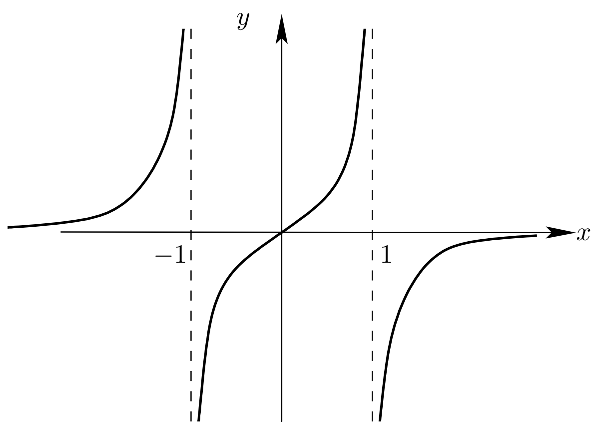
Stony Brook University
Mathematics Department
J. Viro

Introduction to Calculus
MAT 123, Fall 2012
December 12th, 2012

Final Exam. Solutions

1. A function $y = f(x)$ is given by its graph below. Determine the domain and the range of f, intervals of increasing and intervals of decreasing, indicate all asymptotes. Is f even or odd or nether even nor odd? Explain! Is f invertible? Explain!



Solution: The domain is $(-\infty,-1)\cup(-1,1)\cup(1,\infty) = \mathbb{R} \setminus \{\pm 1\}$. The range in $\mathbb{R}$. Function f increases on $(-\infty,-1)\cup(-1,1)\cup(1,\infty)$, that is on the whole domain. There are no intervals of decreasing. The x-axis is a horizontal asymptote, the lines $x = -1$ and $x = 1$ are vertical asymptotes. Function f is odd since its graph is symmetric about the origin. Function f is not invertible since it fails a horizontal line test.

2. Find the point where the function $f(x) = -\frac{1}{2}x^2 + x + 1$ attains its maximum. What is the maximal value of the function?

Solution: The graph of this quadratic function is a parabola with horns turned down. So the point where the function attains its maximum is the vertex of the parabola. To find the vertex, we use the formula $x = -\frac{b}{2a} = -\frac{1}{2\cdot(-\frac{1}{2})} = 1$, $y(1) = -\frac{1}{2}\cdot 1^2 + 1 + 1 = \frac{3}{2}$. Therefore, the vertex is located at $(1, 3/2)$.

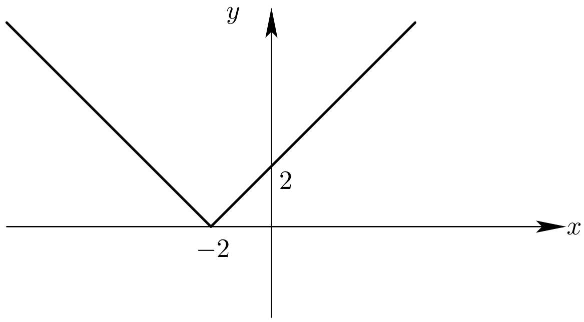
Alternatively, we can complete the square:

$$-\frac{1}{2}x^2 + x + 1 = -\frac{1}{2}(x^2 - 2x - 2) = -\frac{1}{2}((x-1)^2 - 3) = -\frac{1}{2}(x-1)^2 + \frac{3}{2}.$$

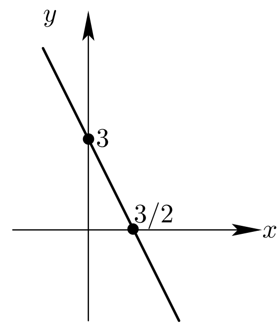
So the maximum of the function is $3/2$, it is attained at $x = 1$.

3. Draw the graphs of the following functions. On your pictures, indicate asymptotes (if there are any), and x- and y-intercepts.

