Name ______________________________

Problem 1 (10pt). Show that the linear system

$$\begin{cases} 3x & + & 2y & = & 4 \\ 4x & - & 3y & = & 3 \end{cases}$$

has a unique solution. Find the solution in three different ways: by the Gauss-Jordan elimination, by Cramer's rule and using the inverse of the coefficient matrix. Give a geometric interpretation of the system and its solution.

3pt • by Gauss-Jordan elimination:

$$\left(\begin{array}{cc|c} 3 & 2 & 4 \\ 4 & -3 & 3 \end{array}\right) \sim \left(\begin{array}{cc|c} 3 & 2 & 4 \\ 1 & -5 & -1 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -5 & -1 \\ 0 & 17 & 7 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 18/17 \\ 0 & 1 & 7/17 \end{array}\right)$$

$(x,y) = (18/17, 7/17)$

3pt • by Cramer's rule:

$$x = \frac{\begin{vmatrix} 4 & 2 \\ 3 & -3 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ 4 & -3 \end{vmatrix}} = \frac{-18}{-17} = \frac{18}{17}, \quad y = \frac{\begin{vmatrix} 3 & 4 \\ 4 & 3 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ 4 & -3 \end{vmatrix}} = \frac{-7}{-17} = \frac{7}{17}$$

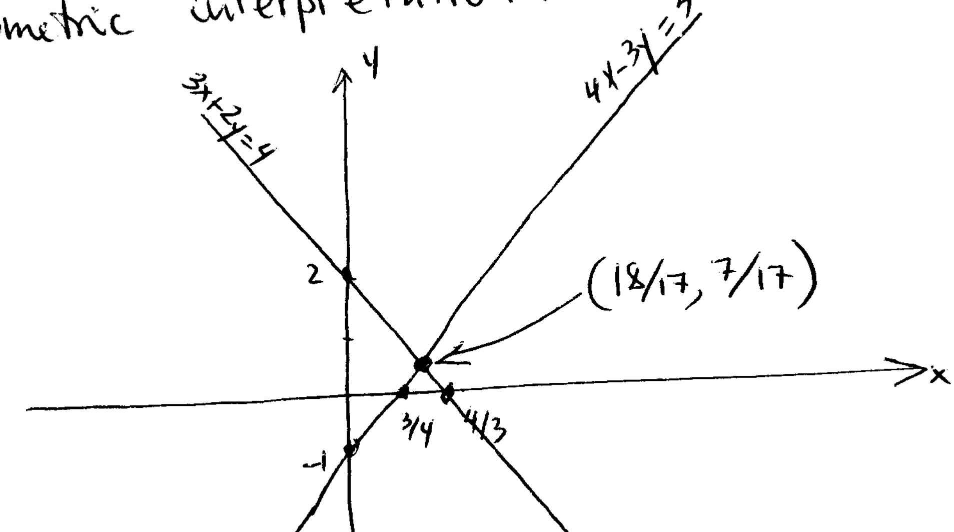
$(x,y) = (18/17, 7/17)$

3pt • by using the inverse matrix:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 4 & -3 \end{pmatrix}^{-1} \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \frac{1}{-17} \begin{pmatrix} -3 & -2 \\ -4 & 3 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \frac{1}{17} \begin{pmatrix} 3 & 2 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 18/17 \\ 7/17 \end{pmatrix}$$

$(x,y) = (18/17, 7/17)$

1pt • Geometric interpretation:



Name ______________________

Problem 2 (15pt). Prove that the set of 2×2 matices A such that

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A = A \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

forms a subspace of the vector space M_2 of all 2×2 matrices. Find a basis and the dimension of this subspace.

Let $U = \{ A \in M_2 \mid \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A = A \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \}$.

1) $\forall A, B \in U$

$$\underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}(A+B)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} B = A\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + B\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \underbrace{(A+B)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$$

$\Rightarrow A+B \in U$

2) $\forall A \in U \quad \forall k \in \mathbb{R}$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}(kA) = k\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A = (kA)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Rightarrow kA \in U$$

5pt 1)+2) $\Rightarrow$ U is a subspace of M_2.

