{x^4}$ b) $\sqrt{y^6}$ c) $\sqrt{16x^2y^8}$ d) $\sqrt{x^3y^{10}}$ e) $\sqrt{20x^9y^7}$ f) $\sqrt{54xy^5}$

Solution. a) $\sqrt{x^4} = \sqrt{(x^2)^2} = x^2$

d) $\sqrt{x^3y^{10}} = \sqrt{x^2 \cdot x \cdot (y^5)^2} = xy^5 \cdot \sqrt{x}$

□

Problem 215. Explain why the following statement is **incorrect**: $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$ for arbitrary non-negative x and y .

Solution. Take, for example, $x = 1$ and $y = 1$. Then the left hand side is $\sqrt{x+y} = \sqrt{1+1} = \sqrt{2}$, and the right hand side is $\sqrt{x} + \sqrt{y} = \sqrt{1} + \sqrt{1} = 1 + 1 = 2$. Since $\sqrt{2} \neq 2$, then $\sqrt{x+y} \neq \sqrt{x} + \sqrt{y}$ for **arbitrary** x and y .

Notice, that for **some** values of x and y , $\sqrt{x+y}$ is equal to $\sqrt{x} + \sqrt{y}$. Take, for example, $x = 0$ and $y = 1$.

□

Problem 216. Find all values of x and y for which $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$.

Solution. Let us square both sides of the equation:

$$\begin{aligned} (\sqrt{x+y})^2 &= (\sqrt{x} + \sqrt{y})^2 \\ x+y &= x + 2\sqrt{x}\sqrt{y} + y \\ 0 &= \sqrt{x}\sqrt{y} \\ x=0 \text{ or } y &= 0 \end{aligned}$$

It follows, that $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$ if and only if $x = 0$ or $y = 0$.

□

Problem 217. Simplify

a) $\sqrt{36+64x^2}$ b) $\sqrt{x^2-x^3y^2}$ c) $\sqrt{x^2+2xy+y^2}$ d) $\sqrt{9x^2-6xy+y^2}$

Solution. a) $\sqrt{36+64x^2} = \sqrt{4(9+16x^2)} = 2\sqrt{9+16x^2}$. Notice that the latter expression can't be simplified further and $\sqrt{9+16x^2} \neq 3+4x$.

b) $\sqrt{x^2 - x^3y^2} = \sqrt{x^2(1 - xy^2)} = |x|\sqrt{1 - xy^2}$

c) $\sqrt{x^2 + 2xy + y^2} = \sqrt{(x + y)^2} = |x + y|$ □

Problem 218. Simplify the following expressions

a) $3\sqrt{2} + 5\sqrt{2}$ b) $\sqrt{2}(1 - 3\sqrt{2})$ c) $(1 + \sqrt{2})(1 - \sqrt{2})$ d) $(\sqrt{2} + \sqrt{3})(\sqrt{2} - 3\sqrt{6})$

Problem 219. The period T of an oscillating pendulum of length l is given by the formula

$$T = 2\pi\sqrt{\frac{l}{g}},$$

where g is the acceleration of gravity. Use this equation to find l in terms of T and g .

Solution. We have to solve the equation for l :

$$T = 2\pi\sqrt{\frac{l}{g}} \implies T^2 = \left(2\pi\sqrt{\frac{l}{g}}\right)^2 \implies T^2 = 4\pi^2\frac{l}{g} \implies \frac{T^2}{4\pi^2} = \frac{l}{g} \implies \frac{T^2g}{4\pi^2} = l.$$

So $l = \frac{T^2g}{4\pi^2}$. □

Problem 220. The **root-mean-square speed** v_{rms} of particles in a gas is defined by

$$v_{rms} = \sqrt{\frac{3RT}{M}},$$

where R is the universal gas constant, T is the temperature, and M is the molar mass of the gas. Use this identity to solve M in terms of R , T and v_{rms} .

Problem 221. According to Newton's law of universal gravitation,

$$F = G \frac{m_1m_2}{R^2},$$

where F is the force between the masses of m_1 and m_2 , G is the gravitational constant, and R is the distance between the centers of the masses. Use this equation to find R in terms of F , G , m_1 and m_2 .

7.3. Cube roots and other radicals

Problem 222. a) What is $\sqrt[3]{64}$ equal to and why?

b) What is $\sqrt[3]{-64}$ equal to and why?

Problem 223. Evaluate without calculator

a) $\sqrt[3]{1}$ b) $\sqrt[3]{-1}$ c) $\sqrt[3]{8}$ d) $\sqrt[3]{-8}$ e) $\sqrt[3]{27}$ f) $\sqrt[3]{-27}$ g) $\sqrt[3]{64}$ h) $\sqrt[3]{-64}$

i) $\sqrt[3]{125}$ j) $\sqrt[3]{-125}$ k) $\sqrt[3]{\frac{8}{27}}$ l) $\sqrt[3]{-\frac{125}{64}}$ m) $\sqrt[3]{0.001}$ n) $\sqrt[3]{-0.000008}$

Problem 224. Evaluate without calculator

a) $\sqrt[4]{16}$ b) $\sqrt[5]{-32}$ c) $\sqrt[6]{64}$ d) $\sqrt[7]{-128}$ e) $\sqrt[8]{256}$ f) $\sqrt[9]{-512}$ g) $\sqrt[10]{1024}$

Problem 225. Explain why the following identities are **incorrect**:

a) $\sqrt[3]{-8} = 2$ b) $\sqrt[3]{8} = -2$ c) $\sqrt[4]{16} = -2$ d) $\sqrt[4]{-16} = -2$

Problem 226. Simplify

a) $\sqrt[3]{x^3}$ b) $(\sqrt[3]{x})^3$ c) $\sqrt[4]{x^4}$ d) $(\sqrt[4]{x})^4$ e) $\sqrt[5]{x^5}$ f) $(\sqrt[5]{x})^5$ g) $\sqrt[6]{x^6}$ h) $(\sqrt[6]{x})^6$

Solution. a) $\sqrt[3]{x^3} = x$ b) $(\sqrt[3]{x})^3 = x$ c) $\sqrt[4]{x^4} = |x|$

d) The expression $\sqrt[4]{x}$ makes sense only for non-negative x . For such x , $(\sqrt[4]{x})^4 = x$. □

7.4. Radicals as powers with rational exponents

Problem 227. Express in terms of powers with rational exponents

a) $\sqrt{2}$ b) $\sqrt{2^2}$ c) $\sqrt{2^3}$ d) $\sqrt{2^4}$ e) $\sqrt{2^5}$ f) $\sqrt{2^6}$ g) $\sqrt{2^7}$

Solution. a) $\sqrt{2} = 2^{1/2}$ b) $\sqrt{2^2} = 2^{2/2} = 2^1 = 2$ c) $\sqrt{2^3} = 2^{3/2}$ □

Problem 228. Express in terms of powers with rational exponents

a) $\sqrt[3]{2}$ b) $\sqrt[3]{2^2}$ c) $\sqrt[3]{2^3}$ d) $\sqrt[3]{2^4}$ e) $\sqrt[4]{2^5}$ f) $\sqrt[4]{2^6}$ g) $\sqrt[5]{2^7}$

Problem 229. Evaluate

a) $27^{1/3}$ b) $25^{3/2}$ c) $81^{3/4}$ d) $8^{4/3}$ e) $32^{3/5}$

Problem 230. Simplify

a) $(2^{3/4})^{4/9}$ b) $\left(\frac{25}{81}\right)^{3/2}$ c) $(x^6y^3)^{2/3}$ d) $\sqrt[3]{(x^6y^3)^2}$ e) $\sqrt[3]{(x^6y^3)^{2/3}}$

8. Quadratic life

8.1. Quadratic trinomials and quadratic equations

Problem 231. Which of the following are quadratic expressions?

a) $3x^2 + 1$ b) $1 - x^2$ c) $2x - 1$ d) $x^{-2} + 4x + 1$ e) $\frac{x^2}{3}$ f) $\sqrt{\pi}x^2 + x - 2$

Solution. Expressions **a)**, **b)**, **e)** and **f)** are quadratic, namely, **e)** is a quadratic monomial, **a)** and **b)** are quadratic binomials, **f)** is a quadratic trinomial. Expressions **c)** and **d)** are not quadratic. \square

Problem 232. Which of the following are quadratic equations?

a) $\sqrt{3}x^2 - \pi x + 2015 = 0$ b) $x^2 = 0$ c) $-a^2 + 5a - 7 = 0$ d) $x^2 - x(2x + 1) = 0$
e) $(x + 1)(x - 2) = x^2$ f) $x^2(x + 1) = x^3 - x$ g) $x^2 + x + 1$

Solution. An equation is called *quadratic*, if it is equivalent to the equation $ax^2 + bx + c = 0$, where a, b, c are given constants and $a \neq 0$. This equation is called a quadratic equation in the *standard form*.

Equation **a)** is quadratic in standard form, with “ugly” coefficients though. Equation **b)** is also quadratic, it is the simplest quadratic equation that one can make up. Equation **c)** is quadratic with a as an unknown instead of x .

Equation **d)** can be simplified

$$x^2 - x(2x + 1) = 0 \iff x^2 - 2x^2 - x = 0 \iff -x^2 - x = 0$$

to the standard form. So it is quadratic.

Equation **e)** can be written:

$$(x + 1)(x - 2) = x^2 \iff x^2 - x - 2 = x^2 \iff -x - 2 = 0.$$

It is a linear equation, not quadratic.

Equation **f)** doesn't look as a quadratic, but a simplification

$$x^2(x + 1) = x^3 - x \iff x^3 + x^2 = x^3 - x \iff x^2 + x = 0$$

gives a standard form of a quadratic equation. So the original equation is quadratic.

The quadratic expression in **g)** is not an equation. \square

8.2. How to solve quadratic binomial equations

Problem 233. Solve the following quadratic equations:

a) $x^2 - 9 = 0$ b) $x^2 + 4 = 0$ c) $3x^2 - 1 = 0$ d) $-2x^2 + 3 = 0$ e) $4x^2 - \pi = 0$

Solution. a) $x^2 = 9 \iff \sqrt{x^2} = \sqrt{9} \iff |x| = 3 \iff x = 3$ or $x = -3$.

Alternatively, one can solve the equation by factoring:

$$x^2 - 9 = 0 \iff x^2 - 3^2 = 0 \iff (x - 3)(x + 3) = 0 \iff$$

$$x - 3 = 0 \text{ or } x + 3 = 0 \iff x = 3 \text{ or } x = -3.$$

d) $-2x^2 + 3 = 0 \iff x^2 = \frac{3}{2} \iff x = \pm\sqrt{\frac{3}{2}} = \pm\frac{\sqrt{6}}{2}$. Solving quadratic equations, always rationalize the answer. \square

Problem 234. The kinetic energy E of a rigid body of mass m moving at the speed v is $E = \frac{mv^2}{2}$. Use this equation to find the speed v in terms of E and m .

Solution. $E = \frac{mv^2}{2} \implies 2E = mv^2 \implies \frac{2E}{m} = v^2 \implies v = \sqrt{\frac{2E}{m}}$.

Riddle: In our solution, why can we go from v^2 to v without having an absolute value in the answer? \square

Problem 235. The surface area A of a sphere of radius r is given by the formula $A = 4\pi r^2$. Use this formula to find the radius r in terms of the surface area A .

Problem 236. The total surface area A of a cylinder of radius r and a height h is given by the formula

$$A = 2\pi rh + 2\pi r^2.$$

Use this formula to find the radius r in terms of A and h .

Problem 237. Solve the following quadratic equations:

a) $x^2 - 3x = 0$ b) $-3x^2 + 4x = 0$ c) $\pi x^2 - x = 0$ d) $\frac{x^2}{2} + 4x = 0$ e) $-x^2 + \frac{2x}{3} = 0$

Solution. a) $x^2 - 3x = 0$
 $x(x - 3) = 0$
 $x = 0$ or $x = 3$

d) $\frac{x^2}{2} + 4x = 0$
 $x^2 + 8x = 0$
 $x(x + 8) = 0$
 $x = 0$ or $x = -8$

\square

8.3. How to solve quadratic equations by completing the square

Problem 238. Solve the following quadratic equations by completing the square:

a) $x^2 + 6x + 5 = 0$ b) $x^2 - 4x + 1 = 0$ c) $3x^2 - 4x - 2 = 0$

Solution. a) $x^2 + 6x + 5 = 0$

$$x^2 + 6x = -5$$

$$x^2 + 2 \cdot x \cdot 3 + 3^2 = -5 + 3^2$$

$$(x + 3)^2 = 4$$

$$x + 3 = 2 \text{ or } x + 3 = -2$$

$$x = -1 \text{ or } x = -5$$

c) $3x^2 - 4x - 2 = 0$

$$3 \left(x^2 - \frac{4}{3}x \right) = 2$$

$$x^2 - \frac{4}{3}x = \frac{2}{3}$$

$$x^2 - 2 \cdot x \cdot \frac{2}{3} + \left(\frac{2}{3} \right)^2 = \frac{2}{3} + \left(\frac{2}{3} \right)^2$$

$$\left(x - \frac{2}{3} \right)^2 = \frac{10}{9}$$

$$x - \frac{2}{3} = \sqrt{\frac{10}{9}} \text{ or } x - \frac{2}{3} = -\sqrt{\frac{10}{9}}$$

$$x = \frac{2 + \sqrt{10}}{3} \text{ or } x = \frac{2 - \sqrt{10}}{3}$$

□

8.4. How to solve quadratic equations using the quadratic formula

Problem 239. Solve the following quadratic equations using the quadratic formula

a) $x^2 + 6x + 5 = 0$ b) $-2x^2 - 8x + 2 = 0$ c) $3x^2 - 4x + 2 = 0$

Solution. The quadratic equation $ax^2 + bx + c = 0$ has solutions

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This formula may be used if $b^2 - 4ac \geq 0$. Otherwise the equation has no real solutions.

a) To use the quadratic formula, we have to identify the coefficients a, b, c in our equation $x^2 + 6x + 5 = 0$: $a = 1, b = 6, c = 5$. So the quadratic formula gives

$$x_{1,2} = \frac{-6 \pm \sqrt{6^2 - 4 \cdot 1 \cdot 5}}{2 \cdot 1} = \frac{-6 \pm \sqrt{36 - 20}}{2} = \frac{-6 \pm \sqrt{16}}{2} = \frac{-6 \pm 4}{2} = -3 \pm 2.$$

The latter expression -3 ± 2 is not the answer yet. We need to split it: $x_1 = -3 + 2 = -1$, $x_2 = -3 - 2 = -5$.