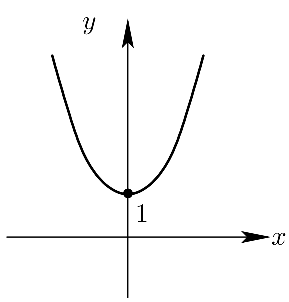
a) $y = |x + 2|$



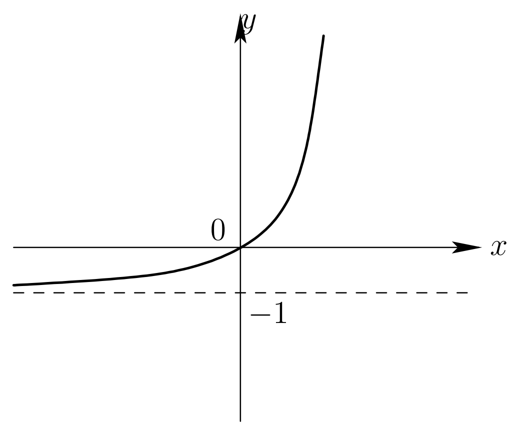
b) $y = -2x + 3$



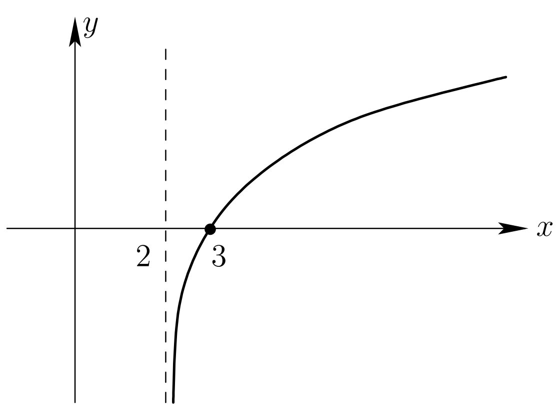
c) $y = x^2 + 1$



d) $y = 2^x - 1$



e) $y = \ln(x - 2)$



4. Explain why the graph of the function $y = x^2 + \cos x$ is symmetric about the y-axis. (There is no need to draw the graph.)

Solution: The function is even since $y(-x) = (-x)^2 + \cos(-x) = x^2 + \cos x = y(x)$ for any x. The graph of an even function is symmetric about the y-axis. Therefore, the graph of the function $y = x^2 + \cos x$ is symmetric about the y-axis.

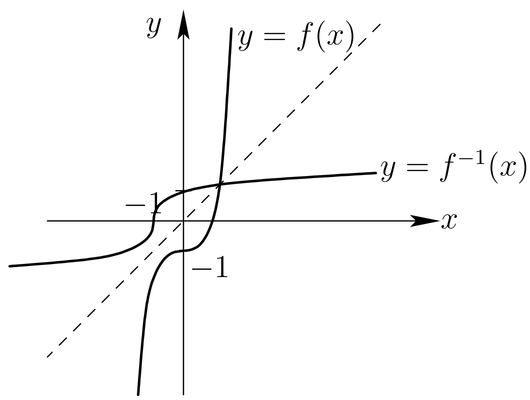
5. Let $f(x) = e^{\sin(3x)}$. **a)** Present f as a composition of two functions. **b)** Present f as a composition of three functions. (None of the function is allowed to be the identity function.)

Solution: a) Let $g(x) = \sin(3x)$ and $h(x) = e^x$. Then $f(x) = (h \circ g)(x)$.

b) Let $k(x) = 3x$, $g(x) = \sin(x)$ and $h(x) = e^x$. Then $f(x) = (h \circ g \circ k)(x)$.

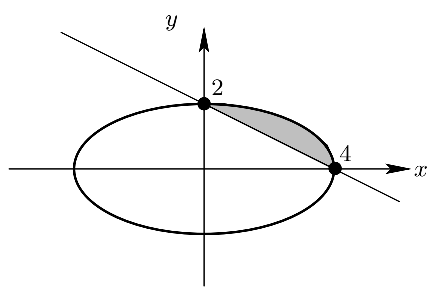
6. Let $f(x) = x^3 - 1$. Find a formula for the inverse function $f^{-1}(x)$. Determine the domain and the range of f and f^{-1}. On the same coordinate system, draw the graphs of $y = f(x)$ and $y = f^{-1}(x)$.

