Solutions for Midterm I

Problem 1. Solve the system using the Gauss-Jordan elimination and verify your answer

\[
\begin{align*}
\begin{cases}
    x_1 + 2x_3 - x_4 &= 1 \\
    x_2 + 2x_4 &= -1 \\
    x_1 - x_2 + 2x_3 - 3x_4 &= 2
\end{cases}
\end{align*}
\]

Solution. Elementary row transformations of the augmented matrix of the system give rise to the reduced row-echelon form:

\[
\begin{pmatrix}
1 & 0 & 2 & -1 & 1 \\
0 & 1 & 0 & 2 & -1 \\
1 & -1 & 2 & -3 & 2
\end{pmatrix} \sim \begin{pmatrix}
1 & 0 & 2 & -1 & 1 \\
0 & 1 & 0 & 2 & -1 \\
0 & -1 & 0 & -2 & 1
\end{pmatrix} \sim \begin{pmatrix}
1 & 0 & 2 & -1 & 1 \\
0 & 1 & 0 & 2 & -1 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

Obviously, the rank of the matrix is 2. The number of free variables is \(4 - 2 = 2\) (the number of unknowns minus the rank). We write down the solution starting from the back:

- \(x_4 = t\) (choose \(x_4\) as a free variable),
- \(x_3 = s\) (choose \(x_3\) as a free variable),
- \(x_2 = -1 - 2t\) (from the second row in the rref),
- \(x_1 = 1 + t - 2s\) (from the first row in the rref).

So the solution is

\[(x_1, x_2, x_3, x_4) = (1 + t - 2s, -1 - 2t, s, t) = (1, -1, 0, 0) + t(1, -2, 0, 1) + s(-2, 0, 1, 0),\]

where \(t\) and \(s\) are arbitrary real numbers. Geometrically, the solution is a plane in \(\mathbb{R}^4\).

To verify the solution, we substitute it into the three equations of the system:

\[
\begin{align*}
1 + t - 2s + 2s - t &= 1 \\
-1 - 2t + 2t &= -1 \\
1 + t - 2s + 1 + 2t + 2s - 3t &= 2
\end{align*}
\]

Since all the equations are satisfied, our solution is correct.

Answer: \((x_1, x_2, x_3, x_4) = (1 + t - 2s, -1 - 2t, s, t), \ t, s \in \mathbb{R}\).
Problem 2. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be an orthogonal projection onto the line $x - 2y = 0$ followed by a counterclockwise rotation by $45^\circ$. Find a matrix of $T$ (with respect to the standard basis). Describe geometrically and show on a picture the kernel and the image of $T$. Is $T$ invertible? Explain!

Solution. Denote by $P$ the orthogonal projection onto the line $x - 2y = 0$ and by $R$ the counterclockwise rotation by $45^\circ$. Then $T = R \circ P$. Let $A$ and $B$ be standard matrices of $P$ and $R$ respectively. Then the standard matrix of $T$ is $BA$. Let us evaluate matrices $A$ and $B$.

Matrix $A$ consists of two columns representing the coordinates of the images of the standard basis vectors $\overline{e}_1, \overline{e}_2$ under the transformation $P$:

$$ A = \begin{pmatrix} | & | \hline P\overline{e}_1 & P\overline{e}_2 \end{pmatrix}. $$

To evaluate the images we use a formula defining the orthogonal projection $P$:

$$ P\overline{x} = \frac{\overline{x} \cdot \overline{u}}{\|\overline{u}\|^2} \overline{u}, $$

where $\overline{x}$ is an arbitrary vector in $\mathbb{R}^2$, $\overline{u}$ is a vector along the line of projection, $\|\overline{u}\|$ is its length and $\cdot$ is a dot product. Take $\overline{u} = (2, 1)$. (Note that one can take any vector $\overline{u} = (x, y)$ whose coordinate satisfy the equation of the line $x - 2y = 0$.) Its length is $\|\overline{u}\| = \sqrt{2^2 + 1^2} = \sqrt{5}$.

Calculate the images of $\overline{e}_1 = (1, 0)$ and $\overline{e}_2 = (0, 1)$ under projection $P$:

$$ P\overline{e}_1 = \frac{(1, 0) \cdot (2, 1)}{5} (2, 1) = \frac{2}{5} (2, 1) = \left( \frac{4}{5}, \frac{2}{5} \right), $$

$$ P\overline{e}_2 = \frac{(0, 1) \cdot (2, 1)}{5} (2, 1) = \frac{1}{5} (2, 1) = \left( \frac{2}{5}, \frac{1}{5} \right). $$

It gives us the matrix $A$:

$$ A = \begin{pmatrix} 4/5 & 2/5 \\ 2/5 & 1/5 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}. $$

The standard matrix of a counterclockwise rotation by $45^\circ$ is

$$ B = \begin{pmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{pmatrix} = \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix} = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}. $$

The standard matrix of $T$ is

$$ BA = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \frac{1}{5} \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} = \frac{\sqrt{2}}{10} \begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix}. $$

The kernel of $T$ is a subspace which is annihilated by $T$. This is a line passing through the origin which is orthogonal to the line $x - 2y = 0$. The equation of this line is $2x + y = 0$. Note that $\text{Ker} T$ is spanned by vector $(1, -2)$ which is annihilated by $T$.

The image of $T$ is the line $x - 2y = 0$ rotated counterclockwise by $45^\circ$ around the origin. The equation of this line is $3x - y = 0$. Note that $\text{Im} T$ is spanned by a column vector $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ of the matrix of $T$. 


Transformation $T$ is not invertible. It can be explained in many different ways. For example, $\text{Ker } T \neq \mathbb{0}$ or $\text{Im } T \neq \mathbb{R}^2$ or the determinant of the matrix of $T$ is 0.

The picture makes all calculations crystal clear:

\[
\begin{align*}
&\text{Ker } T \\
&\text{Im } T \\
x - 2y = 0 \\
3x - y = 0 \\
2x + y = 0
\end{align*}
\]

**Answer:**

the standard matrix of $T$ is $\frac{\sqrt{2}}{10} \begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix}$,

$\text{Ker } T = \text{span } \{(1, -2)\}$,

$\text{Im } T = \text{span } \{(1, 3)\}$,

$T$ is **not** invertible.
Problem 3.  

A secret agent has got an encoded message 
\[-4, 2, -19, 0, 3, -9\]
representing the time of the beginning of a secret mission. He knows that the encoding matrix is
\[
\begin{pmatrix}
1 & 0 & -1 \\
0 & 1 & 0 \\
2 & -3 & -3
\end{pmatrix}
\]
but nevertheless cannot decode since he is not good in Linear Algebra. Help him to decode the secret time!

Solution. Oh, boy! First, invert the matrix:

\[
\begin{pmatrix}
1 & 0 & -1 \\
0 & 1 & 0 \\
2 & -3 & -3
\end{pmatrix}
\xrightarrow{R_3-2R_1}
\begin{pmatrix}
1 & 0 & -1 \\
0 & 1 & 0 \\
0 & -3 & -1
\end{pmatrix}
\xrightarrow{R_3+3R_2}
\begin{pmatrix}
1 & 0 & -1 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{pmatrix}
\xrightarrow{R_1-R_3}
\begin{pmatrix}
1 & 0 & 3 & -3 & -1 \\
0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 2 & -3 & -1
\end{pmatrix}.
\]

Second, multiply the inverse matrix by the two vectors \((-4, 2, -19)\) and \((0, 3, -9)\) from the encoded message:

\[
\begin{pmatrix}
3 & -3 & -1 \\
0 & 1 & 0 \\
2 & -3 & -1
\end{pmatrix}
\begin{pmatrix}
-4 \\
2 \\
-19
\end{pmatrix}
= \begin{pmatrix}
1 \\
2 \\
5
\end{pmatrix},
\begin{pmatrix}
3 & -3 & -1 \\
0 & 1 & 0 \\
2 & -3 & -1
\end{pmatrix}
\begin{pmatrix}
0 \\
3 \\
-9
\end{pmatrix}
= \begin{pmatrix}
0 \\
3 \\
0
\end{pmatrix}.
\]

Third, read the secret time: 1, 2, 5, 0, 3, 0 or 12:50:30. (The secret mission is Midterm I : o)

Answer: \boxed{12:50:30}
Problem 4. For each value of a constant $a$, find the dimension of a subspace generated by vectors $(a, 1, 1)$, $(2, -3, 5)$ and $(1, 0, 1)$.

Solution. Let $V = \text{span}\{(a, 1, 1), (2, -3, 5), (1, 0, 1)\}$. The dimension of $V$ is equal to the rank of the matrix

\[
\begin{pmatrix}
a & 2 & 1 \\
1 & -3 & 0 \\
1 & 5 & 1 \\
\end{pmatrix} \sim \begin{pmatrix}
1 & -3 & 0 \\
1 & 5 & 1 \\
a & 2 & 1 \\
\end{pmatrix} \sim \begin{pmatrix}
1 & -3 & 0 \\
0 & 8 & 1 \\
0 & 2 + 3a & 1 \\
\end{pmatrix} \sim \begin{pmatrix}
1 & -3 & 0 \\
0 & 8 & 1 \\
0 & -6 + 3a & 0 \\
\end{pmatrix}.
\]

If $-6 + 3a = 0$, that is $a = 2$, then the rank is 2. If $-6 + 3a \neq 0$, that is $a \neq 2$, then the rank is 3.

Answer: If $a = 2$ then the dimension is 2. If $a \neq 2$ then the dimension is 3.
Problem 5. A linear transformation \( T : \mathbb{R}^3 \rightarrow \mathbb{R}^4 \) is defined by
\[
T(x, y, z) = (x + y - z, -y + z, -2x - 2y + 2z, 3y - 3z).
\]
a) Find the matrix of \( T \) with respect to the standard bases.
b) Find a basis in the kernel of \( T \) and a basis in the image of \( T \).
c) Find the dimensions of the kernel and the image.
d) Find the rank of \( T \).
e) Verify the Kernel-Image theorem for \( T \).

Solution. The matrix of \( T \) with respect to the standard bases in \( \mathbb{R}^3 \) and \( \mathbb{R}^4 \) is
\[
A = \begin{pmatrix}
1 & 1 & -1 \\
0 & -1 & 1 \\
-2 & -2 & 2 \\
0 & 3 & -3
\end{pmatrix}_{4 \times 3}
\]
We perform elementary row transformations to get the reduced row-echelon form of \( A \):
\[
A = \begin{pmatrix}
1 & 1 & -1 \\
0 & -1 & 1 \\
-2 & -2 & 2 \\
0 & 3 & -3
\end{pmatrix}
\sim \begin{pmatrix}
1 & 1 & -1 \\
0 & -1 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\sim \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & -1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} = \text{rref}(A).
\]
Obviously, the rank of \( A \) is 2. It is the dimension of the image of \( T \). The image of \( T \) is generated by the first and the second columns of \( A \), since the leading ones in the \( \text{rref}(A) \) stay in the first and the second columns: \( \text{Im}T = \text{span}\{(1, 0, -2, 0), (1, -1, -2, 3)\} \). Since the spanning vectors are linearly independent they comprise a basis of \( T \).

The Kernel-Image Theorem says that
\[
\dim \ker T + \dim \text{Im} T = \dim \mathbb{R}^3
\]
or
\[
\dim \ker T + 2 = 3.
\]
So \( \dim \ker T = 1 \). A basis of \( \ker T \) can be found by solving a homogenous linear system with coefficient matrix \( A \). It is easy to read the solution from the \( \text{rref}(A) \): \( x = 0, y = t, z = t \), where \( t \) is an arbitrary real number. Hence
\[
\ker T = \{(0, t, t) \mid t \in \mathbb{R} \} = \text{span}\{(0, 1, 1)\}.
\]

Answer: The standard matrix of \( T \) is \( A = \begin{pmatrix}
1 & 1 & -1 \\
0 & -1 & 1 \\
-2 & -2 & 2 \\
0 & 3 & -3
\end{pmatrix},
\]
a basis of \( \ker T \) is \( \{(0, 1, 1)\} \),
a basis of \( \text{Im} T \) is \( \{(1, 0, -2, 0), (1, -1, -2, 3)\} \),
\( \dim \ker T = 1 \),
\( \dim \text{Im} T = \text{rk} T = 2 \).