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Introduction to Linear Algebra
MAT 211, Spring 2009

Exercises for the Final Exam

The Final Exam will contain 7 problems. The following sections will be covered: systems of linear equations, vector spaces and their subspaces, bases and dimension, linear transformations, kernel and image, isomorphism, inner product spaces, orthogonal projections, orthogonal compliment, orthogonal transformations, determinants, eigenvalues and eigenvectors, diagonalization.

1. Show that the linear system

$$\begin{cases} -4x + 5y = 4 \\ 2x - 3y = 1 \end{cases}$$

has a unique solution. Find the solution in three different ways: by the Gauss-Jordan elimination, by Cramer's rule and using the inverse of the coefficient matrix. Give a geometric interpretation of the system and its solution.

Answer: $(x, y) = (-17/2, -6)$

2. Solve the system and verify the solution.

$$\begin{cases} 2x_1 - x_2 + x_3 + 2x_4 + 3x_5 = 2 \\ 6x_1 - 3x_2 + 2x_3 + 4x_4 + 5x_5 = 3 \\ 6x_1 - 3x_2 + 4x_3 + 8x_4 + 13x_5 = 9 \\ 4x_1 - 2x_2 + x_3 + x_4 + 2x_5 = 1 \end{cases}$$

Answer: $(x_1, x_2, x_3, x_4, x_5) = (s,\ 1 + 2s - t,\ 3 - 4t,\ 0,\ t), \quad s, t \in \mathbb{R}$

3. Show that the set of traceless 2×2 matices $U = \{A \in M_2 \,|\, \mathrm{tr}A = 0\}$ is a subspace of the space of 2×2 matices M_2. Find a basis of U. Let $< A, B >= \mathrm{tr}(A^T B)$ be the inner product on M_2. Find the orthogonal complement $U^\perp$.

Answer: $U = \mathrm{span}\left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}, \quad U^\perp = \mathrm{span}\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

4. Let $\mathcal{P}_4$ be the vector space of polynomials in one variable of degree ≤ 4. Show that even polynomials $(p(-x) = p(x)\ \forall x)$ form a subspace of $\mathcal{P}_4$. Find the dimension of this subspace.

Let $< p, q >= \int_{-1}^{1} p(x)q(x)\, dx$ be the inner product on $\mathcal{P}_4$. Find the orthogonal complement of the subspace of even polynomials.

Answer: dimension is 3, the orthogonal complement is spanned by x and x^3.

5. Let $W = \{(x, y, z, w) \in \mathbb{R}^4 \mid x+y+z+w = 0,\ x-y-z+w = 0\}$ be a subspace of Euclidean space $\mathbb{R}^4$. Find orthonormal bases of W and $W^\perp$. Find the orthogonal projections of the vector $(1, 2, 3, 4)$ onto W and $W^\perp$.

Answer: $\{\frac{1}{\sqrt{2}}(1, 0, 0, -1),\ \frac{1}{\sqrt{2}}(0, 1, -1, 0)\},\ \{\frac{1}{\sqrt{2}}(1, 0, 0, 1),\ \frac{1}{\sqrt{2}}(0, 1, 1, 0)\},\ \frac{1}{\sqrt{2}}(-3, -1, 1, 3),$ $\frac{5}{\sqrt{2}}(1, 1, 1, 1)$

6. Show that the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by the formula

$$T(x, y, z) = \frac{1}{\sqrt{6}}\left(\sqrt{2}x + \sqrt{3}y + z,\ \sqrt{2}x - \sqrt{3}y + z,\ \sqrt{2}x - 2z\right)$$

is orthogonal.

Answer: Check that the rows (or columns) are orthonormal.

7. Does there exist an eigenbasis of $\mathbb{R}^4$ for the linear operator $T : \mathbb{R}^4 \to \mathbb{R}^4$ given by the formula

$$T(x, y, z, w) = (x, 0, x, w).$$

If so, find this basis and the diagonal form of T.

Answer: T is diagonalized in eigenbasis $\{(0, 1, 0, 0), (0, 0, 1, 0), (1, 0, 1, 0), (0, 0, 0, 1)\}$. The diagonal form is $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

8. Show that the matrix

$$A = \begin{pmatrix} -1 & 3 & -1 \\ -3 & 5 & -1 \\ -3 & 3 & 1 \end{pmatrix}$$

is diagonalizable. Find a diagonal matrix D and an invertible matrix S such that $S^{-1}AS = D$.

Answer: $D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $S = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 3 & 1 \end{pmatrix}$

9. Calculate

$$\begin{pmatrix} 6 & 3 \\ 2 & 7 \end{pmatrix}^{100}.$$

Answer: $\dfrac{1}{5}\begin{pmatrix} 3\cdot 2^{200}+2\cdot 3^{200} & -3\cdot 2^{200}+3^{201} \\ -2^{201}+2\cdot 3^{200} & 2^{201}+3^{201} \end{pmatrix}$

10. Evaluate the following determinants

$$\textbf{a)}\ \begin{vmatrix} 5 & 4 & 2 & 1 \\ 2 & 3 & 1 & -2 \\ -5 & -7 & -3 & 9 \\ 1 & -2 & -1 & 4 \end{vmatrix} \qquad \textbf{b)}\ \begin{vmatrix} a & b & b & \dots & b \\ b & a & b & \dots & b \\ b & b & a & \dots & b \\ \dots & \dots & \dots & \dots & \dots \\ b & b & b & \dots & a \end{vmatrix}_{n\times n}$$

Answer: 38, $[a+b(n-1)](a-b)^n$

11. For which values of a constant a is the matrix

$$\begin{pmatrix} a & 1 & 1 & \dots & 1 \\ 1 & a & 1 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & \dots & a \end{pmatrix}_{n\times n}$$

invertible? Hint: use your result from **11 b)**.

Answer: $a \neq 1$, $a \neq 1-n$