Solution: Let $y = x^3 - 1$. Then $y + 1 = x^3$ and $\sqrt[3]{y+1} = x$. Therefore, $f^{-1}(x) = \sqrt[3]{x+1}$. The domain of f and the range of f^{-1} is $\mathbb{R}$, the range of f and the domain of f^{-1} is $\mathbb{R}$.



7. The line $x + 2y = 4$ cuts the region inside the ellipse $\dfrac{x^2}{16} + \dfrac{y^2}{4} = 1$ into two parts. Find the area of the smallest part.

Solution: Let us draw a picture:



The area we are interested in is equal to one-quater of the area inside the ellipse minus the area of the triangle with sides of length 4 and 2.

The area of the region inside the ellipse $\dfrac{x^2}{4^2} + \dfrac{y^2}{2^2} = 1$ is 8π, the area of the triangle is 4, so the area of the shaded region is $2\pi - 4$.

8. Find the minimal annual interest rate compounded continuously which will guarantee the doubling of invested amount after 50 years. (You may use an approximation $\ln 2 \approx 0.69$.)

Solution: The model for an interest rate compounded continuously is $P(t) = P\,e^{rt}$, where P is the principal (original amount), r is the annual interest rate, and $P(t)$ is the amount after t years.

According to the problem, $t = 50$ years and $P(50) = 2P$. We plug in these values into the equation:

$P(50) = Pe^{50r}$, that is $2P = Pe^{50r}$, which gives $2 = e^{50r}$ and, after taking natural logarithm from both sides, $\ln 2 = 50r$. From here, $r = \dfrac{\ln 2}{50} \approx \dfrac{0.69}{50} = 0.0138$. So the minimal interest rate is 1.38%.

9. Solve the following equations:

a) $\sin\theta = \dfrac{1}{2}$ **b)** $\cos\theta = \dfrac{\pi}{2}$

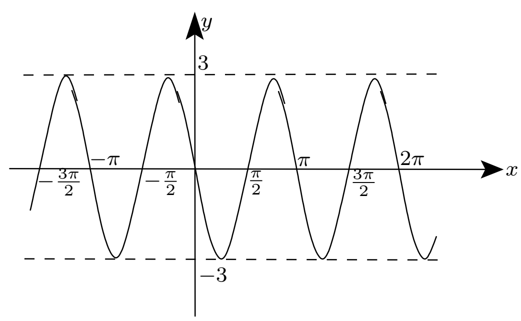
Solution.

a) $\sin\theta = 1/2$
$\theta = \pi/6 + 2\pi n$ or
$\theta = 5\pi/6 + 2\pi n$,
where n is integer

b) $\cos\theta = \pi/2$
Since $\pi/2 > 1$,
the equation has no solutions.

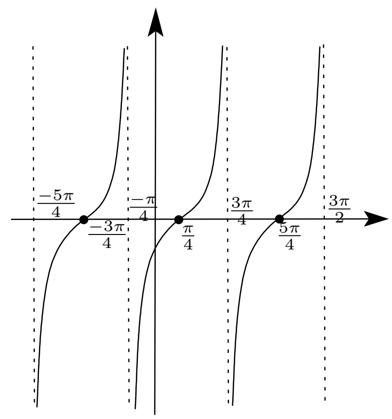
10. For the function $y = -3\sin(2x)$, find the period and the amplitude. (Your graph should contain at least three periods.) Draw the graph. Indicate x-intercepts on the graph.

Solution: The period is $2\pi/2 = \pi$, the amplitude is $|-3| = 3$. x-intercepts are at $\pi/2 + \pi n$, where n is an integer.

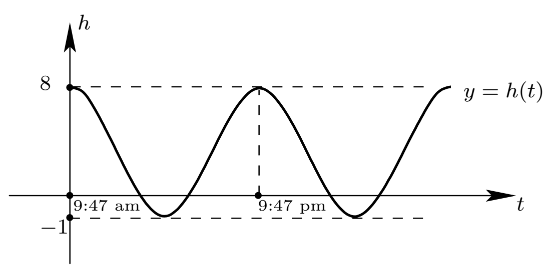


11. Draw the graph of the function $y = \tan(x - \dfrac{\pi}{4})$. (Your graph should contain at least three periods.) Indicate asymptotes and x-intercepts on your picture.