Answer: $x_1 = -1, x_2 = -5$.

b) First, let us simplify the equation by division of both sides by -2 :

$$-2x^2 - 8x + 2 = 0 \iff x^2 + 4x - 1 = 0.$$

Identify the coefficients: $a = 1, b = 4, c = -1$, and solve:

$$x_{1,2} = \frac{-4 \pm \sqrt{4^2 - 4 \cdot 1 \cdot (-1)}}{2 \cdot 1} = \frac{-4 \pm \sqrt{20}}{2} = \frac{-4 \pm 2\sqrt{5}}{2} = -2 \pm \sqrt{5}.$$

Hence $x_1 = -2 + \sqrt{5}, x_2 = -2 - \sqrt{5}$.

c) In $3x^2 - 4x + 2 = 0$, the coefficients are $a = 3$, $b = -4$, $c = 2$. So

$$x_{1,2} = \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 3 \cdot 2}}{2 \cdot 3} = \frac{4 \pm \sqrt{16 - 24}}{6} = \frac{4 \pm 2\sqrt{-8}}{6}.$$

In this case, there are no real solutions, since $\sqrt{-8}$ is not a real number. \square

Problem 240. Solve the following quadratic equations using the quadratic formula

a) $x^2 - 5 = 0$ b) $3x^2 - 2x = 0$ c) $2x^2 - x - 1 = 0$ d) $x^2 - x - 1 = 0$

e) $3x^2 - x + 1 = 0$ f) $40x^2 - 60x + 20 = 0$ g) $-\frac{x^2}{2} + 3x - 2 = 0$

8.5. Factoring quadratic trinomials and roots of quadratic equations

Problem 241. Factor the following quadratic trinomials:

a) $x^2 + x - 6$ b) $x^2 - x - 1$ c) $3x^2 - x + 1$

Solution. If the quadratic equation $ax^2 + bx + c = 0$ has solutions x_1 and x_2 , then the trinomial $ax^2 + bx + c$ is factored as follows:

$$ax^2 + bx + c = a(x - x_1)(x - x_2)$$

a) To factor $x^2 + x - 6$, let us first solve the equation $x^2 + x - 6 = 0$:

$$x_{1,2} = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 1 \cdot (-6)}}{2 \cdot 1} = \frac{-1 \pm \sqrt{25}}{2} = \frac{-1 \pm 5}{2}.$$

Hence $x_1 = 2$, $x_2 = -3$. Therefore, a factoring is

$$x^2 + x - 6 = a(x - x_1)(x - x_2) = 1 \cdot (x - 2)(x - (-3)) = (x - 2)(x + 3).$$

Notice that the factoring may be easily obtained by guessing, since the roots of the equation are integers.

b) To factor $x^2 - x - 1$, find the roots:

$$x_{1,2} = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \cdot 1 \cdot (-1)}}{2 \cdot 1} = \frac{1 \pm \sqrt{5}}{2}.$$

Hence $x_1 = \frac{1 + \sqrt{5}}{2}$, $x_2 = \frac{1 - \sqrt{5}}{2}$ and a factoring is

$$x^2 - x - 1 = a(x - x_1)(x - x_2) = \left(x - \frac{1 + \sqrt{5}}{2}\right) \left(x - \frac{1 - \sqrt{5}}{2}\right).$$

Notice that the factoring doesn't look appealing. Since the roots of the equation are **irrational** numbers, we can't get any better result.

c) $3x^2 - x + 1 = 3(x - x_1)(x - x_2)$, where x_1, x_2 are the roots of $3x^2 - x + 1 = 0$. Let us find the roots:

$$x_{1,2} = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \cdot 3 \cdot 1}}{2 \cdot 3} = \frac{1 \pm \sqrt{-11}}{6}.$$

In this situation, there are **no** real roots. It means that the factoring is impossible. Polynomials which can't be factored, are called *irreducible* (over real numbers). The polynomial $3x^2 - x + 1$ is irreducible. \square

Problem 242. Factor the following quadratic trinomials:

a) $x^2 + 5x + 4$ b) $2x^2 - 3x + 1$ c) $2x^2 + 3x + 2$

8.6. Vieta's theorem and secrets of factoring quadratic trinomials

Problem 243. a) Find the roots of the equation $(x - 2)(x + 5) = 0$. Find the sum and the product of the roots.

b) Rewrite the equation $(x - 2)(x + 5) = 0$ in the standard form $ax^2 + bx + c = 0$.

c) How is the sum of the roots of the equation related to the coefficient b ?

d) How is the product of the roots related to the coefficient c ?

Problem 244. Use Vieta's theorem to solve the following equations:

a) $x^2 - 7x + 10 = 0$ b) $x^2 - x - 6 = 0$ c) $x^2 + x - 2 = 0$

Solution. According to Vieta's theorem, the roots x_1, x_2 of the quadratic equation $ax^2 + bx + c = 0$ are related to the coefficients as follows:

$$x_1 + x_2 = -\frac{b}{a}, \quad x_1x_2 = \frac{c}{a}.$$

In particular, if $a = 1$, then

$$x_1 + x_2 = -b, \quad x_1x_2 = c.$$

Vieta's theorem gives a shortcut to finding the roots of a quadratic equation (if the roots are integers).

a) If x_1, x_2 are the roots of $x^2 - 7x + 10 = 0$, then, by Vieta's theorem, $x_1 + x_2 = 7$ and $x_1x_2 = 10$. It's easy to guess such roots: $x_1 = 2, x_2 = 5$.

b) If x_1, x_2 are the roots of $x^2 - x - 6 = 0$, then, by Vieta's theorem, $x_1 + x_2 = 1$ and $x_1x_2 = -6$. So $x_1 = -2, x_2 = 3$.

c) For the roots x_1, x_2 of $x^2 + x - 2 = 0$ we have $x_1 + x_2 = -1$ and $x_1x_2 = -2$. So $x_1 = 1, x_2 = -2$. \square

Problem 245. Factor the following quadratic trinomials:

a) $x^2 - 5x - 6$ b) $4x^2 + 12x + 9$ c) $x^2 - 3x + 1$ d) $x^2 + x + 1$

Solution. If the quadratic trinomial $ax^2 + bx + c$ has real roots x_1 and x_2 , then a factoring is $ax^2 + bx + c = a(x - x_1)(x - x_2)$. Therefore, to factor a quadratic trinomial, we need to find its roots.

a) The quadratic trinomial $x^2 - 5x - 6$ has roots $x_1 = -1$ and $x_2 = 6$. They may be found by Vieta's theorem ($x_1 + x_2 = 5$ and $x_1 \cdot x_2 = -6$) or by the quadratic formula. A factoring is $x^2 - 5x - 6 = (x + 1)(x - 6)$.

b) If we recognize $4x^2 + 12x + 9$ as a perfect square trinomial, then a factoring is easy: $4x^2 + 12x + 9 = (2x + 3)^2$. If we don't see a perfect square, then we have to go a long way: to find the roots by the quadratic formula

$$x_{1,2} = \frac{-12 \pm \sqrt{12^2 - 4 \cdot 4 \cdot 9}}{2 \cdot 4} = \frac{-12 \pm \sqrt{144 - 144}}{8} = \frac{-12}{8} = -\frac{3}{2},$$

and factor the trinomial:

$$4x^2 + 12x + 9 = 4 \left(x + \frac{3}{2}\right) \left(x + \frac{3}{2}\right) = (2x + 3)(2x + 3) = (2x + 3)^2.$$

Notice that a factoring is not unique, it is defined up to multiplication by a constant. So the answer to the problem may be given in different forms:

$$(2x + 3)^2 \quad \text{or} \quad 4 \left(x + \frac{3}{2}\right)^2 \quad \text{or} \quad 2(2x + 3) \left(x + \frac{3}{2}\right).$$

c) For the trinomial $x^2 - 3x + 1$, guessing the roots by Vieta's theorem won't be successful, since the roots are irrational. The only way to factor is to find the roots:

$$x_{1,2} = \frac{3 \pm \sqrt{(-3)^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1} = \frac{3 \pm \sqrt{5}}{2}$$

and use the factoring formula

$$x^2 - 3x + 1 = 1 \cdot (x - x_1)(x - x_2) = \left(x - \frac{3 + \sqrt{5}}{2}\right) \left(x - \frac{3 - \sqrt{5}}{2}\right).$$

d) The trinomial $x^2 + x - 1$ has **no** real roots, so it **can't** be factored. This trinomial is irreducible.

□

8.7. Equations reducible to quadratic

Problem 246. Solve the following equations:

a) $2x^2 - 3 = x(x - 1)$ **b)** $x^3 - 3x^2 = 4x$ **c)** $x^4 + 2x^2 - 3 = 0$

d) $x^6 - 2x^3 - 3 = 0$ **e)** $\frac{1}{x} + \frac{2}{x+1} = 1$ **f)** $\frac{2x+4}{2-x} = \frac{3-x}{x-1}$

Solution. **a)** To solve the equation $2x^2 - 3 = x(x - 1)$, we have to bring it to the standard form:

$$2x^2 - 3 = x(x - 1) \iff 2x^2 - 3 = x^2 - x \iff x^2 + x - 3 = 0,$$

and then use the quadratic formula:

$$x_{1,2} = \frac{-1 \pm \sqrt{1 + 12}}{2} = \frac{-1 \pm \sqrt{13}}{2}.$$

c) The equation $x^4 + 2x^2 - 3 = 0$ is called *biquadratic*. It is solved by a substitution $t = x^2$ (notice that new variable t should be non-negative, that is $t \geq 0$):

$$x^4 + 2x^2 - 3 = 0 \iff t^2 + 2t - 3 = 0 \iff (t - 1)(t + 3) = 0 \iff t = 1 \text{ or } t = -3.$$

Since $t \geq 0$, we reject the negative root $t = -3$. By this, the only solution is $t = 1$, so $x^2 = t = 1$ and $x = \pm 1$.

e) We have to assume that $x \neq 0$ and $x \neq -1$.

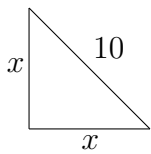
$$\begin{aligned} \frac{1}{x} + \frac{2}{x+1} = 1 &\iff \frac{1}{x} + \frac{2}{x+1} - 1 = 0 \iff \frac{(x+1) + 2x - x(x+1)}{x(x+1)} = 0 \iff \\ &-x^2 + 2x + 1 = 0 \iff x_{1,2} = 1 \pm \sqrt{2}. \end{aligned}$$

□

8.8. Word problems leading to quadratic equations

Problem 247. The hypotenuse of an isosceles triangle is 10 feet long. Find the lengths of the legs of the triangle.

Solution.



Let x ft be the length of each leg. By Pythagorean theorem, $x^2 + x^2 = 10^2$, or, equivalently, $2x^2 = 100$. So $x^2 = 50$ and $x = \pm\sqrt{50} = \pm 5\sqrt{2}$. Since the length is positive, we reject the negative root. So the each leg of the triangle has the length of $5\sqrt{2}$ feet.

□

Problem 248. One side of a rectangle is 2 inches longer than the other side. Find the lengths of both sides if the area of the rectangle is 15 square inches.

Problem 249. A side of a right triangle is 2 cm shorter than its hypotenuse. Find the lengths of all sides of the triangle if its perimeter is 12 cm.

Problem 250. A motorboat spent five hours going downstream from A to B and then going back from B to A . Find the speed of the motorboat in still water, if the speed of the stream is 4 km/h and the distance between A and B is 48 km.

Solution. Let x km/h be the speed of the motorboat in still water. Since the speed of the stream is 4 km/h, the speed of the motorboat going downstream is $(x + 4)$ km/h and its speed going upstream is $(x - 4)$ km/h. We use the formula for a uniform motion

$$\text{time} = \frac{\text{distance}}{\text{speed}}$$

to calculate the time spent traveling downstream from A to B :

$$\frac{48}{x+4} \text{ (hours),}$$

and the time spent traveling upstream from B to A :

$$\frac{48}{x-4} \text{ (hours).}$$

By this, the total travel time is

$$\frac{48}{x+4} + \frac{48}{x-4}.$$

Given that the boat spent 5 hours in total for the travel, we get

$$\frac{48}{x+4} + \frac{48}{x-4} = 5.$$

Let us solve this equation. $\frac{48}{x+4} + \frac{48}{x-4} = 5 \iff \frac{48}{x+4} + \frac{48}{x-4} - 5 = 0 \iff$

$$\frac{48(x-4) + 48(x+4) - 5(x+4)(x-4)}{(x+4)(x-4)} = 0 \iff$$

$$96x - 5(x^2 - 16) = 0 \iff$$

$$5x^2 - 96x - 80 = 0 \iff$$

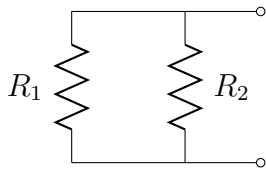
$$x_{1,2} = \frac{96 \pm \sqrt{96^2 + 4 \cdot 5 \cdot 80}}{10} = \frac{96 \pm 8\sqrt{169}}{10} = \frac{48 \pm 52}{5}, \quad x_1 = 20; \quad x_2 = -\frac{4}{5}$$

We reject negative root and get the answer: the speed of the motorboat in still water is 20 km/h. \square

Problem 251. Two parallel resistors provide the total resistance of 2 Ohms. Find the value of each resistor if one of them is 3 Ohms more than the other. Use the law for parallel resistors:

$$\frac{1}{R_{total}} = \frac{1}{R_1} + \frac{1}{R_2}.$$

Solution.



Plug in $R_{total} = 2$ and $R_2 = R_1 + 3$ in the given identity:
 $\frac{1}{2} = \frac{1}{R_1} + \frac{1}{R_1 + 3} \iff 2(R_1 + 3 + R_1) = R_1(R_1 + 3) \iff$
 $R_1^2 - R_1 - 6 = 0$. The roots are $R_1 = 3$ or $R_1 = -2$ (reject the latter root since a negative resistance makes no sense).

Answer: the resistors have the resistances of 3 Ohms and 6 Ohms. \square

8.9. Parabolas

Problem 252. For the parabola $y = x^2 - x - 2$, find the vertex, the axis of symmetry, the x - and y -intercepts, and draw the graph.

Solution.

- (1) The vertex of the parabola $y = ax^2 + bx + c$ is located at the point with x -coordinate $x = \frac{-b}{2a}$. In our case, $a = 1$ and $b = -1$, so $x = \frac{-b}{2a} = \frac{-(-1)}{2} = \frac{1}{2}$. The y -coordinate of the vertex is obtained by plugging $x = \frac{-b}{2a}$ into the formula $y = ax^2 + bx + c$. In our

$$\text{case, } y \Big|_{x=1/2} = y \left(\frac{1}{2} \right) = \left(\frac{1}{2} \right)^2 - \frac{1}{2} - 2 = \frac{1}{4} - \frac{1}{2} - 2 = -\frac{9}{4}.$$

So the vertex is located at $\left(\frac{1}{2}, -\frac{9}{4} \right)$. We place this point on the coordinate system.

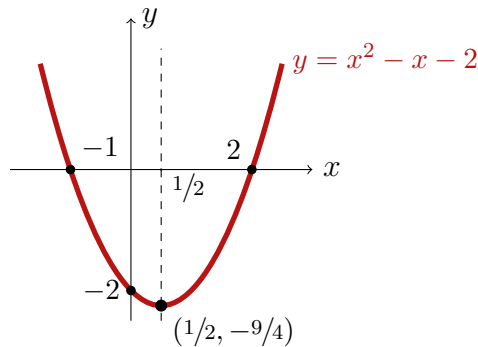
- (2) The axis of symmetry is the vertical line passing through the vertex. It has the equation $x = \frac{-b}{2a}$. In our case, this line is $x = \frac{1}{2}$. Draw this vertical line through the vertex.

- (3) The leading coefficient $a = 1$ is positive, hence the parabola opens upward. From the vertex, draw a small sprout of parabola growing upward.
- (4) The y -intercept is located at the point $(0, c)$. In our case, it is $(0, -2)$. We place the y -intercept on the y -axis at $(0, -2)$.
- (5) The x -intercepts are obtained by solving the equation $x^2 - x - 2 = 0$:

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{1 \pm \sqrt{1 + 8}}{2} = \frac{1 \pm 3}{2} \implies x_1 = 2, x_2 = -1.$$

Alternatively, we may find the roots by factoring: $x^2 - x - 2 = (x - 2)(x + 1)$. So the x -intercepts are $(2, 0)$ and $(-1, 0)$. We place these points on the x -axis.

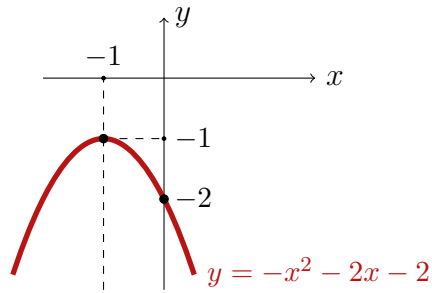
- (6) Now draw the parabola. From the vertex, let the horns grow up through the the y - and x -intercepts. Be neat: the parabola should be smooth and symmetric.



Problem 253. For the parabola $y = -x^2 - 2x - 2$, find the vertex, the axis of symmetry, the x - and y -intercepts, and draw the graph.

Solution.

- (1) The vertex is at $x = -b/2a = -(-2)/2 \cdot (-1) = -1$. The y -coordinate of the vertex is $y(-1) = -(-1)^2 - 2 \cdot (-1) - 2 = -1 + 2 - 2 = -1$. So the vertex is located at $(-1, -1)$. Place the vertex on the coordinate system.
- (2) The axis of symmetry is the vertical line $x = -1$. Draw this line through the vertex.
- (3) Since $a = -1$ is negative, the parabola opens downward. From the vertex, draw a small sprout of parabola growing downward. At this moment, we see that the parabola doesn't intersect the x -axis and we are free from calculating the x -intercepts since there are none of them!
- (4) Place the y -intercept $(0, -2)$ on the y -axis.
- (5) Draw the parabola, smooth and symmetric.



Problem 254. For the parabola $y = -x^2 - 4x + 5$, find the vertex, the axis of symmetry, the x - and y -intercepts, and draw the graph.

Problem 255. For the parabola $y = \frac{1}{2}x^2 + x + 3$, find the vertex, the axis of symmetry, the x - and y -intercepts, and draw the graph.

Problem 256. Consider the parabola $y = ax^2 + bx + c$. Determine the **signs** of the coefficients a and c if it is known that the vertex of the parabola has negative y -coordinate, and the parabola never intersects the x -axis.

Problem 257. How many x -intercepts may a parabola have? How many y -intercepts may a parabola have?

Problem 258. Find the maximal value of the function $y = -3x^2 + x + 1$.

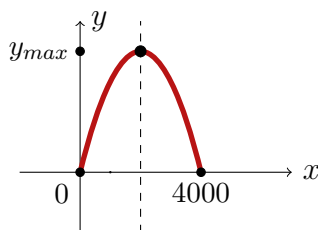
Solution. The graph of the given quadratic function is a parabola. It opens down since $-3 < 0$. The highest point on the graph is its vertex, which is located at $x = \frac{-b}{2a} = \frac{-1}{2 \cdot (-3)} = \frac{1}{6}$.

The corresponding y -value of the vertex is $y \Big|_{x=1/6} = -3 \cdot \left(\frac{1}{6}\right)^2 + \frac{1}{6} + 1 = \frac{13}{12}$.

By this, the maximal value of the function is $\frac{13}{12}$, it is attained at $x = \frac{1}{6}$. □

Problem 259. The rate at which a disease is spread in a village is given by the model $y(x) = 10^{-5} \cdot x(4000 - x)$, where 4000 is the population of the village, x in the number of infected persons, $4000 - x$ is the number of non-infected persons, and y is the rate of spread of the disease (in persons per day). According to this model, the rate of the spread is jointly proportional to the number of infected people and the number of non-infected people with the coefficient of proportionality 10^{-5} . Find the maximal rate of the spread and the number of infected people at this moment.

Solution. Let us draw the graph of the function $y(x) = 10^{-5} \cdot x(4000 - x)$. Drawing is easy, since we identify the x -intercepts directly from the formula, $x = 0$ and $x = 4000$. The axis of symmetry goes vertically in between the intercepts.



As we see from the picture, the maximal rate y_{max} is attained at $x = 2000$, that is at the midpoint of the interval $[0, 4000]$. To calculate the maximal rate, plug in $x = 2000$ in the equation: $y(2000) = 10^{-5} \cdot 2000(4000 - 2000) = 10^{-5} \cdot 4 \cdot 10^6 = 40$. Therefore, the the maximal rate of the spread is 40 people per day, and the number of infected people at this moment is 2000. □

8.10. Quadratic inequalities

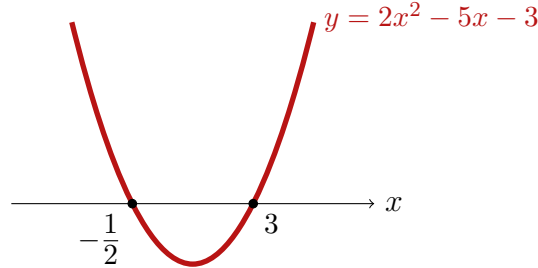
Problem 260. Solve the quadratic inequalities along the following lines: first, determine the roots of the quadratic polynomial. Second, sketch how the parabola is located with respect to the x -axis. Finally, determine where the parabola is above/below the x -axis.

a) $2x^2 - 5x - 3 \leq 0$ **b)** $2x^2 - 5x - 3 > 0$ **c)** $-x^2 - x + 2 < 0$ **d)** $-x^2 - x + 2 \geq 0$

Solution. **a)** The roots of $2x^2 - 5x - 3 = 0$ are

$$x_{1,2} = \frac{5 \pm \sqrt{25 - 4 \cdot 2 \cdot (-3)}}{2 \cdot 2} = \frac{5 \pm \sqrt{49}}{4} = \frac{5 \pm 7}{4} \implies x_1 = 3, \quad x_2 = -\frac{1}{2}.$$

The leading coefficient of $2x^2 - 5x - 3$ is positive, so the parabola $2x^2 - 5x - 3 = 0$ opens upward:



In order to find all x for which $2x^2 - 5x - 3 \leq 0$, we identify the interval on the x -axis where the parabola is situated **below** the x -axis. It is the interval $\left[-\frac{1}{2}, 3\right]$. Both endpoints are included since the original inequality is non-strict.

b) Since the parabola $y = 2x^2 - 5x - 3$ is the same as in **a)**, to solve the inequality $2x^2 - 5x - 3 > 0$, we have to find the intervals on the x -axis where the parabola is **above** the x -axis. There are two such intervals, $\left(-\infty, -\frac{1}{2}\right)$ and $(3, \infty)$. Notice that the endpoints $x = -\frac{1}{2}$ and $x = 3$ are **not** included since the inequality is strict. The solution of the inequality then is the **union** $\left(-\infty, -\frac{1}{2}\right) \cup (3, \infty)$. □

Problem 261. Solve the quadratic inequalities by the *method of intervals*.

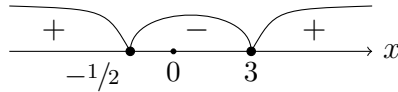
a) $2x^2 - 5x - 3 \leq 0$ **b)** $2x^2 - 5x - 3 > 0$ **c)** $-x^2 - x + 2 < 0$ **d)** $-x^2 - x + 2 \geq 0$

Solution. **a)** First, find the roots of $2x^2 - 5x - 3 = 0$ and place them on the number line:



These two roots divide the number line into three intervals. On each of these intervals the quadratic polynomial $2x^2 - 5x - 3$ keeps its sign unchanged. To determine the signs, inspect

the sign of the polynomial at any point, for example at $x = 0$: $2x^2 - 5x - 3 \Big|_{x=0} = -3 < 0$. Therefore, the signs are distributed as follows:



Now find the interval corresponding to negative sign. It is $\left[-\frac{1}{2}, 3\right]$. Both endpoints are included since the original inequality is non-strict.

b) To solve $2x^2 - 5x - 3 > 0$, find on the picture above the intervals corresponding to positive sign. They are $\left(-\infty, -\frac{1}{2}\right)$ and $(3, \infty)$. Notice that the endpoints $x = -\frac{1}{2}$ and $x = 3$ are **not** included since the inequality is strict. The solution of the inequality then is the **union** $\left(-\infty, -\frac{1}{2}\right) \cup (3, \infty)$. \square

Problem 262. Solve the quadratic inequalities by factoring:

a) $2x^2 - 5x - 3 \leq 0$ b) $2x^2 - 5x - 3 > 0$ c) $-x^2 - x + 2 < 0$ d) $-x^2 - x + 2 \geq 0$

Solution. a)

$$2x^2 - 5x - 3 \leq 0 \iff 2\left(x + \frac{1}{2}\right)(x - 3) \leq 0 \iff \begin{cases} x + \frac{1}{2} \geq 0 \\ x - 3 \leq 0 \end{cases} \text{ or } \begin{cases} x + \frac{1}{2} \leq 0 \\ x - 3 \geq 0 \end{cases} \iff$$

$$\begin{cases} x \geq -\frac{1}{2} \\ x \leq 3 \end{cases} \text{ or } \begin{cases} x \leq -\frac{1}{2} \\ x \geq 3 \end{cases} \iff x \in \left[-\frac{1}{2}, 3\right] \text{ or } x \in \emptyset \iff x \in \left[-\frac{1}{2}, 3\right].$$

b)

$$2x^2 - 5x - 3 > 0 \iff 2\left(x + \frac{1}{2}\right)(x - 3) > 0 \iff \begin{cases} x + \frac{1}{2} > 0 \\ x - 3 > 0 \end{cases} \text{ or } \begin{cases} x + \frac{1}{2} < 0 \\ x - 3 < 0 \end{cases} \iff$$

$$\begin{cases} x > -\frac{1}{2} \\ x > 3 \end{cases} \text{ or } \begin{cases} x < -\frac{1}{2} \\ x < 3 \end{cases} \iff x \in \left(-\infty, -\frac{1}{2}\right) \text{ or } x \in (3, \infty) \iff$$

$$x \in \left(-\infty, -\frac{1}{2}\right) \cup (3, \infty). \quad \square$$

Problem 263. Solve the quadratic inequalities by completing the square:

a) $2x^2 - 5x - 3 \leq 0$ b) $2x^2 - 5x - 3 > 0$ c) $-x^2 - x + 2 < 0$ d) $-x^2 - x + 2 \geq 0$

Solution. a) First, complete the square for the quadratic trinomial:

$$\begin{aligned} 2x^2 - 5x - 3 &= 2\left(x^2 - \frac{5}{2}x\right) - 3 = 2\left(x^2 - 2x \cdot \frac{5}{4} + \frac{25}{16} - \frac{25}{16}\right) - 3 = \\ &= 2\left(x - \frac{5}{4}\right)^2 - \frac{25}{8} - 3 = 2\left(x - \frac{5}{4}\right)^2 - \frac{49}{8} \end{aligned}$$

Then

$$\begin{aligned} 2x^2 - 5x - 3 \leq 0 &\iff 2\left(x - \frac{5}{4}\right)^2 - \frac{49}{8} \leq 0 \iff \left(x - \frac{5}{4}\right)^2 \leq \frac{49}{16} \iff \\ &\left|x - \frac{5}{4}\right| \leq \frac{7}{4} \iff -\frac{7}{4} \leq x - \frac{5}{4} \leq \frac{7}{4} \iff -\frac{1}{2} \leq x \leq 3 \end{aligned}$$

So $2x^2 - 5x - 3 \leq 0 \iff x \in \left[-\frac{1}{2}, 3\right]$.

b) Analogously,

$$2x^2 - 5x - 3 > 0 \iff \left|x - \frac{5}{4}\right| > \frac{7}{4} \iff x - \frac{5}{4} > \frac{7}{4} \quad \text{or} \quad x - \frac{5}{4} < -\frac{7}{4} \iff x > 3 \quad \text{or} \quad x < -\frac{1}{2}.$$

So $2x^2 - 5x - 3 > 0 \iff x \in \left(-\infty, -\frac{1}{2}\right) \cup (3, \infty)$. □

9. Polynomials

9.1. Monomials and polynomials

Problem 264. Which of the following are **monomials** in variable x ?

- a) x^3 b) $\frac{4}{x}$ c) $2x^4 + 1$ d) x^{-2} e) $|x|$ f) $-3x^7$ g) $x^{3/5}$ h) yx^2
i) x^π j) \sqrt{x} k) $x \cdot x \cdot x$ l) 4

Solution. A *monomial* in x is an expression of the form ax^n , where a is a coefficient (independent on x) and n is a degree. The degree is always a non-negative integer.

a) x^3 is a monomial, since it is in the form of ax^n , where $a = 1$ and $n = 3$.

b) $\frac{4}{x}$ is not a monomial: $\frac{4}{x} = 4x^{-1}$ and the exponent -1 is negative.

c) $2x^4 + 1$ is not a monomial, since it is a sum.

d) x^{-2} is not a monomial, since the variable has a negative exponent.

e) $|x|$ is not a monomial, since it can't be written in the form ax^n .

f) $-3x^7$ is a monomial: it is of the form of ax^n , where $a = -3$ and $n = 7$.

g) $x^{3/5}$ is not a monomial. Fractional exponents for variables are not allowed in the monomials.

h) yx^2 is a monomial. The coefficient here is y . This is a variable, but it's OK as long as it does not depend on x .

i) x^π is not a monomial, since the exponent π is not an integer.

j) \sqrt{x} is not a monomial: $\sqrt{x} = x^{1/2}$ and $\frac{1}{2}$ is not an integer.

k) $x \cdot x \cdot x$ is a monomial, it is nothing else but x^3 .

l) 4 is a monomial, since $4 = 4x^0$, so it is in the form of ax^n , where $a = 4$ and $n = 0$.

□

Problem 265. Which of the following are polynomials?

- a) $-3x^2 + x^{10} + \frac{1}{2}$ b) $2x - \sqrt{x} + 4$ c) $x^2 + x^{-1} + 1$ d) $\pi x + 5$ e) $(x+1)(x^3 - 2)$
f) $(x+1) \div (x^3 - 2)$ g) $\frac{x^3 + x}{-3}$ h) $x^4 \div 9$ i) $9 \div x^4$ j) 2015

Solution. a) $-3x^2 + x^{10} + \frac{1}{2}$ is a polynomial as a sum of monomials.

b) $2x - \sqrt{x} + 4$ is not a polynomial, because taking roots in monomials is not allowed.

- c) $x^2 + x^{-1} + 1$ is not a polynomial, because negative exponents are not allowed.
- d) $\pi x + 5$ is a polynomial. Having a coefficient of π for monomials is allowed.
- e) $(x + 1)(x^3 - 2)$ is a polynomial as a product of two polynomials.
- f) $(x + 1) \div (x^3 - 2)$ is not a polynomial.
- g) $\frac{x^3 + x}{-3} = -\frac{1}{3}x^3 - \frac{1}{3}x$ is a polynomial. Alternatively, $\frac{x^3 + x}{-3}$ is a polynomial as a product of $x^3 + x$ by $-\frac{1}{3}$.
- h) $x^4 \div 9 = \frac{1}{9}x^4$ is a polynomial.
- i) $9 \div x^4 = 9x^{-4}$ is not a polynomial.
- j) 2015 is a polynomial, since it is a monomial. □

Problem 266. Which of the following are polynomials?

- a) $4x^{-2} + 3x - 1$ b) $\sqrt{2}x^3 + 3x^2 - x$ c) $\frac{4x^2 + 3x + 5}{x}$ d) $x^3(2x - 1)$ e) $\frac{\pi}{2}$

9.2. Operations on polynomials

Problem 267. Let $p(x) = 3x - 7$ and $q(x) = x^3 + 2x^2$. Find:

- a) $p\left(\frac{1}{2}\right)$ b) $q(2)$ c) $p(x)q(x)$ d) $3p(x) - 2q(2)$ e) $p(q(x))$

Solution.

- a) To calculate $p\left(\frac{1}{2}\right)$, we have to plug in $\frac{1}{2}$ for x :

$$p(x)\Big|_{x=\frac{1}{2}} = (3x - 7)\Big|_{x=\frac{1}{2}} = 3 \cdot \left(\frac{1}{2}\right) - 7 = \frac{3}{2} - 7 = \frac{3 - 2 \cdot 7}{2} = -\frac{11}{2}.$$

So $p\left(\frac{1}{2}\right) = -\frac{11}{2}$.

- b) To calculate $q(2)$, we have to plug in 2 for x :

$$q(x)\Big|_{x=2} = (x^3 + 2x^2)\Big|_{x=2} = 2^3 + 2 \cdot 2^2 = 8 + 2 \cdot 4 = 8 + 8 = 16.$$

So $q(2) = 16$.

c) $p(x)q(x) = (3x - 7)(x^3 + 2x^2) = 3x^4 + 6x^3 - 7x^3 - 14x^2 = 3x^4 - x^3 - 14x^2$.

d) $3p(x) - 2q(2) = 3(3x - 7) - 2(2^3 + 2(2)^2) = 3(3x - 7) - 2(16) = 9x - 21 - 32 = 9x - 53$.

- e) To find $p(q(x))$, we have to plug in $q(x)$ for x in $p(x)$:

$$p(q(x)) = 3q(x) - 7 = 3(x^3 + 2x^2) - 7 = 3x^3 + 6x^2 - 7.$$



Problem 268. Let $p(x) = -x^2 + 1$ and $q(x) = 2x + 3$. Find:

- a) $p(-3)$ b) $p(3x)$ c) $q\left(-\frac{2}{3}\right)$ d) $p(x)q(x)$ e) $q(p(x))$ f) $p(q(x))$

9.3. Polynomials in several variables

Problem 269. Which of the following are monomials in variables x and y ?

- a) $-\frac{2}{3}xy^2$ b) $\frac{y}{x}$ c) $\sqrt{y}x$ d) x^3y^{-2} e) $|x|y$ f) x^y g) $5x$ h) -7

Solution. a) $-\frac{2}{3}xy^2$ is a monomial since it is the form of ax^ny^m with $a = -\frac{2}{3}$, $n = 1$, $m = 2$.

b) $\frac{y}{x}$ is not a monomial. As in the single-variable case, dividing by a variable is not allowed for monomials.

c) $\sqrt{y}x$ is not a monomial. Taking roots of a variable is not allowed for monomials.

d) x^3y^{-2} is not a monomial. Taking variables to negative exponents is the same as dividing by a variable. It is not allowed for monomials.

e) $|x|y$ is not a monomial: taking absolute value of a variable is another operation that is not allowed for monomials.

f) x^y is not a monomial. As in the single-variable case, variables may only be raised to non-negative integer exponents.

g) $5x$ is a monomial, since $5x = 5xy^0$.

h) -7 is a monomial, since $-7 = -7x^0y^0$. □

Problem 270. Find the value of the polynomial $-3x^2y + xy - 1$ if $x = -2$ and $y = 4$.

Solution.

$$(-3x^2y + xy - 1)\Big|_{x=-2, y=4} = -3 \cdot (-2)^2 \cdot 4 + (-2) \cdot 4 - 1 = -3 \cdot 4 \cdot 4 - 8 - 1 = -48 - 8 - 1 = -57.$$



Problem 271. Find the value of the polynomial $-xy^2 + x^2 - y$ if $x = -\frac{1}{3}$ and $y = 2$.

Problem 272. Perform the indicated polynomial operations and simplify your answer:

- a) $(3xy^2 - 4xy) \cdot (x + 2y)$ b) $xy(x^2 + 2y) - y(3x^3 + 2y^2 - 1)$

Solution. a) $(3xy^2 - 4xy) \cdot (x + 2y) = 3x^2y^2 + 6xy^3 - 4x^2y - 8xy^2$

b) $xy(x^2 + 2y) - y(3x^3 + 2y^2 - 1) = \underline{x^3y} + 2xy^2 - \underline{3x^3y} - 2y^3 + y = -2x^3y + 2xy^2 - 2y^3 + y.$ □

Problem 273. Perform the indicated polynomial operations and simplify your answer:

- a) $-xy^2(x - 3x^2y) + xy(xy - y^2)$ b) $3x^2y(-xy^4 - 2y^2) + 2y^3(2x^2 - x^3y^2)$

10. Algebra to go

We present a collection of problems from calculus, physics and chemistry, which are algebraic in their nature and require **nothing** but algebra to solve them. These problems often cause difficulties, which can be mistaken as difficulties in calculus, physics or chemistry, respectively. But if you are algebraically savvy, these problems won't slow down your future studies of math and science.

10.1. Calculus

Problem 274. Let $f(x) = x^2$. Calculate and simplify the following expression:

$$\frac{f(x+h) - f(x)}{h},$$

where h is a non-zero number.

Solution. Since $f(x+h) = (x+h)^2$,

$$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^2 - 2x^2}{h} = \frac{x^2 + 2xh + h^2 - x^2}{h} = \frac{2xh + h^2}{h} = \frac{2h(x+h)}{h} = 2(x+h)$$

Note. The expression $\frac{f(x+h) - f(x)}{h}$ is called the *difference quotient* or *average rate of change* of the function. It plays a fundamental role in Calculus, being involved in the definition of *derivative*. \square

Problem 275. Find the difference quotients for the following functions

a) $f(x) = 5$ b) $f(x) = 3x + 2$ c) $f(x) = x^2 + x$ d) $f(x) = \frac{1}{x}$

Problem 276. Prove that $\frac{1}{\sqrt{x^2+1}+x} = \sqrt{x^2+1} - x$.

Solution.

$$\frac{1}{\sqrt{x^2+1}+x} = \frac{\sqrt{x^2+1}-x}{(\sqrt{x^2+1}+x)(\sqrt{x^2+1}-x)} = \frac{\sqrt{x^2+1}-x}{(\sqrt{x^2+1})^2 - x^2} = \frac{\sqrt{x^2+1}-x}{x^2+1-x^2} = \sqrt{x^2+1}-x$$

Note. This trick is called *rationalizing the denominator*. You will exploit it while calculating limits. \square

Problem 277. Rationalize the denominator in $\frac{x-9}{\sqrt{x}-3}$.

Problem 278. Rationalize the numerator in $\frac{\sqrt{x^2+4}-2}{x^2}$.

Problem 279. Express $\frac{1}{x^2-x-2}$ as a sum of simpler fractions.

Solution. Let us factor the denominator. Remember that $ax^2 + bx + c$ is factored as follows:

$$ax^2 + bx + c = a(x - x_1)(x - x_2),$$

where x_1, x_2 are the roots of the quadratic polynomial $ax^2 + bx + c$.

In our case, $x^2 - x - 2$ has roots $x_1 = 1$ and $x_2 = -2$. They may be found by using the quadratic formula or, in a short cut, by Vieta's theorem. The factoring is

$$x^2 - x - 2 = (x - 1)(x + 2).$$

To express $\frac{1}{x^2 - x - 2}$ as a sum of simpler fractions means to find a decomposition

$$\frac{1}{x^2 - x - 2} = \frac{A}{x - 1} + \frac{B}{x + 2},$$

where A, B are unknown coefficients which we have to find. The right hand side of the decomposition is

$$\frac{A}{x - 1} + \frac{B}{x + 2} = \frac{A(x + 2) + B(x - 1)}{(x - 1)(x + 2)} = \frac{Ax + 2A + Bx - B}{(x - 1)(x + 2)} = \frac{(A + B)x + (2A - B)}{(x - 1)(x + 2)}.$$

Therefore,

$$\frac{1}{x^2 - x - 2} = \frac{(A + B)x + (2A - B)}{(x - 1)(x + 2)}.$$

Hence

$$(A + B)x + (2A - B) = 1.$$

Let us present 1 as a polynomial: $1 = 0 \cdot x + 1$. Then

$$(A + B)x + (2A - B) = 0 \cdot x + 1.$$

Comparing the corresponding coefficients for two equal polynomials, we get $A + B = 0$ and $2A - B = 1$. Let us solve this linear system:

$$\begin{cases} A + B = 0 \\ 2A - B = 1 \end{cases} \iff \begin{cases} A + B = 0 \\ 3A = 1 \end{cases} \iff \begin{cases} A + B = 0 \\ A = \frac{1}{3} \end{cases} \iff \begin{cases} A = \frac{1}{3} \\ B = -\frac{1}{3} \end{cases}.$$

From this we get the desired decomposition:

$$\frac{1}{x^2 - x - 2} = \frac{A}{x - 1} + \frac{B}{x + 2} = \frac{1/3}{x - 1} + \frac{1/3}{x + 2}.$$

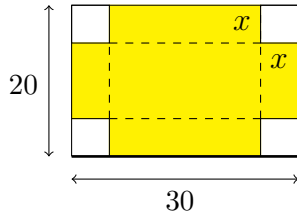
□

Note. This presentation of a fraction as a sum of simpler fractions is called *partial fraction decomposition* and is used for integration.

Problem 280. Use partial fraction decomposition to express $\frac{1}{x^2 - x - 12}$ as a sum of simpler fractions.

Problem 281. An open box is made from a rectangular piece of cardboard by cutting a square with a side of length x from each corner and bending the sides. Find an algebraic expression for the volume of the box, if the cardboard measures 20 in by 30 in.

Solution.



The folded box has the length of $(30 - 2x)$ in, the width of $(20 - 2x)$ in and the height of x in. The volume of a rectangular box is equal to the product of the three dimensions (length \times width \times height). Therefore, the volume of the folded box is given by

$$(30 - 2x)(20 - 2x)x.$$

If we want the expression in the expanded form, then we have to distribute:

$$(30 - 2x)(20 - 2x)x = 4x^3 - 100x^2 + 600x.$$

□

Problem 282. A stadium has the shape of a rectangle with two attached half-circles on opposite sides. Find an algebraic expressions for the perimeter and the area of the stadium, if the width of the rectangle is x meters and the length is 6 times longer than the width.

10.2. Physics

Problem 283. A car moving at a constant speed of 40 mi/h starts to accelerate with a constant acceleration of 16 ft/sec^2 . How long will it take for the car to increase the speed up to 80 mi/h?

Solution. The speed v of an object moving with a constant acceleration a is given by

$$v = at + v_0,$$

where t is the time and v_0 is the initial speed.

In the problem, we are given $v_0 = 40 \text{ mi/h}$, $a = 16 \text{ ft/sec}^2$ and $v = 80 \text{ mi/h}$. We have to find time t . It means that we have to solve the equation for t :

$$v = at + v_0 \implies v - v_0 = at \implies \frac{v - v_0}{a} = t \text{ or } t = \frac{v - v_0}{a}.$$

Before we plug in the numbers in this formula, we have to convert mi/h to ft/sec:

$$\frac{\text{mi}}{\text{h}} = \frac{5280 \text{ ft}}{3600 \text{ sec}} = \frac{22}{15} \frac{\text{ft}}{\text{sec}}$$

So

$$t = \frac{v - v_0}{a} = \frac{(80 - 40) \frac{\text{mi}}{\text{h}}}{16 \frac{\text{ft}}{\text{sec}^2}} = \frac{(80 - 40) \cdot \frac{22}{15} \frac{\text{ft}}{\text{sec}}}{16 \frac{\text{ft}}{\text{sec}^2}} = \frac{40 \cdot 22 \text{ sec}}{16 \cdot 15} = \frac{11 \text{ sec}}{3} = \frac{11}{3} \text{ sec}.$$

Therefore, it will take $\frac{11}{3}$ sec for the car to accelerate to 80 mi/h. □

Problem 284. A ball is thrown down from a cliff with an initial speed of 20 ft/sec. How long will it take for the ball before it hits the ground, if the height of the cliff is 500 ft?

Solution. The height $h(t)$ of an object moving vertically down under gravity is given by

$$h(t) = -\frac{gt^2}{2} - v_0t + h_0,$$

where $g = 32 \text{ ft/sec}^2$ is the acceleration of gravity, v_0 is the initial speed and h_0 is the initial height.

In our case, $v_0 = 20 \text{ ft/sec}$, $h_0 = 500 \text{ ft}$, and $h(t) = 0 \text{ ft}$. We have to find time t . Plug in these numerical values in the equation:

$$0 = -\frac{32t^2}{2} - 20t + 500.$$

This is a quadratic equation to solve:

$$0 = -\frac{32t^2}{2} - 20t + 500 \iff 16t^2 + 20t - 500 = 0 \iff 4t^2 + 5t - 125 = 0.$$

Hence

$$t = \frac{-5 \pm \sqrt{25 - 4 \cdot 4 \cdot 125}}{8} = \frac{-5 \pm 45}{8}.$$

Since the negative solution makes no sense, $t = \frac{-5 + 45}{8} = 5$.

Therefore, it will take 5 sec for the ball before it hits the ground. \square

Problem 285. A ball is thrown down from a cliff with an initial speed of $v_0 \text{ ft/sec}$. How long will it take for the ball to reach the height h , if the height of the cliff is $h_0 \text{ ft}$?

Problem 286. The relativistic mass m of a particle moving with velocity v is given by

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}},$$

where m_0 is the mass of the particle at rest and c is the speed of light. Find the velocity of a particle if its relativistic mass is m and the rest mass is m_0 .

Problem 287. According to the lens equation,

$$\frac{1}{f} = \frac{1}{d_o} + \frac{1}{d_i},$$

where f is the focal length, d_o is the distance to the object and d_i is the distance to the image. Find the distance to the image given the focal length f and the distance to the object d_o .

10.3. Chemistry

Problem 288. In biochemistry, the rate v of enzymatic reaction is related to the concentration $[S]$ of a substance as follows:

$$v = \frac{V_{\max} [S]}{K_m + [S]},$$

where V_{\max} is the maximum rate at maximum concentration. Use this equation to find $[S]$ in terms of other variables.

Solution. We have to solve the equation for $[S]$:

$$v = \frac{V_{\max} [S]}{K_m + [S]} \implies v(K_m + [S]) = V_{\max} [S] \implies vK_m + v[S] = V_{\max} [S] \implies vK_m = V_{\max} [S] - v[S] \implies vK_m = (V_{\max} - v)[S] \implies \frac{vK_m}{V_{\max} - v} = [S].$$

Therefore, $[S] = \frac{vK_m}{V_{\max} - v}$. □

Problem 289. Van der Waals equation for fluids says that

$$\left(p + \frac{n^2 a}{V^2}\right) (V - nb) = nRT.$$

Solve this equation for p .