- A basis of U? Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U \Rightarrow$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} c & d \\ a & b \end{pmatrix} = \begin{pmatrix} a & -b \\ c & -d \end{pmatrix} \Rightarrow \begin{matrix} c = a \\ d = -b \end{matrix} \Rightarrow A = \begin{pmatrix} a & b \\ a & -b \end{pmatrix} =$$

$$= a\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$$

So $\forall A \in U \quad A = a\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$ for some $a, b \in \mathbb{R}$

Hence $U = \text{span}\{\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}\}$ and since $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$ are lin. independent, a basis of U is

10pt $\boxed{\{\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}\}}$. $\boxed{\dim U = 2}$

Name ________________________________

Problem 3 (15pt). Let T be a linear operator in the space $\mathcal{P}_2$ of polynomials of degree 2 in one variable defined by the formula

$$T : p \mapsto xp'(x) + p(0), \quad \text{where } p(x) \in \mathcal{P}_2.$$

Is T an isomorphism? Explain!

$$1 \xmapsto{T} 1$$

$$x \xmapsto{T} x$$

$$x^2 \xmapsto{T} x \cdot 2x = 2x^2$$

The matrix of T wrt the basis $\{1, x, x^2\}$ of $\mathbb{P}_2$ is

$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$; $\det A = 2 \neq 0 \Rightarrow T$ is an <u>isomorphism</u>.

10pt

5pt

Name ______________________

Problem 4 (15pt). Let W be a subspace of Euclidean space $\mathbb{R}^4$ spanned by vectors $(1,1,0,1)$ and $(0,1,1,0)$. Find a an **orthonormal** basis of $W^\perp$.

Let $w_1 = (1,1,0,1)$, $w_2 = (0,1,1,0)$

$W = \operatorname{span}\{w_1, w_2\}$

$W^\perp = \{v \in \mathbb{R}^4 \mid v \perp w_1 \text{ and } v \perp w_2\} =$

$= \{(x_1,x_2,x_3,x_4) \in \mathbb{R}^4 \mid (x_1,x_2,x_3,x_4)\cdot(1,1,0,1) = 0,$
$(x_1,x_2,x_3,x_4)\cdot(0,1,1,0) = 0\} =$

5pt $\{(x_1,x_2,x_3,x_4) \mid x_1+x_2+x_4 = 0,\ x_2+x_3 = 0\} =$

$= \operatorname{Ker}\begin{pmatrix}1&1&0&1\\0&1&1&0\end{pmatrix} = \operatorname{Ker}\begin{pmatrix}1&0&-1&1\\0&1&1&0\end{pmatrix} =$

$\left\{\begin{pmatrix}-t+s\\-s\\s\\t\end{pmatrix} \in \mathbb{R}^4,\ t,s \in \mathbb{R}\right\} = \operatorname{span}\left\{\begin{pmatrix}-1\\0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\1\\0\end{pmatrix}\right\}$ 5pt

Let $v_1 = \begin{pmatrix}-1\\0\\0\\1\end{pmatrix}$, $v_2 = \begin{pmatrix}1\\-1\\1\\0\end{pmatrix}$. So $\{v_1, v_2\}$ is a basis of $W^\perp$.

Make it orthonormal by Gram-Schmidt orthogonalization:

$\{v_1, v_2\} \longrightarrow \{u_1, u_2\} \longrightarrow \{e_1, e_2\}$

$u_1 = v_1 = (-1,0,0,1)$

$u_2 = v_2 - \operatorname{pr}_{v_1} v_2 = v_2 - \frac{v_1\cdot v_2}{v_1\cdot v_1} v_1 = (1,-1,1,0) - \frac{-1}{2}(-1,0,0,1) =$

$= \left(\frac{1}{2}, -1, 1, \frac{1}{2}\right) = \frac{1}{2}(1,-2,2,1)$

$e_1 = \frac{u_1}{\|u_1\|} = \frac{1}{\sqrt{2}}(-1,0,0,1)$, $e_2 = \frac{u_2}{\|u_2\|} = \frac{1}{\sqrt{10}}(1,-2,2,1)$

So an ON basis of $W^\perp$ is

$\left\{\frac{1}{\sqrt{2}}(-1,0,0,1), \frac{1}{\sqrt{10}}(1,-2,2,1)\right\}$ 5pt

Name ______________________________

Problem 5 (15pt).

The former secret agent has retired. He is now a big fan of sudoku. Help him to solve an orthogonal sudoku, insert missing entries in an **orthogonal** matrix.

$$\begin{pmatrix} \frac{1}{3\sqrt{2}} & * & \frac{2}{3} \\ * & * & \frac{2}{3} \\ -\frac{4}{3\sqrt{2}} & 0 & * \end{pmatrix}$$

There are 2 such matrices:

$$\begin{pmatrix} \frac{1}{3\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{2}{3} \\ \frac{1}{3\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{2}{3} \\ -\frac{4}{3\sqrt{2}} & 0 & \frac{1}{3} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \frac{1}{3\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{2}{3} \\ \frac{1}{3\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{2}{3} \\ -\frac{4}{3\sqrt{2}} & 0 & \frac{1}{3} \end{pmatrix}$$

Name ______________________________

Problem 6 (10pt). Calculate the following determinant

$$\begin{vmatrix} 4 & 0 & -3 & 5 \\ 2 & 9 & 0 & -1 \\ 3 & -2 & 1 & 1 \\ 0 & 2 & 3 & -4 \end{vmatrix}.$$

$$\begin{vmatrix} 4 & 0 & -3 & 5 \\ 2 & 9 & 0 & -1 \\ 3 & -2 & 1 & 1 \\ 0 & 2 & 3 & -4 \end{vmatrix} \times 3 = \begin{vmatrix} 13 & -6 & 0 & 8 \\ 2 & 9 & 0 & -1 \\ 3 & -2 & 1 & 1 \\ -9 & 8 & 0 & -7 \end{vmatrix} = \begin{vmatrix} 13 & -6 & 8 \\ 2 & 9 & -1 \\ -9 & 8 & -7 \end{vmatrix} \begin{matrix} \times 8 \\ \\ \times 7 \end{matrix} =$$

$$= \begin{vmatrix} 29 & 66 & 0 \\ 2 & 9 & -1 \\ -23 & -55 & 0 \end{vmatrix} = -\begin{vmatrix} 29 & 66 \\ 23 & 55 \end{vmatrix} = -\begin{vmatrix} 6 & 11 \\ 23 & 55 \end{vmatrix} = -11 \begin{vmatrix} 6 & 1 \\ 23 & 5 \end{vmatrix} =$$

$$= \boxed{-77}$$

Name ______________________________

Problem 7 (20pt). Find the eigenvalues and a basis for eigenspaces of the matrix

$$A = \begin{pmatrix} 4 & -1 \\ 5 & -2 \end{pmatrix}.$$

Does there exists an invertible matrix S such that $D = S^{-1}AS$ is a diagonal matrix? If so, find S and D.

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 4-\lambda & -1 \\ 5 & -2-\lambda \end{vmatrix} = 0$$

$$-(4-\lambda)(2+\lambda) + 5 = 0$$

$$\lambda^2 - 2\lambda - 3 = 0$$

$$(\lambda+1)(\lambda-3) = 0$$

$$\lambda_1 = -1, \ \lambda_2 = 3$$ Eigenvalues 5 pt

$$E_{\lambda_1=-1} = \ker(A+I) = \ker\begin{pmatrix} 5 & -1 \\ 5 & -1 \end{pmatrix} = \ker\begin{pmatrix} 5 & -1 \\ 0 & 0 \end{pmatrix} = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 5 \end{pmatrix}\right\}$$ 10 pts

$$E_{\lambda_2=3} = \ker(A-3I) = \ker\begin{pmatrix} 1 & -1 \\ 5 & -5 \end{pmatrix} = \ker\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$$

$$S = \begin{pmatrix} 1 & 1 \\ 5 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}$$ 5 pt

Check: $D = S^{-1} A S$

$$\begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix} \stackrel{?}{=} -\frac{1}{4}\begin{pmatrix} 1 & -1 \\ -5 & 1 \end{pmatrix}\begin{pmatrix} 4 & -1 \\ 5 & -2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 5 & 1 \end{pmatrix} =$$

$$= \frac{1}{4}\begin{pmatrix} -1 & 1 \\ 5 & -1 \end{pmatrix}\begin{pmatrix} -1 & 3 \\ -5 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}$$ yes!

Name ______________________

Problem 8 (This is an extra problem, 25pt). Let T be a linear operator in $\mathbb{R}^3$ given in the standard basis by the matrix

$$A = \begin{pmatrix} 2 & 1 & 2 \\ a & 2 & a-3 \\ 1 & -1 & 1 \end{pmatrix},$$

where a is a real constant. For which values of a is T diagonalizable? For each such a find en eigenbasis for T.

$$\det(A-\lambda I) = \begin{vmatrix} 2-\lambda & 1 & 2 \\ a & 2-\lambda & a-3 \\ 1 & -1 & 1-\lambda \end{vmatrix} = \begin{vmatrix} 3-\lambda & 0 & 3-\lambda \\ a & 2-\lambda & a-3 \\ 1 & -1 & 1-\lambda \end{vmatrix} =$$

$$= (3-\lambda)\begin{vmatrix} 1 & 0 & 1 \\ a & 2-\lambda & a-3 \\ 1 & -1 & 1-\lambda \end{vmatrix} \times a = (3-\lambda)\begin{vmatrix} 1 & 0 & 1 \\ 0 & 2-\lambda & -3 \\ 0 & -1 & -\lambda \end{vmatrix} =$$

$$= (3-\lambda)(-\lambda(2-\lambda)-3) = (3-\lambda)(\lambda^2-2\lambda-3) = (3-\lambda)(\lambda-3)(\lambda+1) =$$

10pt $$= -(\lambda-3)^2(\lambda+1)$$

Eigenvalues are $\lambda = -1$ and $\lambda = 3$.

$$E_{\lambda=3} = \operatorname{Ker}(A-3I) = \operatorname{Ker}\begin{pmatrix} -1 & 1 & 2 \\ a & -1 & a-3 \\ 1 & -1 & -2 \end{pmatrix} = \operatorname{Ker}\begin{pmatrix} -1 & 1 & 2 \\ a-1 & 0 & a-1 \\ 0 & 0 & 0 \end{pmatrix}$$

5pt $\boxed{a=1} \Rightarrow \dim E_{\lambda=3} = 2$ and T is diagonalizable

$a \neq 1 \Rightarrow \dim E_{\lambda=3} = 1$ and T is not diagonalizable.

So for $a=1$:

$$E_{\lambda=3} = \operatorname{Ker}\begin{pmatrix} -1 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}\right\}$$

$$E_{\lambda=-1} = \operatorname{Ker}(A+I) = \operatorname{Ker}\begin{pmatrix} 3 & 1 & 2 \\ 1 & 3 & -2 \\ 1 & -1 & 2 \end{pmatrix} = \operatorname{Ker}\begin{pmatrix} 0 & 4 & -4 \\ 0 & 4 & -4 \\ 1 & -1 & 2 \end{pmatrix} = \operatorname{Ker}\begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} =$$

$$= \operatorname{Ker}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} = \operatorname{span}\left\{\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}\right\}$$

An eigenbasis is $\left\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}\right\}$ 10pt

A diagonal form of T is $\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix}$