Solution: Asymptotes are $x = -\pi/4 + \pi n$, x-intercepts are at $\pi/4 + \pi n$, where n is an integer.



12. The graph below shows the level of water (in feet) in tides on Long Island Sound today, December 12th 2012, starting at 9:47 am. Assuming that the level of water is described by a function $h(t) = A\cos(Bt) + C$ (where t is the time measured in hours and h is the level of water measured in feet), find the coefficients A, B and C. When the level of water is minimal during this day?



Solution: Let A be the amplitude. Then $2A = 8 - (-1) = 9$, so $A = 4.5$ (feet). Let T be the period. Then $T = 9:47\,\text{pm} - 9:47\,\text{am} = 12$ (hours). Period T and coefficient B (circular frequency) are related by the formula $T = 2\pi/B$, so $B = 2\pi/T = 2\pi/12 = \pi/6$.

Therefore, $h(t) = 4.5\cos\left(\frac{\pi t}{6}\right) + C$. To find coefficient C, we have a look on the first point of maximum on the picture: $h(0) = 8$, from which $8 = h(0) = 4.5\cos\left(\frac{\pi \cdot 0}{6}\right) + C = 4.5 + C$ and $C = 8 - 4.5 = 3.5$.

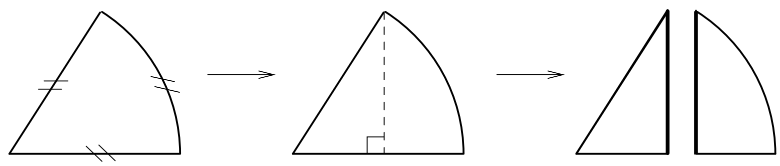
Finally, $h(t) = 4.5\cos\left(\frac{\pi t}{6}\right) + 3.5$.

The water level is minimal at $t = T/2 = 6$, that is at 9:47am+6 hours=3:47pm.

Bonus problems

1. Explain for a little brother what "sin 1" is.

Solution: Imagine a slice of pizza (a circular sector) with all three sides of the same length of 1 feet. Cut this slice along the dashed line as shown:



The length of the cut is called in mathematics "$\sin 1$" (feet).

2. Solve the equation $\sin(e^x) = 0$.

Solution: $\sin(e^x) = 0 \implies e^x = \pi n$, where n is a positive integer (since $e^x > 0$). Therefore, the solution is $x = \ln(\pi n)$, where n is a positive integer.

3. Show that $1 + \tan^2\theta = \sec^2\theta$ for any θ.

Solution: $1 + \tan^2\theta = 1 + \dfrac{\sin^2\theta}{\cos^2\theta} = \dfrac{\cos^2\theta}{\cos^2\theta} + \dfrac{\sin^2\theta}{\cos^2\theta} = \dfrac{\cos^2\theta + \sin^2\theta}{\cos^2\theta} = \dfrac{1}{\cos^2\theta} = \sec^2\theta.$

4. Prove that the area of the region under the hyperbola $y = 1/x$, above the x-axis, and between the lines $x = a$ and $x = b$ (where $1 < a < b$) is equal to $\ln\dfrac{b}{a}$.

Solution: This area is the difference between the area of the region under the hyperbola $y = 1/x$, above the x-axis and between the lines $x = 1$ and $x = b$ and the area of the region under the hyperbola $y = 1/x$, above the x-axis and between the lines $x = 1$ and $x = a$, that is $\ln b - \ln a = \ln\dfrac{b}{a}$.

5. Show that $f(x) = e^{\cos x}$ is a periodic function. Find its domain and the range and sketch the graph.

Solution: $f(x + 2\pi n) = e^{\cos(x+2\pi n)} = e^{\cos x} = f(x)$ for any x and for any integer n. Therefore, f is periodic with period $2\pi n$.

The domain is $\mathbb{R}$. To find the range, we observe that the exponential function $y = e^x$ is increasing, therefore $-1 \le \cos x \le 1 \implies e^{-1} \le e^{\cos x} \le e^1$. So the range is $[1/e, e]$.

Based on the information above, we may conclude that the graph of f oscillates periodically between $1/e$ and e. Take a couple of support points: $f(0) = e$, $f(\pm\pi) = 1/e$ and sketch the graph: