CURVES OF SMALL DEGREE ON CUBIC THREEFOLDS

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Abstract. In this article we consider the spaces $\mathcal{H}^{d,g}(X)$ parametrizing smooth curves of degree d and genus g on a smooth cubic threefold $X \subset \mathbb{P}^4$. For $1 \le d \le 5$, we show that each variety $\mathcal{H}^{d,g}(X)$ is irreducible of dimension $2d$.

1. Introduction

Suppose that $X \subset \mathbb{P}^4$ is a smooth cubic hypersurface in complex projective 4-space. In this article we consider the space $\mathcal{H}^{d,g}(X)$ parametrizing smooth curves of degree d and genus g on a smooth cubic threefold $X \subset \mathbb{P}^4$. For $1 \le d \le 5$ we show that each variety $\mathcal{H}^{d,g}(X)$ is irreducible of dimension $2d$.

For the *Fano scheme* of lines $F = \mathcal{H}^{1,0}(X)$, this is a classical result, c.f. [1]. We bootstrap from this case by residuation: in each case we show that for a general point $[C] \in \mathcal{H}^{d,g}(X)$ there is a surface $\Sigma \subset \mathbb{P}^4$ which contains C and such that every irreducible component of the residual to C in $\Sigma \cap X$ has degree $e < d$. In this way we inductively prove that for $1 \le d \le 5$ the space $\mathcal{H}^{d,g}(X)$ is irreducible, and in several cases we also show smoothness. In a forthcoming paper [8], we use similar methods to describe the *Abel-Jacobi maps* $u_{d,g} : \mathcal{H}^{d,g}(X) \to J(X)$ for $1 \le d \le 5$.

1.1. **Notation.** All schemes in this paper will be schemes over $\mathbb{C}$. All absolute products will be understood to be fiber products over $\mathrm{Spec}(\mathbb{C})$.

For a projective variety X and a numerical polynomial $P(t)$, $\mathrm{Hilb}_{P(t)}(X)$ denotes the corresponding Hilbert scheme. For integers d, g, $\mathcal{H}^{d,g}(X) \subset \mathrm{Hilb}_{dt+1-g}(X)$ denotes the open subscheme parametrizing smooth, connected curves of degree d and genus g.

2. Preliminaries

In this section we gather some preliminary facts about deformation theory, residuation, and Abel-Jacobi maps.

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2.1. **Deformation Theory.**
All of the irreducibility arguments in this paper follow the same pattern, and the linchpin of these arguments is the infinitesimal analysis of the Hilbert scheme in [10, section I.2], in particular [10, theorem I.2.15]. The part of this theorem which we shall use most often is the following:

Proposition 2.1. *Let Y be a smooth complex variety with canonical divisor class K_Y and let $C \subset Y$ be a connected, local complete intersection curve with normal bundle $N_{C/Y} = \mathcal{I}_C/\mathcal{I}_C^2$ and with arithmetic genus p_a. Every irreducible component of the Hilbert scheme at $[C]$ has dimension at least*

$$\chi(N_{C/Y}) = h^0(N_{C/Y}) - h^1(N_{C/Y}) = -K_Y.[C] + (1-p_a)(\dim Y - 3). \tag{1}$$

The Zariski tangent space has dimension $h^0(N_{C/Y})$, therefore the Hilbert scheme is smooth at $[C]$ if $h^1(N_{C/Y}) = 0$.

Although this is technically inaccurate, we will say that the curve $C \subset Y$ is *unobstructed* if $h^1(N_{C/Y}) = 0$.

Another condition closely related to smoothness of the Hilbert scheme at $[C]$ is the question of whether deformations of C *smooth* the singularities of C, i.e., whether or not C is in the closure of the open set parametrizing smooth curves. Suppose that C is a *nodal* curve, i.e., every singular point is formally isomorphic to the formal neighborhood of $0 \in \mathrm{Spec}\ \mathbb{C}[x,y]/xy$. Then [2, lemma 9.2.2] the deformation space of the nodes $p_1, \dots, p_\delta$ is canonically identified with

$$H^0(C, \mathrm{Ext}^1_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C)) \ = \ \oplus_{i\,=\,1}^{\delta} T_i' \otimes T_i'' \tag{2}$$

where T_i', T_i'' are the tangent spaces of the two branches of C at p_i. In the case that C is unobstructed we have a short exact sequence:

$$H^0(C, N_{C/X}) \longrightarrow H^0(C, \mathrm{Ext}^1_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C)) \longrightarrow H^1(C, T_Y|_C) \longrightarrow 0 \tag{3}$$

This calculation leads to the following:

Lemma 2.2. *When $h^1(C, T_Y|_C) = 0$ the morphism from the formal neighborhood of $[C]$ in the Hilbert scheme to the deformation space of the nodes is smooth at $[C]$, thus deformations of C smooth the nodes.*

A different approach to smoothing nodes is as follows (at some level it is equivalent to the last paragraph). Suppose Y is a smooth variety and $Z \subset Y$ is a simple normal crossings subscheme with no triple points, in particular each irreducible component of Z is smooth. Let Z_i be an irreducible component of Z and let $D_1, \dots, D_r$ be the connected components of $\mathrm{sing}(Z) \cap Z_i$. For each $j = 1, \dots, r$, let Z_j' be the second irreducible component of Z which contains D_i (if an irreducible component intersects itself, make an étale base change such that the preimage of Z_i decomposes into a union of irreducible components in a neighborhood of the preimage of D_i). Consider the diagram of sheaves:

$$0 \longrightarrow (I_{Z/Y}) \,/\, (I_{Z/Y} I_{Z_i/Y}) \longrightarrow (I_{Z_i/Y}) \,/\, \left(I^2_{Z_i/Y}\right) \longrightarrow (I_{Z_i/Y}) \,/\, (I_{Z/Y}) \longrightarrow 0 \tag{4}$$

where $I_{A/B}$ is the ideal sheaf of A in B. By passing to formal neighborhoods and using the canonical form for a simple normal crossings variety, one sees that this is a short exact sequence. Moreover one can identify the last term with $\oplus_{j\,=\,1}^r I_{D_j/Z_j'}/I^2_{D_j/Z_j'}$. Dualizing this short exact sequence leads to the short exact sequence:

$$0 \longrightarrow N_{Z_i/Y} \longrightarrow N_{Z/Y}|_{Z_i} \longrightarrow \oplus_{j\,=\,1}^r N_{D_j/Z_i} \otimes N_{D_j/Z_j'} \longrightarrow 0 \tag{5}$$

Now suppose that Z is a curve with two irreducible components Z_1 and Z_2 intersecting at a node p. We have an obvious short exact sequence of sheaves:

$$0 \longrightarrow N_{Z/Y}|_{Z_1}(-p) \longrightarrow N_{Z/Y} \longrightarrow N_{Z/Y}|_{Z_2} \longrightarrow 0, \tag{6}$$

and the map in equation 3 is simply the composite map

$$H^0(Z, N_{Z/Y}) \longrightarrow H^0(Z_2, N_{Z/Y}|_{Z_2}) \longrightarrow T_{Z_1,p} \otimes T_{Z_2,p} \tag{7}$$

where the second map comes from equation 5. Again using equation 5 and combining this with the long exact sequence in cohomology associated to a short exact sequence of sheaves, we conclude the following

Lemma 2.3. *Suppose that $Z \subset X$ is a nodal curve and Z_1, Z_2 are two closed nodal subcurves of Z which intersect transversally in a single point $p \in Z_1 \cap Z_2$. Then Z is unobstructed and the node of Z smooths when $H^1(Z_1, N_{Z_1/Y}(-p)) \ = \ H^1(Z_2, N_{Z_2/Y}) \ = \ 0$.*

Let us return now to the strategy of proving that $\mathcal{H}^{d,g}(X)$ is irreducible. The first case will be showing that *Fano scheme* of lines $F := \mathcal{H}^{1,0}(X)$ is irreducible, in fact a smooth, projective surface. The analysis of this case is classical. For each $1 < d \leq 5$, we define an incidence correspondence

$$f_{d,g} : I^{d,g} \rightarrow \mathcal{H}^{d,g}(X) \tag{8}$$

parametrizing curves $C \subset X$ along with some extra data and such that $f_{d,g}$ is dominant of constant fiber dimension. The extra data will allow us to associate a surface $S \subset \mathbb{P}^4$ which contains C and such that the residual of C in $S \cap X$ is made up of curves of strictly smaller degree. We stratify $I_{d,g}$ according to the behavior of the residual curve. By studying the residual curves in each case, we prove that there is a unique irreducible component of $I_{d,g}$ whose image in $\mathcal{H}^{d,g}(X)$ has dimension $\geq 2d$, and that this image has dimension precisely $2d$. Then it follows that $\mathcal{H}^{d,g}(X)$ is irreducible of dimension $2d$.

2.2. **Residuation.** In this section we review a few basic facts about residuation of subschemes in a Gorenstein scheme.

Definition 2.4. Suppose that D is a Gorenstein scheme and $D_1 \subset D$ is a closed subscheme of codimension 0. Let $\mathcal{I}$ denote the ideal sheaf of D_1 in D. Define

$$\mathcal{J} = (0 :_{\mathcal{O}_D} \mathcal{I}) = \underline{\mathrm{Hom}}_{\mathcal{O}_D}(\mathcal{O}_{D_1}, \mathcal{O}_D). \tag{9}$$

Denote by $D_2 \subset D$ the closed subscheme associated to the ideal sheaf $\mathcal{J}$. We define $D_2 \subset D$ to be the *residual subscheme* to $D_1 \subset D$.

Theorem 2.5. [3, theorem 21.23] *Let D be a Gorenstein scheme, $D_1 \subset D$ a codimension 0 closed subscheme. Let $D_2 \subset D$ be the residual subscheme to $D_1 \subset D$.*

1. *The codimension of $D_2 \subset D$ is zero and D_2 has no embedded components. If D_1 has no embedded components, then $D_1 \subset D$ is the residual subscheme to $D_2 \subset D$.*
2. *If D_1 is Cohen-Macaulay, then D_2 is Cohen-Macaulay.*
3. *If D_1 is Cohen-Macaulay, then $\mathcal{J} \otimes \omega_D$ is a canonical sheaf for D_1. In particular, D_1 is Gorenstein iff $\mathcal{J}$ is locally principal.*

We will often be concerned with flat families of 1-cycles. The question arises when flatness of D and D_1 over B implies that D_2 is also flat over B. The following lemma addresses this issue and also establishes a *base-change result* for residual subschemes.

Lemma 2.6. *Let R be a local Noetherian ring. Let A be a local Noetherian A-algebra (i.e., $R \rightarrow A$ is a local homomorphism) such that A is Gorenstein and flat over R. Let $I \subset A$ be a codimension zero ideal such that A/I is Cohen-Macaulay. Define $J = (0 :_A I)$.*

1. *For any regular sequence $(r_1, \ldots, r_n)$ in R, we have*
$$J/(r_1, \ldots, r_n)J = \left(0 :_{A/(r_1,\ldots,r_n)A} I/(r_1, \ldots, r_n)I\right). \tag{10}$$
2. *If R is regular, then A/I and A/J are flat over R.*

Proof. First we prove (1). Since $I \subset A$ has codimension zero and A/I is Cohen-Macaulay, A/I is a maximal Cohen-Macaulay module. Since A is flat over R, $(r_1, \ldots, r_n)$ is a regular sequence for A. Using [3, proposition 18.13], the result follows by induction on n with [3, proposition 21.12(b)] as the induction step.

Now we prove (2). By (2) of theorem 2.5, we know that A/J is Cohen-Macaulay. By [3, theorem 18.16], A/J is flat over R iff

$$\dim(A/J) = \dim(R) + \dim(A/(J + m_R A)). \tag{11}$$

We always have the inequality

$$\dim(A/J) \leq \dim(R) + \dim(A/(J + m_R A)). \tag{12}$$

We also have the inequality

$$\dim(R) + \dim(A/(J + m_R A)) \leq \dim(R) + \dim(A/m_R A). \tag{13}$$

Now A is flat over R, so we have

$$\dim(R) + \dim(A/m_R A) = \dim(A). \tag{14}$$

Finally, since $J \subset A$ has codimension zero, $\dim(A) = \dim(A/J)$. Putting the inequalities together, we have

$$\dim(A/J) \leq \dim(R) + \dim(A/(J + m_R A)) \leq \dim(A/J). \tag{15}$$

Thus A/J is flat over R. By the same argument A/I is also flat over R. □

Corollary 2.7 (Reformulation)**.** *Let B be a scheme and let $f : D \to B$ be a flat morphism with D Gorenstein. Let $D_1 \subset D$ be a codimension zero closed subscheme which is Cohen-Macaulay. Let $D_2 \subset D$ be the residual subscheme to $D_1 \subset D$.*

(1) *For any closed subscheme $C \subset B$ which is a regular embedding, $D_1 \times_B C \subset D \times_B C$ and $D_2 \times_B C \subset D \times_B C$ are residual to each other.*
(2) *If B is regular, then D_1 and D_2 are flat over B.*

2.3. **Reminder about Abel-Jacobi Maps.** We shall make occasional use of the *Abel-Jacobi maps* associated to families of 1-cycles on X. The reader is referred to [1], [6] for full definitions. Here we recall only a few facts about Abel-Jacobi maps.

Associated to a smooth, projective threefold X there is a complex torus

$$J^2(X) = H^3_{\mathbb{Z}}(X) \backslash H^3(X, \mathbb{C}) / \left(H^{3,0}(X) \oplus H^{2,1}(X)\right). \tag{16}$$

In case X is a cubic hypersurface in $\mathbb{P}^4$ (in fact for any *rationally connected* threefold) then $J^2(X)$ is a principally polarized abelian variety with theta divisor Θ. Given an algebraic 1-cycle $\gamma \in A_1(X)$ which is *homologically equivalent to zero* [6, 13], one can associate a point $u_2(\alpha)$. The construction is analogous to the Abel-Jacobi map for a smooth, projective algebraic curve C which associates to each 0-cycle $\gamma \in A_0(C)$ which is homologically equivalent to zero a point $u_1(\alpha) \in J^1(C)$, the Jacobian variety of C. In particular $u_2 : A_1(X)^{hom} \to J^2(X)$ is a group homomorphism.

Suppose that B is a normal, connected variety of dimension n and $\Gamma \in A_{n+1}(B \times X)$ is an $(n+1)$-cycle such that for each closed point $b \in B$ the corresponding cycle $\Gamma_b \in A_1(X)$ [4, §10.1] is homologically equivalent to zero. Then in this case the set map $b \mapsto u_2(\Gamma_b) \in J^2(X)$ comes from a (unique) algebraic morphism $u = u_\Gamma : B \to J^2(X)$. We call this morphism the *Abel-Jacobi map* determined by Γ.

More generally, suppose B as above, $\Gamma \in A_{n+1}(B \times X)$ is any $(n+1)$-cycle, and suppose $b_0 \in B$ is some base-point. Then we can form a new cycle $\Gamma' = \Gamma - \pi_2^* \Gamma_{b_0}$, and for all $b \in B$ we have $\Gamma'_b = \Gamma_b - \Gamma_{b_0}$ is homologically equivalent to zero. Thus we have an algebraic morphism $u = u_{\Gamma'} : B \to J^2(X)$. Of course this morphism depends on the choice of a base-point, but changing the base-point only changes the morphism by a constant translation. Thus we shall speak of any of the morphisms $u_{\Gamma'}$ determined by Γ and the choice of a base-point as an *Abel-Jacobi map* determined by Γ.

Suppose that $\Gamma_1, \Gamma_2 \in A_{n+1}(B \times X)$ are two $(n+1)$-cycles. Then $u_{\Gamma_1 + \Gamma_2}$ is the pointwise sum $u_{\Gamma_1} + u_{\Gamma_2}$. This trivial observation is frequently useful. Another useful observation is that any Abel-Jacobi morphism α_Γ contracts all rational curves on X, since an Abelian variety contains no rational curves.

3. Lines, Conics and Plane Cubics

We begin our analysis of the spaces $\mathcal{H}^{d,g}(X)$ by recalling known results about the Fano scheme of lines on X, $F := \mathcal{H}^{1,0}(X)$.

Two general lines $L_1, L_2 \subset \mathbb{P}^4$ determine a hyperplane by $\mathrm{span}(L_1, L_2)$. We generalize this as follows: Let $(F \times F - \Delta) \xrightarrow{\Phi} \mathbb{P}^{4\vee}$ denote the following set map:

$$\Phi([L_1, L_2]) = \begin{cases} [\mathrm{span}(L_1, L_2)] & \text{if } L_1 \cap L_2 = \emptyset, \\ [T_p X] & \text{if } p \in L_1 \cap L_2 \end{cases} \tag{17}$$

By [1, lemma 12.16], Φ is algebraic. Let $X^\vee \subset \mathbb{P}^{4\vee}$ denote the dual variety of X, i.e., the variety parametrizing tangent hyperplanes to X. Let $X^\vee_s \subset X^\vee$ denote the subvariety parametrizing hyperplanes H which are tangent to X and such that the singular locus of $H \cap X$ is not simply a single ordinary double point. Let $U_s \subset U \subset F \times F$ denote the open sets $\Phi^{-1}\left(\mathbb{P}^{4\vee} - X^\vee\right) \subset \Phi^{-1}\left(\mathbb{P}^{4\vee} - X^\vee_s\right)$. Finally, let $I \subset F \times F$ denote the divisor parametrizing incident lines, i.e., I is the closure of the set $\{([L_1], [L_2]) : L_1 \neq L_2, L_1 \cap L_2 \neq \emptyset\}$.

In [1], Clemens and Griffiths completely describe both the total Abel-Jacobi map $F \times F \xrightarrow{\psi} J(X)$ and the Abel-Jacobi map $F \xrightarrow{i} J(X)$. Here is a summary of their results

Theorem 3.1.

(1) *The Fano variety F is a smooth surface and the Abel-Jacobi map $F \xrightarrow{u} J(X)$ is a closed immersion* [1, theorem 7.8, theorem 12.37].

(2) *The induced map $Alb(F) = J^2(F) \rightarrow J(X)$ is an isomorphism of principally polarized Abelian varieties* [1, theorem 11.19].

(3) *The class of $u(F)$ in $J(X)$ is $\frac{[\Theta]^3}{3!}$* [1, proposition 13.1].

(4) *The difference of Abel-Jacobi maps*

$$\psi : F \times F \to J(X), \quad \psi([L],[L']) = u([L]) - u([L']) \tag{18}$$

maps $F \times F$ generically 6*-to-*1 *to the theta divisor $\Theta \subset J(X)$* [1, section 13].

(5) *Let $(\Theta - \{0\}) \xrightarrow{\mathcal{G}} \mathbb{P}(H^{1,2}(X)^\vee)$ denote the Gauss map. If we identify $\mathbb{P}(H^{1,2}(X))$ with $\mathbb{P}^4$ via the Griffiths residue calculus* [5]*, then the composite map*

$$(F \times_C F - \Delta) \xrightarrow{\psi} (\Theta - \{0\}) \xrightarrow{\mathcal{G}} \mathbb{P}^{4\vee} \tag{19}$$

is just the map Φ defined above [1, formula 13.6].

(6) *The fibers of the Abel-Jacobi map form a* Schläfli double-six*, i.e., the general fiber of $\psi : F \times F \rightarrow J$ is of the form $\{(E_1, G_1), \ldots, (E_6, G_6)\}$ where the lines E_i, G_j lie in a smooth hyperplane section of X, the E_i are pairwise skew, the G_j are pairwise skew, and E_i and G_j are skew iff $i = j$.*

There is a more precise result than above. Let

$$R' \subset (U \times_{\mathbb{P}^{4\vee}} U) \times F \times Grass(3,V) \times Grass(3,V) \tag{20}$$

be the closed subscheme parametrizing data $(([L_1],[L_2]), ([L_3],[L_4]), [l], [H_1], [H_3])$ such that for each $i = 1, \ldots, 4$, $l \cap L_i \neq \emptyset$ and such that $H_1 \cap X = l \cup L_1 \cup L_4$, $H_2 \cap X = l \cup L_2 \cup L_3$. Let $R \subset U \times_{\mathbb{P}^{4\vee}} U$ be the image of R' under the projection map. Let $\Delta \subset U \times U$ be the diagonal. Then the fiber product $U \times_\Theta U \subset U \times U$ is just the union $R \cup \Delta$ [1, p. 347-348]

(7) *The branch locus of $\Theta \xrightarrow{\mathcal{G}} \mathbb{P}^{4\vee}$ equals the branch locus of $F \times F \xrightarrow{\Phi} \mathbb{P}^{4\vee}$ equals the dual variety of X, i.e., the variety parametrizing the tangent hyperplanes to X. The ramification locus of $U \xrightarrow{\Phi} \mathbb{P}^{4\vee}$ equals the ramification locus of $U \xrightarrow{\psi} \Theta$ equals the divisor I. Each such pair is a simple ramification point of both ψ and Φ* [1, lemma 13,8].

3.1. **Conics.** Next we consider $\mathcal{H}^{2,0}(X)$ which parametrizes plane conics on X. We are mostly interested just in the irreducibility of the spaces $\mathcal{H}^{d,g}(X)$, but in this case we can give a complete description of $\mathcal{H}^{2,0}(X)$. We begin by proving that $\mathcal{H}^{2,0}(X)$ is smooth.

Lemma 3.2. *$\mathcal{H}^{2,0}(X)$ is smooth of dimension* 4.

Proof. Any plane conic C is a local complete intersection. So by lemma 2.2, it suffices to prove that $h^1(N_{C/X}) = 0$. In fact we will prove that for each smooth conic $C \subset X$, either $N_{C/X} \cong \mathcal{O}_C(1) \oplus \mathcal{O}_C(1)$ or else $N_{C/S} \cong \mathcal{O}_C \oplus \mathcal{O}_C(2)$.

We have the standard normal bundle sequence:

$$0 \longrightarrow N_{C/X} \longrightarrow N_{C/\mathbb{P}^4} \longrightarrow N_{X/\mathbb{P}^4}|_C \longrightarrow 0. \tag{21}$$

Of course $N_{X/\mathbb{P}^4} \cong \mathcal{O}_{\mathbb{P}^4}(3)|_C$ and it isn't hard to see that

$$N_{C/\mathbb{P}^4} \cong \mathcal{O}_{\mathbb{P}^4}(2)|_C \oplus \mathcal{O}_{\mathbb{P}^4}(1)^2|_C \cong \mathcal{O}_C(4) \oplus \mathcal{O}_C(2) \oplus \mathcal{O}_C(2). \tag{22}$$

By the Lefschetz hyperplane theorem we know that the 2-plane $P = \mathrm{span}(C)$ is not contained in X. Therefore the induced map $N_{C/P} \rightarrow N_{X/\mathbb{P}^4}$ is injective with length 2 cokernel. It follows then that $N_{C/X}$, considered as a subsheaf of $N_{C/\mathbb{P}^4}$ maps injectively to the quotient $N_{P/\mathbb{P}^4}|_C \cong \mathcal{O}_C(2)\oplus\mathcal{O}_C(2)$ and the cokernel is the length two cokernel above. So $N_{C/X}$ has degree 2 and no summand of $N_{C/X}$ can have degree higher than 2. So either $N_{C/X} \cong \mathcal{O}_C(1) \oplus \mathcal{O}_C(1)$ or else $N_{C/X} \cong \mathcal{O}_C \oplus \mathcal{O}_C(2)$.

Since $h^1(\mathcal{O}_C(d)) = 0$ for all $d > -2$, we conclude that $\mathcal{H}^{2,0}(X)$ is smooth. □

Every plane conic $C \subset \mathbb{P}^4$ is contained in a unique 2-plane $\text{span}(C) \subset \mathbb{P}^4$. Therefore over $\mathcal{H}^{2,0}(X)$ we have a flat family of 2-planes, $\Pi \subset \mathcal{H}^{2,0}(X) \times \mathbb{P}^4$ such that $\Pi_{[C]} = \text{span}(C)$. Of course the projection morphism $\Pi \to \mathcal{H}^{2,0}(X)$ is smooth. By lemma 3.2, it follows that Π is smooth. Now consider the intersection $D \subset \mathcal{H}^{2,0}(X)\times X$ of Π with $\mathcal{H}^{2,0}(X)\times X$ in $\mathcal{H}^{2,0}(X)\times\mathbb{P}^4$. First of all note that $D \to \mathcal{H}^{2,0}(X)$ has constant fiber dimension 1 over $\mathcal{H}^{2,0}(X)$, since by the Lefschetz hyperplane theorem [7, p. 156] X contains no 2-planes. Since $\mathcal{H}^{2,0}(X) \times X$ is a Cartier divisor in $\mathcal{H}^{2,0}(X) \times \mathbb{P}^4$, also $D \subset \Pi$ is a Cartier divisor. In particular D is a local complete intersection. Therefore $D \to \mathcal{H}^{2,0}(X)$ is flat.

Now let $\mathcal{C} \subset \mathcal{H}^{2,0}(X) \times X$ denote the universal smooth family of plane conics. Then $\mathcal{C}$ is smooth and $\mathcal{C} \subset D$ is a codimension zero closed subscheme. Let $D_2 \subset D$ be the residual to $\mathcal{C}$ in D. Then by corollary 2.7, we conclude that $D_2 \to \mathcal{H}^{2,0}(X)$ is flat and the fiber of D_2 over a closed point $[C] \in \mathcal{H}^{2,0}(X)$ is simply the residual of C in $\text{span}(C)\cap X$. But $\text{span}(C)\cap X$ is a plane cubic curve, so the fiber of D_2 is just a line. So we have an induced morphism $g : \mathcal{H}^{2,0}(X) \to F$ which associates to each $[C]$ the residual line in $\text{span}(C) \cap X$.

Define Q to be the rank 3 vector bundle on F which is the quotient of $\mathcal{O}_F^5$ by the universal sub-bundle. Let $\pi : \mathbb{P}(Q) \to F$ be the projective bundle associated to the rank 3 vector bundle. The points of $\mathbb{P}(Q)$ correspond to pairs $([L], [P])$ where $L \subset X$ is a line and $P \subset \mathbb{P}^4$ is a 2-plane such that $L \subset P$. Therefore over $\mathbb{P}(Q)$ we have a flat family of 2-planes $\Pi' \subset \mathbb{P}(Q) \times \mathbb{P}^4$. Let $D' \subset \Pi'$ denote the intersection of Π' with $\mathbb{P}(Q) \times X$ and let $\mathcal{C}' \subset \mathbb{P}(Q) \times X$ denote the pullback from F of the universal family of lines. Then, $\mathcal{C}' \subset D'$ and the residual D'_2 is a flat family of conics. Thus there is an induced morphism $h : \mathbb{P}(Q) \to \text{Hilb}_{2t+1}(X)$. It is easy to see that h is a bijection of closed points over the open subset $\mathcal{H}^{2,0}(X) \subset \text{Hilb}_{2t+1}(X)$. Since both $\mathbb{P}(Q)$ and $\mathcal{H}^{2,0}(X)$ are smooth, it follows by Zariski's main theorem [12, p. 288-289] that $\mathcal{H}^{2,0}(X)$ is isomorphic to an open subset of $\mathbb{P}(Q)$ and g corresponds to the projection morphism $\mathbb{P}Q \to F$.

But we can say more: since the Abel-Jacobi morphism $u : F \to J(X)$ is an embedding, F contains no rational curves. Thus, all the rational curves in $\mathbb{P}(Q)$ lie in fibers. Since h is finite over $\mathcal{H}^{2,0}(X)$, no fiber of $\mathbb{P}(Q) \to F$ is contracted by h, thus no rational curve in $\mathbb{P}(Q)$ is contracted by h (since all rational curves in $\mathbb{P}(Q)$ are numerically equivalent, if one is contracted they all are). But by [10, theorem VI.1.2], the exceptional locus of h is ruled. Thus we conclude that h is a finite morphism. It follows by Zariski's main theorem that $h : \mathbb{P}(Q) \to \text{Hilb}_{2t+1}(X)$ is the normalization of $\text{Hilb}_{2t+1}(X)$. We summarize the results as follows:

Proposition 3.3. *The morphism $\mathcal{H}^{2,0}(X) \to F$ is isomorphic to an open subset of a $\mathbb{P}^2$-bundle $\mathbb{P}(Q) \to F$. In particular, $\mathcal{H}^{2,0}(X)$ is smooth and connected of dimension 4. Moreover $\mathbb{P}(Q)$ is the normalization of $Hilb_{2t+1}(X)$.*

3.2. **Plane Cubics.** Every curve $C \subset \mathbb{P}^4$ with Hilbert polynomial $3t$ is a plane cubic, and the 2-plane $P = \text{span}(C)$ is unique; we have that $C = X \cap P$. Therefore the Hilbert scheme $\text{Hilb}_{3t}(X)$ is just the Grassmannian $\mathbb{G}(2, 4)$ of 2-planes in $\mathbb{P}^4$ and $\mathcal{H}^{3,1}(X)$ is just an open subset of $\mathbb{G}(2, 4)$.

4. Twisted Cubics

In this section we prove the irreducibility of $\mathcal{H}^{3,0}(X)$. But first we prove an enumerative result about the number of 2-secant lines to a curve $C \subset X$.

Given a smooth curve $C \subset X$ we want to consider the set of 2-secant lines to C which lie in X.

Definition 4.1. For a smooth curve $C \subset X$ we define $B_C \subset F$ to be the scheme parametrizing lines in X which intersect C in a scheme of degree 2 or more.

A dimension count leads one to expect that B_C is a 0-dimensional scheme. What is the degree of this scheme?

Lemma 4.2. *Suppose that $C \subset X$ is a smooth curve of genus g and degree d. Define $b(C) = \frac{5d(d-3)}{2} + 6 - 6g$. If B_C is not positive dimensional and if $b(C) \geq 0$, then the degree of B_C is $b(C)$.*

Proof. This is a standard Chern class argument. We work in the Chow ring of $C \times C$. Let $\omega \in A^*(C)$ denote the first Chern class of $\mathcal{O}_{\mathbb{P}^4}(1)|_C$ so that ω is algebraically equivalent to d times the class of a point. Let $\omega_1, \omega_2 \in A^*(C \times C)$ denote the pullbacks of ω by the two projection maps. Let $C \xrightarrow{\Delta} C \times C$ denote the diagonal morphism. Also let Δ, $\Delta_*\omega \in A^*(C \times C)$ denote the class of the image of Δ and the class of the pushforward by Δ of ω respectively.

Let V be the underlying vector space of $\mathbb{P}^4$, and $A_C \subset \mathrm{Grass}_{\mathbb{C}}(2, V)$ be the scheme parametrizing chords to C in $\mathbb{P}^4$. We adopt the following convention: for $p \in C$ we denote by $\mathrm{span}(p, p)$ the tangent line to C at p. Then we have a morphism $C \times C \xrightarrow{f} A_C$ by $(p, q) \mapsto [\mathrm{span}(p, q)]$. Let S be the universal rank two subbundle of $V \otimes \mathcal{O}_{C\times C}$ whose fibre over a point (p, q) corresponds to the line $\mathrm{span}(p, q)$. The inclusion $S \rightarrow V \otimes_{\mathbb{C}} \mathcal{O}_{C\times C}$ induces a morphism of schemes $P := \mathbb{P}(S) \rightarrow (C \times C) \times \mathbb{P}^4$. We have two sections of P determined by $(p, q) \mapsto p \in \mathrm{span}(p, q)$ and $(p, q) \mapsto q \in \mathrm{span}(p, q)$. Let $\mathcal{I}_1$ and $\mathcal{I}_2$ denote the ideal sheaves of these sections in P. Since both of these sections are divisors, the ideal sheaf of their scheme theoretic union is just $\mathcal{I}_1 \cdot \mathcal{I}_2 \cong \mathcal{I}_1 \otimes_{\mathcal{O}_P} \mathcal{I}_2$. Let $g : P \to \mathbb{P}^4$ be the inclusion of P into $(C \times C) \times \mathbb{P}^4$ followed by projection onto $\mathbb{P}^4$. Let D be the set of points of P which are sent into X under this map, with ideal sheaf $\mathcal{I}_D = g^*\mathcal{I}_X$. The two sections are two subvarieties of D, and therefore we have that $\mathcal{I}_D \hookrightarrow \mathcal{O}_P$ factors through the subsheaf $\mathcal{I}_1 \cdot \mathcal{I}_2 \hookrightarrow \mathcal{O}_P$, i.e., we have $\mathcal{I}_D \hookrightarrow \mathcal{I}_1 \cdot \mathcal{I}_2$. The ideal sheaf of the residual to these sections inside of D is just what we obtain when we twist this last map, namely $\mathcal{I}_D \otimes_{\mathcal{O}_P} (\mathcal{I}_1 \cdot \mathcal{I}_2)^\vee \hookrightarrow \mathcal{O}_P$. We wish to determine when this residual subscheme contains fibers of the projection map $P \xrightarrow{\pi} C \times C$. Let us assume that a general chord to C does not lie in X. Then $\mathcal{I}_D$ is isomorphic to the locally free sheaf $\mathcal{O}_S(-3)$. So we may twist our inclusion to get $\mathcal{O}_P \rightarrow (\mathcal{I}_D)^\vee \otimes_{\mathcal{O}_P} \mathcal{I}_1 \otimes_{\mathcal{O}_P} \mathcal{I}_2$. The pushforward of this map yields a map

$$\mathcal{O}_{C\times C} \xrightarrow{\phi} \pi_* \left((\mathcal{I}_D)^\vee \otimes_{\mathcal{O}_P} \mathcal{I}_1 \otimes_{\mathcal{O}_P} \mathcal{I}_2 \right). \tag{23}$$

It is clear that the fiber $\pi^{-1}(p, q)$ will be contained in D iff the image of the constant section 1 under this map vanishes at the stalk of (p, q). Therefore we conclude that the fiber product $(C \times C) \times_{A_C} B_C$ is precisely the zero scheme of ϕ. One sees that $(\mathcal{I}_D)^\vee \otimes_{\mathcal{O}_P} \mathcal{I}_1 \otimes_{\mathcal{O}_P} \mathcal{I}_2$ is a locally free sheaf of fiber degree 1, in particular it is relatively ample. Therefore the pushforward $E := \pi_*((\mathcal{I}_D)^\vee \otimes_{\mathcal{O}_P} \mathcal{I}_1 \otimes_{\mathcal{O}_P} \mathcal{I}_2)$ is a locally free sheaf of rank 2. So, if the zero locus of ϕ is zero dimensional, then we see that the class of this locus in $A^*(C \times C)$ is just $c_2(E)$. So we are reduced to a Chern class calculation.

What is the Chern class of S? The two sections in the last paragraph yield a map of locally free sheaves $pr_1^*(\mathcal{O}_{\mathbb{P}^4}(-1)|_C) \oplus pr_2^*(\mathcal{O}_{\mathbb{P}^4}(-1)|_C) \rightarrow S$. This is an injective map and the cokernel is supported on the diagonal. Using the fact that the cokernel of S in $V \otimes_{\mathbb{C}} \mathcal{O}_{C\times C}$ is locally free and a simple snake lemma argument, one deduces that the cokernel is isomorphic to the coherent sheaf $\mathcal{O}_{C\times C}(\Delta) \otimes_{\mathcal{O}_{C\times C}} \Delta_*(\mathcal{O}_{\mathbb{P}^4}(-1)|_C)$. So we deduce that the Chern class of S is $1 - \omega_1 - \omega_2 + \Delta + \omega_1 \cdot \omega_2 - \Delta_*\omega$. Let η denote the first Chern class of $\mathcal{O}_S(1)$. One has exact sequences

$$0 \longrightarrow \mathcal{O}_S(1) \otimes_{\mathcal{O}_P} (pr_i \circ \pi)^*(\mathcal{O}_{\mathbb{P}^4}(-1)|_C) \longrightarrow \mathcal{O}_S(1) \otimes_{\mathcal{O}_P} \pi^*(S) \longrightarrow \mathcal{I}_i^\vee \longrightarrow 0 \tag{24}$$

for $i = 1, 2$. Thus one deduces that the Chern classes of $\mathcal{I}_1$ and $\mathcal{I}_2$ are $1 - \eta + \omega_2 - \Delta$ and $1 - \eta + \omega_1 - \Delta$ respectively. Of course the Chern class of $\mathcal{I}_D$ is simply $1 - 3\eta$. Since $\mathcal{I}_D^\vee \otimes_{\mathcal{O}_P} \mathcal{I}_1 \otimes_{\mathcal{O}_P} \mathcal{I}_2$ is relatively ample, its higher direct images vanish. Thus we may calculate the second Chern class of E by a simple application of the Grothendieck-Riemann-Roch theorem [4]. It turns out to be $5\omega_1 \cdot \omega_2 - 15\Delta_*\omega + 6\Delta \cdot \Delta$. If we work modulo algebraic equivalence and omitting the phrase "class of a point", we have $\omega_1 \cdot \omega_2 = d^2$, $\Delta_*\omega = d$ and $\Delta \cdot \Delta = \chi(C) = 2 - 2g$. Using the fact that the map f is generically 2-to-1, we deduce that the degree of B_C is $\frac{5d(d-3)}{2} + 6 - 6g$. □

Lemma 4.3. *$\mathcal{H}^{3,0}(X)$ is smooth of dimension* 6.

Proof. By lemma 2.2 we need to prove that $h^1(N_{C/X}) = 0$ for all $[C] \in \mathcal{H}^{3,0}(X)$. Consider the normal bundle sequence

$$0 \longrightarrow N_{C/X} \longrightarrow N_{C/\mathbb{P}^4} \longrightarrow N_{X/\mathbb{P}^4}|_C \longrightarrow 0. \tag{25}$$

Of course for any twisted cubic C, we have that $H = \mathrm{span}(C)$ is a hyperplane, and $N_{C/H} \cong \mathcal{O}_C(5)^2$. Thus we conclude that $N_{C/\mathbb{P}^4} \cong \mathcal{O}_C(5)^2 \oplus \mathcal{O}_C(3)$, and $N_{X/\mathbb{P}^4}|_C \cong \mathcal{O}_C(9)$. So $N_{C/X}$ is a rank 2 vector bundle of degree 4. By Grothendieck's lemma about vector bundles on $\mathbb{P}^1$, we conclude $N_{C/X} \cong \mathcal{O}_C(a) \oplus \mathcal{O}_C(4 - a)$

for some $a \geq 2$. But since $N_{C/X}$ is a subbundle of $\mathcal{O}_C(5)^2 \oplus \mathcal{O}_C(3)$, we conclude that $a \leq 5$. In all four cases $a = 2, 3, 4$ and 5, we see that $4 - a > -2$ so that $h^1(N_{C/X}) = 0$. □

Define

$$I = I_{3,0} \subset \mathcal{H}^{3,0}(X) \times F \tag{26}$$

to be the closed subset parametrizing pairs $([C], [L])$ where L is a 2-secant line to C, and define

$$f = f_{3,0} : I \to \mathcal{H}^{3,0}(X) \tag{27}$$

to be the projection. By lemma 4.2, we know that $f_{3,0}$ is surjective. Notice also that none of the lines L is a 3-secant line, because any 3 points on a twisted cubic are linearly independent.

Now given $([C], [L]) \in I$, the reducible curve $C \cup L$ lies on a pencil of quadric surfaces in the 3-plane $P = \text{span}(C)$, and the general member of this pencil is smooth. Let $J \subset I \times \text{Hilb}_{t^2+2t+1}(\mathbb{P}^4)$ denote the locally closed subset parametrizing triples $([C], [L], [S])$ where S is a smooth quadric surface containing $C \cup L$. Then $J \to I$ is birational to a $\mathbb{P}^1$-bundle, in particular given an irreducible component J_i of J with image $I_i \subset I$, we have $\dim(I_i) = \dim(J_i) - 1$. By the Lefschetz hyperplane theorem, X does not contain the surface S, thus $S \cap X \subset S$ is a Cartier divisor of type $(3, 3)$ on S. The residual to $C \cup L \subset S \cap X$ is a divisor of type $(1, 1)$ on S, i.e., a conic $D \subset S$.

Theorem 4.4. *The space $\mathcal{H}^{3,0}(X)$ is a smooth, irreducible 6-dimensional variety.*

Proof. By lemma 4.3, every irreducible component of $\mathcal{H}^{3,0}(X)$ has dimension 6. We will prove that there is a unique irreducible component of I of dimension $d \geq 6$. Since $I \to \mathcal{H}^{3,0}(X)$ is surjective, this implies that $\mathcal{H}^{3,0}(X)$ is irreducible. In order to show this, we will prove that J has a unique irreducible component of dimension 7.

We stratify J into locally closed subsets J_1, J_2, according to the type of the residual curve D. If D is a smooth conic, we say that D is the first type. If D is a reducible conic, we say that D is the second type. Notice that D cannot be a *double line* because it is a divisor of type $(1, 1)$ on a smooth quadric surface.

Second Type: First consider J_2 parametrizing triples $([C], [L], [S])$ such that D is the second type. Let $H \subset F \times F \times F$ denote the locally closed subset parametrizing triples $([L], [D_1], [D_2])$ such that L and D_1 intersect transversally in one point, D_1 and D_2 intersect transversally in one point, and L is skew to D_2. There is a morphism $J_2 \to H$ defined by decomposing $D = D_1 \cup D_2$ so that $L \cap D_1$ is non-empty. Given a triple $([L], [D_1], [D_2])$, every quadric surface S containing $L \cup D_1 \cup D_2$ is contained in the 3-plane $\text{span}(L, D_1, D_2)$. Moreover, there is a 2-dimensional linear system of quadrics S containing $L \cup D_1 \cup D_2$. Thus the fiber dimension of $J_2 \to H$ is at most 2. We can also see that the dimension of H is 4: there is a 2-parameter family of choices for the line D_1, and given D_1 there is a 1-parameter family of lines intersecting D_1. Thus the dimension of H is $2 + 1 + 1 = 4$. So every irreducible component of J_2 has dimension at most 6, which is less than 7.

Next we consider J_1 parametrizing triples $([C], [L], [S])$ such that the residual curve D is a smooth conic. Let $K \subset F \times \mathcal{H}^{2,0}(X)$ denote the closed subset parametrizing pairs $([L], [D])$ such that L and D intersect transversally in one point p. There is a morphism $J_1 \to K$ by sending $([C], [L], [S])$ to $([L], [D])$ with D the residual curve. For a point $([L], [D]) \in K$ and a point $([C], [L], [S])$ in the fiber over $([L], [D])$, we have that S is contained in the 3-plane $\text{span}(L, D)$. There is a 2-parameter linear system of quadric surfaces $S \subset \text{span}(L, D)$ which contain $L \cup D$. The collection of quadric surfaces $S \subset \text{span}(L, D)$ containing $L \cup D$ and such that also the residual curve C of $L \cup D \subset S \cap X$ is a smooth twisted cubic forms an open subset of the collection of all quadric surfaces $S \subset \text{span}(L, D)$ containing $L \cup D$. So every (non-empty) fiber of $J_1 \to K$ is irreducible of dimension 2.

Since $J_1 \to K$ has irreducible fibers of dimension 2 (when they are non-empty), we see that for each irreducible component K_i of K, there is at most one irreducible component of J_1 which fibers over K_i with fiber dimension 2. So we are reduced to proving that K is irreducible of dimension 5. In order to specify a pair $([L], [D])$ intersecting at the point p, it is equivalent to specify L, a point $p \in L$, and the line N residual to D since then D is determined as the conic residual to $N \subset X \cap \text{span}(N, p)$. So K is isomorphic to an open subscheme of the product of the universal line over F (parametrizing pairs (L, p)) with another copy of F (parametrizing N), and this is an irreducible 5-fold. Thus there is at most one irreducible component of J_1

of dimension at least 7, and such an irreducible component is exactly 7 dimensional. All that remains is to show that at least one such component exists.

Since $\mathcal{H}^{3,0}(X)$ is nonempty, and $J \to \mathcal{H}^{3,0}(X)$ is surjective with fibre dimension one, we conclude that J_1 has such a component, and therefore that $\mathcal{H}^{3,0}(X)$ is an irreducible 6-dimensional variety. □

4.1. **The Abel-Jacobi map for $\mathcal{H}^{3,0}(X)$.** In order to analyze $\mathcal{H}^{4,0}(X)$ we will need to understand the Abel-Jacobi map $u : \mathcal{H}^{3,0}(X) \to J(X)$.

We have a morphism

$$\mathcal{H}^{3,0}(\mathbb{P}^4) \xrightarrow{\sigma^{3,0}} \mathbb{P}^{4\vee} \tag{28}$$

defined by sending $[C]$ to $\operatorname{span}(C)$. This morphism makes $\mathcal{H}^{3,0}(\mathbb{P}^4)$ into a locally trivial bundle over $\mathbb{P}^{4\vee}$ with fiber $\mathcal{H}^{3,0}(\mathbb{P}^3)$. Recall from section 3 that we defined $X^\vee \subset \mathbb{P}^{4\vee}$ to be the dual variety of X which parametrizes tangent hyperplanes to X and we defined U to be the complement of $X^\vee$ in $\mathbb{P}^{4\vee}$. Then we define $\mathcal{H}^{3,0}_U(X)$ to be the open subscheme of $\mathcal{H}^{3,0}(X)$ which parametrizes twisted cubics, C, in X such that $\sigma^{3,0}([C]) \in U$. By the graph construction we may consider $\mathcal{H}^{3,0}_U(X)$ as a locally closed subvariety of $U \times \mathrm{Hilb}_{3t+1}(X)$. Let $\overline{\mathcal{H}} \subset U \times \mathrm{Hilb}_{3t+1}(X)$ denote the closure of $\mathcal{H}^{3,0}_U(X)$ with the reduced induced scheme structure. Denote by $\overline{\mathcal{H}} \xrightarrow{f} U$ the projection map.

Theorem 4.5. *Let $\overline{\mathcal{H}} \xrightarrow{f''} U' \xrightarrow{f'} U$ be the Stein factorization of $\overline{\mathcal{H}} \xrightarrow{f} U$. Then $\overline{\mathcal{H}} \xrightarrow{f''} U'$ is isomorphic to a $\mathbb{P}^2$-bundle $\mathbb{P}_{U'}(E) \longrightarrow U'$ with E a locally free sheaf of rank 3, and $U' \xrightarrow{f'} U$ is an unramified finite morphism of degree 72. Moreover, the Abel-Jacobi map $\overline{\mathcal{H}} \xrightarrow{i} J(X)$ factors as $\overline{\mathcal{H}} \xrightarrow{f''} U' \xrightarrow{i'} J(X)$ where $U' \xrightarrow{i'} J(X)$ is a birational morphism of U' to a translate of Θ.*

Proof. We need to use the following lemma:

Lemma 4.6. *Let S be a smooth cubic surface in $\mathbb{P}^3$. Then there are exactly 72 line bundles L on S such that $L^2 = 1$, and $L.K_S = -3$ (where K_S is the canonical class). Furthermore, each of them satisfies $H^1(S, L) = H^2(S, L) = 0$, and the general member of $H^0(S, L)$ is a smooth curve.*

We will explicitly describe such bundles L below, and this lemma will be a straightforward consequence. Note that if $C \subset S$ is a curve with Hilbert polynomial $3t + 1$, then $C.K_S = -3$, since K_S is minus the hyperplane class, and since the curve has arithmetic genus zero, adjunction shows that $C^2 = 1$. This shows that all the curves in $\mathcal{H}^{3,0}_U(X)$ give line bundles L satisfying the conditions above. Conversely, given any effective divisor $C \in |L|$, with L a line bundle as above, we see that C has degree three, and arithmetic genus zero, and hence Hilbert polynomial $3t + 1$.

Now, let $\mathcal{X} \xrightarrow{\pi} U$ be the universal family of smooth hyperplane sections of X. For any $[H] \in U$, we use $S_H := H \cap X$ to denote the smooth cubic surface which is the fibre of π. Let $\mathrm{Pic}^{3,0}(\mathcal{X}/U)$ be the subscheme of the relative Picard scheme parameterizing line bundles L_H on the fibers S_H of π such that $L_H^2 = 1$ and $L_H \cdot K_{S_H} = -3$. For any such L_H, we have $\chi_{S_H}(L_H) = 1/2(L_H^2 - L \cdot K_{S_H}) + \chi(\mathcal{O}_{S_H}) = 2 + 1 = 3$. By the above lemma, the line bundle L_H has no higher cohomology on S_H, and so there is a rank three vector bundle E on $\mathrm{Pic}^{3,0}(\mathcal{X}/U)$ whose fibre at a point (H, L_H) of $\mathrm{Pic}^{3,0}(\mathcal{X}/U)$ consists of the global sections $H^0(S_H, L_H)$. Let $P = \mathbb{P}(E)$, be the projectivization of this bundle, with projection map $g : P \longrightarrow \mathrm{Pic}^{3,0}(\overline{X}/U)$. A point of this projectivization consists of the data (H, L_H, C_H) where $[H] \in U$, L_H is a line bundle on S_H satisfying the numerical conditions, and C_H is an effective divisor on S_H with $\mathcal{O}_{S_H}(C_H) = L_H$.

By the remarks after lemma 4.6, we see that C_H has Hilbert polynomial $3t + 1$, and so we have a natural map $P \longrightarrow U \times \mathrm{Hilb}_{3t+1}(X)$ sending $([H], L_H, C_H)$ to $([H], C_H)$. The map is clearly an injection, since we can recover the line bundle L_H from C_H.

The short exact sequence

$$0 \longrightarrow \mathcal{O}_{S_H} \longrightarrow \mathcal{O}_{S_H}(C_H) \longrightarrow \mathcal{O}_{S_H}(C_H)|_{C_H} \longrightarrow 0$$

gives the long exact sequence in cohomology

$$0 \longrightarrow H^0\left(S_H, \mathcal{O}_{S_H}\right) \xrightarrow{\cdot C_H} H^0\left(S_H, \mathcal{O}_{S_H}(C_H)\right) \longrightarrow H^0\left(C_H, N_{C_H/S_H}\right) \longrightarrow H^1\left(S_H, \mathcal{O}_{S_H}\right) = 0.$$

This sequence has the following interpretation. $H^0(S_H, \mathcal{O}_{S_H}(C_H))$ divided by the section C_H is the vertical tangent space (at C_H) for the map $P \longrightarrow \mathrm{Pic}^{3,0}(\mathcal{X}/U)$. $H^0(C_H, N_{C_H/S_H})$ is the tangent space (at C_H) of $\mathrm{Hilb}_{3t+1}(S_H)$. The sequence above shows that the map between these tangent spaces is an isomorphism, and hence that $P \longrightarrow U \times \mathrm{Hilb}_{3t+1}(X)$ is a closed embedding.

The subset $\mathcal{H}_U^{3,0}(X)$ of $U \times \mathrm{Hilb}_{3t+1}(X)$ is contained in the image of P, and, by lemma 4.6, $\mathcal{H}_U^{3,0}(X)$ is dense in each fibre of $P \longrightarrow U$, so we conclude that $P = \overline{\mathcal{H}}$. The map $\rho : \mathrm{Pic}^{3,0}(\mathcal{X}/U) \longrightarrow U$, since it is a finite type subscheme of the the relative Picard scheme, and by lemma 4.6 each fibre consists of 72 points, i.e., the map is finite. We also know that this is unramified since the Picard group of a cubic surface is reduced. Finally, the map $P \longrightarrow \mathrm{Pic}^{3,0}(\mathcal{X}/U)$ is a $\mathbb{P}^2$ bundle by construction.

Therefore, $P \xrightarrow{g} \mathrm{Pic}^{3,0}(\mathcal{X}/U) \xrightarrow{\rho} U$ is the Stein factorization $\overline{\mathcal{H}} \xrightarrow{f''} U' \xrightarrow{f'} U$ of $\overline{\mathcal{H}} \xrightarrow{f} U$, and this factorization has the properties claimed in the theorem.

It only remains to determine the Abel-Jacobi map $\overline{\mathcal{H}} \xrightarrow{i} J(X)$. Since $J(X)$ contains no rational curves, i is a constant map on each fiber of g. Since $\mathrm{Pic}^{3,0}(\mathcal{X}/U)$ is smooth, it follows that i factors through a morphism $i' : \mathrm{Pic}^{3,0}(\mathcal{X}/U) \rightarrow J(X)$. To determine i', we introduce the locus of "Z's of lines" i.e. the subscheme of $\overline{\mathcal{H}}$ parametrizing cubic curves whose irreducible components are lines. To be precise, let $\Sigma \subset \overline{\mathcal{H}} \times X$ be our flat family of cubic curves. We let $\Sigma^s \subset \Sigma$ denote the singular subscheme. We can form the flattening stratification for $\Sigma^s \rightarrow \overline{\mathcal{H}}$, and we define $Z \subset \overline{\mathcal{H}}$ to be the stratum corresponding to the constant Hilbert polynomial 2, i.e., the locus parametrizing curves with two nodes. What are the fibers $Z \cap g^{-1}(q)$ for $q \in \mathrm{Pic}^{3,0}(\mathcal{X}/U)$? Define $H = \rho(q)$. In the analysis below, we will see that we can find a set of 6 mutually skew lines in S_H such that $g^{-1}(q)$ corresponds to the complete linear series of lines in the blown-down $\mathbb{P}^2$. It is clear that a line ℓ in this linear series will correspond to a singular cubic curve iff ℓ intersects one of the 6 special points. Similarly, ℓ will correspond to a cubic curve whose singular locus has degree 2 iff ℓ is one of the 15 lines joining a pair of the 6 special points. Thus each fiber $Z \cap g^{-1}(q)$ consists of 15 points, and $\Sigma_Z^s \rightarrow Z$ is an unramified, finite morphism of degree 2. Thus $\Sigma_Z^s \rightarrow \mathrm{Pic}^{3,0}(\mathcal{X}/U)$ is an unramified, finite morphism of degree 30.

Denote by $i_1 : \Sigma_Z^s \rightarrow J(X)$ the composition of $\Sigma_X^s \rightarrow \mathrm{Pic}^{3,0}(\mathcal{X}/U)$ with the Abel-Jacobi map $\mathrm{Pic}^{3,0}(\mathcal{X}/U) \rightarrow J(X)$. Recall Σ_Z^s parametrizes pairs $([C],[x])$, where C is a completely reducible cubic and x is a node of C. We define a map $h : \Sigma^s \rightarrow F \times F$ as follows. The union of those components of C which intersect x is a completely reducible conic, C'. The residual to C' inside of C is a line ℓ_1. Now C' spans a $\mathbb{P}^2$ in $\mathbb{P}^4$ and the residual to C' in span $(C') \cap X$ is another line ℓ_2. We define h to be the map $([C],[x]) \mapsto ([\ell_1],[\ell_2])$. The point is, since C' and ℓ_2 are residual in a complete intersection which varies in a rational family, it follows by the residuation trick that i_1 is equal to $\psi \circ h$ (up to a fixed translation).

What are the fibers of h? Suppose we are given two skew lines ℓ_1 and ℓ_2 whose span intersects X in a smooth cubic surface, X'. How many reducible conics C' are there which are residual to ℓ_2 and which intersect ℓ_1? One of the lines in C', call it ℓ_3, intersects both ℓ_1 and ℓ_2. The other line of C' is uniquely determined by the condition that it be residual to $\ell_2 \cup \ell_3$ in the $\mathbb{P}^2$ they span. Thus the points in a fiber of h are enumerated by the lines ℓ_3 joining ℓ_1 and ℓ_2. There are 5 such lines. Therefore h is dominant and generically finite of degree 5. We know that ψ maps dominantly and generically finitely to Θ of degree 6, thus Σ_Z^s maps to Θ dominantly and generically finitely of degree $5 \times 6 = 30$. We have already seen that $\Sigma_Z^s \rightarrow \mathrm{Pic}^{3,0}(\mathcal{X}/U)$ is unramified of degree 30. Therefore $\mathrm{Pic}^{3,0}(\mathcal{X}/U) \rightarrow J(X)$ maps generically 1-to-1 and dominates a translate of Θ. □

Corollary 4.7. *The Abel-Jacobi map $i_{3,0} : \mathcal{H}^{3,0}(X) \rightarrow J(X)$ dominates a translate of Θ and is birational to a $\mathbb{P}^2$-bundle over its image.*

Proof. $\mathcal{H}_U^{3,0}(X) \subset \mathcal{H}^{3,0}(X)$ is dense in $\overline{\mathcal{H}}$. □

We now need to examine the line bundles L on a cubic surface S satisfying $L^2 = 1$ and $L.K_S = -3$. We need to establish the facts claimed in lemma 4.6, and also show that for any such L, we can always blow down six lines so that L is pullback of $\mathcal{O}_{\mathbb{P}^2}(1)$ from the resulting $\mathbb{P}^2$. We will follow [9], chapter V, notation 4.7.3 for our notation of the Neron-Severi group of S. Recall that $e_1, \ldots, e_6$ are the linear equivalence classes of 6 mutually skew lines on S, so that the contraction of $e_1, \ldots, e_6$ is a $\mathbb{P}^2$, and l is the linear equivalence class of the total transform of a line in $\mathbb{P}^2$. If we write $L = al - \sum b_i e_i$, then we have $3a - \sum b_i = 3$ and $a^2 - \sum b_i^2 = 1$. Since $(\sum b_i)^2 \leq 6 \sum b_i^2$, we deduce that $(3a-3)^2 \leq 6a^2 - 6$. This implies that either $a = 1, 2, 3, 4$, or 5. One quickly works out all the possibilities for $(a, b_1, \ldots, b_6)$. There is an obvious action of the group S_6 on the set of solutions via permuting $b_1, \ldots, b_6$. Representatives of the orbits of the set of solutions are as follows:

$$(1,0,0,0,0,0,0), \ (2,1,1,1,0,0,0), \ (3,2,1,1,1,1,0), \ (4,2,2,2,1,1,1), \ (5,2,2,2,2,2,2). \tag{29}$$

Counting the size of each orbit shows that there are a total of 72 distinct solutions. With a slight amount of work, one shows that the separate orbits all lie in the same orbit under the action of the full Weyl group of E_6. Thus, for some choice of 6 mutually skew six lines, we have that L is just l. The general member of this linear series is obviously smooth, and the long exact sequence in cohomology coming from

$$0 \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_S(l) \longrightarrow \mathcal{O}_{\mathbb{P}^1}(1) \longrightarrow 0$$

shows that $H^1(S, L) = H^2(S, L) = 0$, which was the last thing to be checked.

5. Quartic Elliptic Curves

Recall that the normalization of $\mathrm{Hilb}_{2t+1}(X)$ is isomorphic to the $\mathbb{P}^2$-bundle $\mathbb{P}(Q) \rightarrow F$ which parametrizes pairs (L, P) which $L \subset X$ a line and $P \subset \mathbb{P}^4$ a 2-plane containing L. Let $A \xrightarrow{g} \mathbb{P}(Q)$ denote the $\mathbb{P}^1$-bundle which parametrizes triples (L, P, H) with H a hyperplane containing P. Let $I_{4,1} \xrightarrow{h} A$ denote the $\mathbb{P}^4$-bundle parametrizing 4-tuples (L, P, H, Q) where $Q \subset H$ is a quadric surface containing the conic $C \subset X \cap P$. Notice that $I_{4,1}$ is smooth and connected of dimension $4 + 1 + 4 = 9$.

Let $D \subset I_{4,1} \times X$ denote the intersection of the universal quadric surface over $I_{4,1}$ with $I_{4,1} \times X \subset I_{4,1} \times \mathbb{P}^4$. Then D is a local complete intersection scheme. By the Lefschetz hyperplane theorem, X contains no quadric surfaces; therefore $D \rightarrow I_{4,1}$ has constant fiber dimension 1 and so is flat. Let $D_1 \subset I_{4,1} \times X$ denote the pullback from $\mathbb{P}(Q) \times X = \mathrm{Hilb}_{2t+1}(X) \times X$ of the universal family of conics. Since $I_{4,1} \times X \rightarrow \mathbb{P}(Q) \times X$ is smooth and the universal family of conics is a local complete intersection which is flat over $\mathbb{P}(Q)$, we conclude that also D_1 is a local complete intersection which is flat over $I_{4,1}$. Clearly $D_1 \subset D$. Thus by corollary 2.7, we see that the residual D_2 of $D_1 \subset D$ is Cohen-Macaulay and flat over $I_{4,1}$.

By the base-change property in corollary 2.7, we see that the fiber of $D_2 \rightarrow I_{4,1}$ over a point (L, P, H, Q) is simply the residual of $C \subset Q \cap X$. If we choose Q to be a smooth quadric, i.e., $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$, then $C \subset Q$ is a divisor of type $(1,1)$ and $X \cap Q \subset Q$ is a divisor of type $(3,3)$. Thus the residual curve E is a divisor of type $(2,2)$, i.e., a quartic curve of arithmetic genus 1. Thus $D_2 \subset I_{4,1} \times X$ is a family of connected, closed subschemes of X with Hilbert polynomial $4t$. So we have an induced map $f : I_{4,1} \rightarrow \mathrm{Hilb}_{4t}(X)$.

Proposition 5.1. *The image of the morphism above $f : I_{4,1} \rightarrow Hilb_{4t}(X)$ is the closure $\overline{\mathcal{H}^{4,1}}(X)$ of $\mathcal{H}^{4,1}(X)$. Moreover the open set $f^{-1}\mathcal{H}^{4,1}(X) \subset I_{4,1}$ is a $\mathbb{P}^1$-bundle over $\mathcal{H}^{4,1}(X)$. Thus $\mathcal{H}^{4,1}(X)$ is smooth and connected of dimension 8.*

Proof. If $E \subset X$ is a smooth, connected curve with Hilbert polynomial $4t$, then E is a quartic elliptic curve in some hyperplane H. Any such curve lies on a pencil of quadric surfaces Q, and the residual of $E \subset Q \cap X$ is a conic. Thus we see that $f(I_{4,1})$ contains the open subscheme $\mathcal{H}^{4,1}(X) \subset \mathrm{Hilb}_{4t}(X)$. Since $f(I_{4,1})$ is closed and irreducible, we conclude that $f(I_{4,1}) = \overline{\mathcal{H}^{4,1}}(X)$. Since the fibre of f over any smooth elliptic quintic E is determined by the $\mathbb{P}^1$ of quadrics Q in $H = \mathrm{span}(E)$, we see that $f^{-1}(\mathcal{H}^{4,1}(X)) \subset I_{4,1}$ is an open subset which is a $\mathbb{P}^1$-bundle over $\mathcal{H}^{4,1}$. In particular, since $I_{4,1}$ is smooth and connected, we conclude that $\mathcal{H}^{4,1}$ is also smooth and connected of dimension 8. $\square$

The surface F of lines contains no rational curves, so in the $\mathbb{P}^1$ fibre of f over $[E] \in \mathcal{H}^{4,1}$, the line L must be constant. Since the hyperplane H is also determined by $[E]$, we have a well-defined morphism $m : \mathcal{H}^{4,1}(X) \rightarrow \mathbb{P}(Q^\vee)$, where $\mathbb{P}(Q^\vee)$ is the $\mathbb{P}^2$-bundle over F parametrizing pairs $([L], [H])$, $L \subset H$. For a general H, the intersection $Y = H \cap X$ is a smooth cubic surface, and the fiber $m^{-1}([L], [H])$ is an open subset of the complete linear series $|\mathcal{O}_Y L + h|$, where h is the hyperplane class on Y. Thus $m : \mathcal{H}^{4,1}(X) \rightarrow \mathbb{P}(Q^\vee)$

is a morphism of smooth connected varieties which is birational to a $\mathbb{P}^4$-bundle. Composing m with the projection $\mathbb{P}(Q^\vee)$ yields a morphism $n : \mathcal{H}^{4,1}(X) \to F$ which is birational to a $\mathbb{P}^4$-bundle over a $\mathbb{P}^2$-bundle.

Corollary 5.2. *The morphism $n : \mathcal{H}^{4,1}(X) \to F$ from above is birational over F to a $\mathbb{P}^4$-bundle over a $\mathbb{P}^2$-bundle over F.*

6. Cubic Scrolls and Applications

6.1. Preliminaries on cubic scrolls.

In the next few sections we will use residuation in a cubic scroll. We start by collecting some basic facts about these surfaces.

There are several equivalent descriptions of cubic scrolls.

(1) A cubic scroll $\Sigma \subset \mathbb{P}^4$ is a connected, smooth surface with Hilbert polynomial $P(t) = \frac{3}{2}t^2 + \frac{5}{2}t + 1$.

(2) A cubic scroll $\Sigma \subset \mathbb{P}^4$ is the determinantal variety defined by the 2×2 minors of a matrix of linear forms:
$$\begin{bmatrix} L_1 & L_2 & L_3 \\ M_1 & M_2 & M_3 \end{bmatrix} \tag{30}$$
such that for each row or column, the linear forms in that row or column are linearly independent

(3) A cubic scroll $\Sigma \subset \mathbb{P}^4$ is the *join* of an isomorphism $\phi : L \to C$. Here $L \subset \mathbb{P}^4$ is a line and $C \subset \mathbb{P}^4$ a conic such that $L \cap \text{span}(C) = \emptyset$. The join of ϕ is defined as the union over all $p \in L$ of the line $\text{span}(p, \phi(p))$.

(4) A cubic scroll $\Sigma \subset \mathbb{P}^4$ is the image of a morphism $f : \mathbb{P}(E) \to \mathbb{P}^4$ where $E = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$ on $\mathbb{P}^1$, the morphism $f : \mathbb{P}(E) \to \mathbb{P}^4$ is such that $f^*\mathcal{O}_{\mathbb{P}^4}(1) = \mathcal{O}_E(1)$, and the pullback map $H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(1)) \to H^0(\mathbb{P}(E), \mathcal{O}_E(1))$ is an isomorphism.

(5) A cubic scroll $\Sigma \subset \mathbb{P}^4$ is as a minimal variety, i.e., $\Sigma \subset \mathbb{P}^4$ is any smooth connected surface with $\text{span}(\Sigma) = \mathbb{P}^4$ which has the minimal possible degree for such a surface, namely $\deg(\Sigma) = 3$.

(6) A cubic scroll $\Sigma \subset \mathbb{P}^4$ is a smooth surface residual to a 2-plane Π in the base locus of a pencil of quadric hypersurfaces which contain Π.

From the fourth description $\Sigma = \mathbb{P}(E)$ we see that $\text{Pic}(\Sigma) = \text{Pic}(\mathbb{P}(E)) \cong \mathbb{Z}^2$. Let $\pi : \mathbb{P}(E) \to \mathbb{P}^1$ denote the projection morphism and let $\sigma : \mathbb{P}^1 \to \mathbb{P}(E)$ denote the unique section whose image $D = \sigma(\mathbb{P}^1)$ has self-intersection $D.D = -1$. Then $f(D)$ is a line on Σ called the *directrix*. For each $t \in \mathbb{P}^1$, $f(\pi^{-1}(t))$ is a line called a *line of ruling* of Σ. Denote by F the divisor class of any $\pi^{-1}(t)$. Then $\text{Pic}(\Sigma) = \mathbb{Z}\{D, F\}$ and the intersection pairing on Σ is determined by $D.D = -1, D.F = 1, F.F = 0$. The hyperplane class is $H = D + 2F$ and the canonical class is $K = -2D - 3F$.

Using the fourth description of a cubic scroll, we see that any two cubic scrolls differ only by the choice of the isomorphism $H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(1)) \to H^0(\mathbb{P}(E), \mathcal{O}_E(1))$. Therefore any two cubic scrolls are conjugate under the action of PGL(5). So the open set $U \subset \text{Hilb}_{P(t)}(\mathbb{P}^4)$ parametrizing cubic scrolls is a homogeneous space for PGL(5), in particular it is smooth and connected.

One possible specialization of a cubic scroll is a reducible surface $\Sigma = \Sigma_1 \cup \Sigma_2$ where Σ_1 is a 2-plane, Σ_2 is a smooth quadric surface, and $\Sigma_1 \cap \Sigma_2$ is a line L. Let $T \subset \text{Hilb}_{P(t)}(\mathbb{P}^4)$ denote the locus parametrizing surfaces Σ of this form.

Lemma 6.1. *The Hilbert scheme $Hilb_{P(t)}(\mathbb{P}^4)$ is smooth along T.*

Proof. Let $[\Sigma]$ be a point of T. Since Σ is a local complete intersection, the normal sheaf $N_{\Sigma/\mathbb{P}^4}$ is locally free. The Zariski tangent space of $\text{Hilb}_{P(t)}(\mathbb{P}^4)$ at $[\Sigma]$ is identified with $H^0(\Sigma, N_{\Sigma/\mathbb{P}^4})$. By [10, theorem I.2.15.2], we see that every irreducible component of $\text{Hilb}_{P(t)}(\mathbb{P}^4)$ through $[\Sigma]$ has dimension at least
$$\dim H^0(\Sigma, N_{\Sigma/\mathbb{P}^4}) - \dim H^1(\Sigma, N_{\Sigma/\mathbb{P}^4}). \tag{31}$$
So once we show that $H^1(\Sigma, N_{\Sigma/\mathbb{P}^4}) = 0$, it will follow that $\text{Hilb}_{P(t)}(\mathbb{P}^4)$ is smooth at $[\Sigma]$.

In order to analyze the normal bundle $N_{\Sigma/\mathbb{P}^4}$ we recall the following result: Suppose that X is a smooth ambient variety and suppose that $Y \subset X$ is a simple normal crossings variety with no triple points. Let

$Y_i \subset Y$ be an irreducible component and let $Z_1, \dots, Z_r$ be the connected components of $\mathrm{Sing}(Y) \cap Y_i$. For each $i = 1, \dots, r$, there is an étale cover $f : W \to V$ of a Zariski neighborhood of $Z_i \subset Y$ such that the preimage $Z = f^{-1}(Z_i)$ of Z_i is connected and such that W is reducible along Z. Denote the two branches of W along Z by W' and W''. Then the line bundle $N_{Z/W'} \otimes N_{Z/W''}$ descends to a line bundle N_i on Z_i. We have a short exact sequence of coherent sheaves:

$$0 \longrightarrow N_{Y_i/X} \longrightarrow N_{Y/X}|_{Y_i} \longrightarrow \oplus_{i\,=\,1}^r N_i \longrightarrow 0. \tag{32}$$

In our particular case, we have the two exact sequences:

$$\begin{array}{l} 0 \longrightarrow N_{\Sigma_1/\mathbb{P}^4} \longrightarrow N_{\Sigma/\mathbb{P}^4}|_{\Sigma_1} \longrightarrow N_{L/\Sigma_1} \otimes N_{L/\Sigma_2} \longrightarrow 0 \\ 0 \longrightarrow N_{\Sigma_2/\mathbb{P}^4} \longrightarrow N_{\Sigma/\mathbb{P}^4}|_{\Sigma_2} \longrightarrow N_{L/\Sigma_1} \otimes N_{L/\Sigma_2} \longrightarrow 0. \end{array} \tag{33}$$

If we identify Σ_1 with $\mathbb{P}^2$, then we have $N_{\Sigma_1/\mathbb{P}^4} \cong \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}_2}(1)$. If we identify Σ_1 with $\mathbb{P}^1 \times \mathbb{P}^1$, then we have $N_{\Sigma_2/\mathbb{P}^4} = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1) \oplus \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(2,2)$. Identifying L with $\mathbb{P}^1$, we have $N_{L/\Sigma_1} \cong \mathcal{O}_{\mathbb{P}^1}(1)$ and $N_{L/\Sigma_2} \cong \mathcal{O}_{\mathbb{P}^1}$. We are more interested in $N_{\Sigma/\mathbb{P}^4}|_{\Sigma_2}(-L)$ than we are in $N_{\Sigma/\mathbb{P}^4}|_{\Sigma_2}$. To relate the two we use the identification $\mathcal{O}_{\Sigma_2}(-L) \cong \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,0)$. With all of these identifications, we get two exact sequences:

$$\begin{array}{l} 0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1) \longrightarrow N_{\Sigma/\mathbb{P}^4}|_{\Sigma_1} \longrightarrow \mathcal{O}_{\mathbb{P}^1}(1) \longrightarrow 0 \\ 0 \longrightarrow \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(0,1) \oplus \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,2) \longrightarrow N_{\Sigma/\mathbb{P}^4}|_{\Sigma_2}(-L) \longrightarrow \mathcal{O}_{\mathbb{P}^1}(1) \longrightarrow 0. \end{array} \tag{34}$$

Applying the long exact sequence in cohomology to these two short exact sequences, we conclude the vanishing result

$$H^1(\Sigma_1, N_{\Sigma/\mathbb{P}^4}|_{\Sigma_1}) = H^2(\Sigma_1, N_{\Sigma/\mathbb{P}^4}|_{\Sigma_1}) = \tag{35}$$

$$H^1(\Sigma_2, N_{\Sigma/\mathbb{P}^4}|_{\Sigma_2}(-L)) = H^2(\Sigma_2, N_{\Sigma/\mathbb{P}^4}|_{\Sigma_2}(-L)) = 0. \tag{36}$$

We also have a short exact sequence:

$$0 \longrightarrow N_{\Sigma/\mathbb{P}^4}|_{\Sigma_2}(-L) \longrightarrow N_{\Sigma/\mathbb{P}^4} \longrightarrow N_{\Sigma/\mathbb{P}^4}|_{\Sigma_1} \longrightarrow 0. \tag{37}$$

Applying the long exact sequence in cohomology to this short exact sequence and combining with the vanishing result of the last paragraph, we conclude that $H^1(\Sigma, N_{\Sigma/\mathbb{P}^4}) = H^2(\Sigma, N_{\Sigma/\mathbb{P}^4}) = 0$. Therefore $\mathrm{Hilb}_{P(t)}(\mathbb{P}^4)$ is smooth along T. □

Lemma 6.2. *The union $V = T \cup U \subset Hilb_{P(t)}(\mathbb{P}^4)$ is a smooth, connected open subset.*

Proof. Given $[\Sigma] \in T$, we will show that every deformation of Σ can be realized as a subvariety of a rank 4 quadric hypersurface $Q \subset \mathbb{P}^4$. Then we will examine the deformations of Σ as a subvariety of Q to prove the lemma.

Let $I \subset \mathrm{Hilb}_{P(t)}(\mathbb{P}^4) \times \mathbb{P}^{14}$ denote the closed subscheme parametrizing pairs (Σ, Q) where $Q \subset \mathbb{P}^4$ is a quadric hypersurface and $\Sigma \subset Q$. The fiber of the projection $I \to \mathrm{Hilb}_{P(t)}(\mathbb{P}^4)$ over a point $[\Sigma]$ is the projective space corresponding to $H^0(\mathbb{P}^4, \mathcal{I}_\Sigma(2))$, where $\mathcal{I}_\Sigma$ is the ideal sheaf of $\Sigma \subset \mathbb{P}^4$.

Let $\widetilde{\Sigma} \subset \mathrm{Hilb}_{P(t)}(\mathbb{P}^4) \times \mathbb{P}^4$ denote the universal closed subscheme, and let $\mathcal{I}$ denote the ideal sheaf of this closed subscheme. Consider the coherent sheaf

$$\mathcal{F} = \mathrm{pr}_{1*}(\mathcal{I} \otimes \mathrm{pr}_2^* \mathcal{O}_{\mathbb{P}^4}(2)). \tag{38}$$

For each $[\Sigma] \in \mathrm{Hilb}_{P(t)}(\mathbb{P}^4)$ there is an evaluation map $\mathcal{F}|_{[\Sigma]} \to H^0(\mathbb{P}^4, \mathcal{I}_\Sigma(2))$. We will show that $H^{i>0}(\mathbb{P}^4, \mathcal{I}_\Sigma(2)) = 0$. Then it follows by cohomology and base change [9, theorem III.12.11] that $\mathcal{F}$ is locally free in a neighborhood of T and that all the evaluation maps are isomorphisms in a neighborhood of T. Thus in a neighborhood of T, $I \to \mathrm{Hilb}_{P(t)}(\mathbb{P}^4)$ is just the projective bundle associated to $\mathcal{F}$.

To show that $H^{i>0}(\mathbb{P}^4, \mathcal{I}_\Sigma(2)) = 0$, we will use the short exact sequence of coherent sheaves:

$$0 \longrightarrow \mathcal{I}_\Sigma(2) \longrightarrow \mathcal{O}_{\mathbb{P}^4}(2) \longrightarrow \mathcal{O}_\Sigma(2) \longrightarrow 0. \tag{39}$$

Applying the long exact sequence in cohomology to this short exact sequence, we see that we need to prove two things:

(1) $H^{i>0}(\Sigma, \mathcal{O}_\Sigma(2)) = 0$
(2) $H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2)) \to H^0(\Sigma, \mathcal{O}_\Sigma(2))$ is surjective.

To prove (1) and (2) we will use the short exact sequence:

$$0 \longrightarrow \mathcal{O}_{\Sigma_1}(2)(-L) \longrightarrow \mathcal{O}_{\Sigma}(2) \longrightarrow \mathcal{O}_{\Sigma_2}(2) \longrightarrow 0. \tag{40}$$

Of course $\mathcal{O}_{\Sigma_1}(2)(-L) \cong \mathcal{O}_{\mathbb{P}^2}(1)$ and $\mathcal{O}_{\Sigma_2}(2) \cong \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(2,2)$. So applying the long exact sequence in cohomology, we conclude that

$$H^1(\Sigma, \mathcal{O}_{\Sigma}(2)) = H^2(\Sigma, \mathcal{O}_{\Sigma}(2)) = 0 \tag{41}$$

i.e., we have established (1).

To see that (2) is true, observe first that the composite map

$$H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2)) \longrightarrow H^0(\Sigma, \mathcal{O}_{\Sigma}(2)) \longrightarrow H^0(\Sigma_2, \mathcal{O}_{\Sigma_2}(2)) \tag{42}$$

is surjective, i.e., the linear system $|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(2,2)|$ on a smooth quadric surface is just the restriction of the linear system $|\mathcal{O}_{\mathbb{P}^3}(2)|$. The kernel of the composite map is the vector space of quadratic polynomials which vanish identically on $\text{span}(\Sigma_2)$. If F is a linear polynomial defining $\text{span}(\Sigma_2)$, this subspace is just the image of the multiplication map

$$H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(1)) \xrightarrow{*F} H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2)). \tag{43}$$

The restriction to $H^0(\Sigma_1, \mathcal{O}_{\Sigma_1}(2)(-L)) = H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$ is the restriction $H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(1)) \to H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$, which is clearly surjective. Thus we have established (2).

We conclude that near T, $\mathcal{F}$ is locally free and $I \to \text{Hilb}_{P(t)}(\mathbb{P}^4)$ is just the projective bundle associated to $\mathcal{F}$. If we let V' be the open subset of $\text{Hilb}_{P(t)}(\mathbb{P}^4)$ where $H^{i>0}(\mathbb{P}^4, \mathcal{I}_\Sigma(2)) = 0$, then we know that V is contained in V', and that over V' the map $I \to \text{Hilb}_{P(t)}(\mathbb{P}^4)$ is smooth (and hence flat). This means that if we have any open subset O of I over V', its image in V', and hence in $\text{Hilb}_{P(t)}(\mathbb{P}^4)$, will be open.

Notice that by the Lefschetz hyperplane theorem, there is no pair $(\Sigma, Q) \in I$ such that $Q \subset \mathbb{P}^4$ is a rank 5 quadric (i.e., a smooth quadric). Denote by $W \subset I$ the open subscheme parametrizing pairs (Σ, Q) such that Q is a rank 4 quadric. Denote by $W_T \subset W$ the locally closed subset such that $\Sigma \in T$ and the singular point of Q is a smooth point of Σ_1.

Claim 6.3. *The map $W_T \to T$ is surjective.*

If $[\Sigma]$ is any point in T, and p any point of Σ_1 not on Σ_2, then letting Q be the cone over Σ_2 with vertex p provides a point of W_T over $[\Sigma]$, which establishes the claim. Now define $O \subset W$ to be the open subset parametrizing pairs (Σ, Q) such that the singular point of Q is a smooth point of Σ.

Claim 6.4. *O is an irreducible open neighborhood of $W_T \subset W$ whose points (Σ, Q) are exactly the points of W_T and the pairs with $[\Sigma] \in U$.*

As part of the proof of the claim, we will see that there *are* points of O with $[\Sigma] \in U$. Since U is a homogeneous space for PGL(5), and since PGL(5) acts on the rank four quadrics Q as well, this means that the image of O is exactly $V = T \cup U$. This will show that V is open in $\text{Hilb}_{P(t)}(\mathbb{P}^4)$, since O is open in I over V', and also that V is irreducible, since O is. Finally, we know that V is a smooth subset of $\text{Hilb}_{P(t)}(\mathbb{P}^4)$ since U is smooth, and the Hilbert scheme is smooth along T (by lemma 6.1). Thus, the only step left in proving lemma 6.2 is to establish claim 6.4 above.

So we are reduced to studying the open neighborhood O. If we let $\tilde{Q} \to Q$ denote the blow-up of Q at p, and if we let $\widetilde{\Sigma} \subset \tilde{Q}$ denote the proper transform of Σ, then this open subset is also the parameter space for pairs $(\widetilde{\Sigma}, \tilde{Q})$.

We will describe the threefold $\tilde{Q}$. Projection from p defines a morphism $\tilde{Q} \to \mathbb{P}^3$ whose image is a smooth quadric surface $S \subset \mathbb{P}^3$. Identifying S with $\mathbb{P}^1 \times \mathbb{P}^1$, the projection $\pi : \tilde{Q} \to S$ is simply the $\mathbb{P}^1$-bundle associated to the vector bundle $G = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1)$. The exceptional divisor E of $f : \tilde{Q} \to Q$ is a section of π. Identifying E with $\mathbb{P}^1 \times \mathbb{P}^1$, the normal bundle of E in $\tilde{Q}$ is identified with $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,-1)$. Let F_1, F_2 denote the divisor classes of $\pi^*\text{pr}_1^*\mathcal{O}_{\mathbb{P}^1}(1), \pi^*\text{pr}_2^*\mathcal{O}_{\mathbb{P}^1}(1)$. Then $\text{Pic}(\tilde{Q}) = \mathbb{Z}\{E, F_1, F_2\}$, and for any $(\Sigma, Q) \in O$, the proper transform $\widetilde{\Sigma}$ is a Cartier divisor with divisor class $E + 2F_1 + F_2$ (up to permuting F_1 and F_2).

Notice that since $p \in \Sigma$ is a smooth point, the intersection $\widetilde{\Sigma} \cap E$ is a (-1)-curve in $\widetilde{\Sigma}$ along which $\widetilde{\Sigma}$ is smooth. Conversely, suppose that $\Gamma \in |E + 2F_1 + F_2|$ is a surface such that $\Gamma \cap E$ is a curve along which Γ

is smooth (actually Γ is automatically smooth along $\Gamma \cap E$ if $\Gamma \cap E$ is a curve, but we won't need this fact). We will show that either Γ is smooth or else Γ is reducible, $\Gamma = \Gamma_1 \cup \Gamma_2$ where Γ_1 is a smooth, connected divisor in the class of F_1, Γ_2 is a smooth section of π in the class of $E + F_1 + F_2$, and $\Gamma_1 \cap \Gamma_2$ is transverse and maps to a line in Q. Then it follows that $f(\Gamma)$ is either a cubic scroll or else the union $f(\Gamma_1) \cup f(\Gamma_2)$ of a 2-plane and a smooth quadric surface along a line, and $p \in f(\Gamma_1)$ is a smooth point.

If Γ is smooth, it is clear that $f(\Gamma)$ is a cubic scroll (it is a smooth connected surface with Hilbert polynomial $P(t)$). Therefore suppose that Γ is singular at some point q. We know that $q \notin E$.

Suppose we pick a line L in the quadric surface S, in the ruling corresponding to F_1. If we restrict the $\mathbb{P}^1$ bundle $\tilde{Q}$ over S to L, the resulting surface is a Hirzebruch surface $\mathbb{F}_1$ over L. A divisor in the class F_2 on $\tilde{Q}$ restricts to the class of a fibre F on $\mathbb{F}_1$, the exceptional divisor E restricts to the unique (-1)-curve D, and a divisor in the class F_1 restricts to the trivial class on $\mathbb{F}_1$.

Now let L_q be the particular line of ruling on S containing $\pi(q)$, and $\mathbb{F}_1$ the surface over L_q. If Γ doesn't contain this $\mathbb{F}_1$, then the intersection $\Gamma \cap \mathbb{F}_1$ is a curve on $\mathbb{F}_1$ in the class $|D + F|$, with a singular point at q, which is not on D. This is a contradiction since the only singularities in the linear system $|D + F|$ occur along D. (In the model of $\mathbb{F}_1$ as the blowup of $\mathbb{P}^2$ at a point, this linear series is the pullback of the lines.)

Therefore the existence of a singular point $q \in \Gamma$ means that Γ is reducible, and can be written $\Gamma_1 \cup \Gamma_2$, with Γ_1 in the class F_1, and Γ_2 in the class $E + F_1 + F_2$. Now since $N_{E/\tilde{Q}} \cong \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1, -1)$, we see that $E \cap \Gamma_2$ is in the linear series $|\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}|$, which means that Γ_2 and E are disjoint (we know that Γ_2 doesn't contain E as a component since Γ intersects E in a curve). This shows both that the point p lies on $f(\Gamma_1)$, and that if Γ_2 were to have a singular point q', this point would not lie on E.

If Γ_2 were to have any singular points, then the same argument as above would show that Γ_2 would be reducible, with one piece in the class of F_1 and one piece in the class of $E + F_2$. However, every element of $|E + F_2|$ contains E as a component, which is again a contradiction. We conclude that Γ_2 is smooth.

The above arguments show that $f(\Gamma)$ is the union of a 2-plane $f(\Gamma_1)$ and a smooth quadric surface $f(\Gamma_2)$ meeting along a line, and that p lies on the 2-plane.

We now know that every point in O is either in W_T or of the form (Σ, Q) with Σ a cubic scroll. Notice also that O fibers over the homogeneous space of rank 4 scrolls Q and the fiber over a point $[Q]$ is an open subset of the linear system $|E + 2F_1 + F_2|$ on $\tilde{Q}$. In particular the fibers are irreducible, so O is irreducible. This finishes the proof of claim 6.4, and hence of lemma 6.2.

□

6.2. **Additional constructions.**

We prove several additional constructions of cubic scrolls which will be needed.

Recall that our fourth description of a cubic scroll was an embedding $f : \Sigma \to \mathbb{P}^4$ where Σ is the Hirzebruch surface $\mathbb{F}_1$, and $f^*\mathcal{O}(1) \sim \mathcal{O}_\Sigma(1) = \mathcal{O}_{\mathbb{P}(E)}(D + 2F)$. The fact that the map is an embedding is equivalent to asking that the map f be given by the complete linear series of $\mathcal{O}_\Sigma(D + 2F)$. In the next sections it will be useful to weaken this condition.

Definition 6.5. A *cubic scroll* in $\mathbb{P}^n$ is a finite morphism $f : \Sigma \to \mathbb{P}^n$ where Σ is isomorphic to the Hirzebruch surface $\mathbb{F}_1$ and such that $f^*\mathcal{O}_{\mathbb{P}^n}(1)$ is isomorphic to $\mathcal{O}_\Sigma(D + 2F)$.

We wish to look at various types of curves C in $\mathbb{P}^4$, and find conditions for them to be enveloped by or contained in a cubic scroll Σ. We always start by looking at the class of the curve on Σ, look at its behavior with respect to the ruling and the directrix, and then seek to reconstruct the scroll out of this type of data. When talking about the "degree" of a curve C on Σ, we always mean with respect to the line bundle $\mathcal{O}_\Sigma(D + 2F)$, which will be used to map Σ into $\mathbb{P}^4$.

Lemma 6.6. *Suppose $L \subset \mathbb{P}^4$ is a line and $T \subset T_{\mathbb{P}^4}|_L$ is a sub-line bundle such that $T \cong \mathcal{O}_L(-1)$. Then there is a unique scroll $f : \Sigma \to \mathbb{P}^4$ with $f(D) = L$ (D the directrix of Σ) and such that the differential map $df_* : T_\Sigma \to f^*T_{\mathbb{P}^4}$ takes the vertical tangent bundle $T_{\Sigma/D} \subset T_\Sigma|_D$ to the pullback $f|_D^*T \subset f^*T_{\mathbb{P}^4}|_D$ on D.*

Proof. We have the restriction to L of the Euler sequence for $\mathbb{P}^4$:

$$0 \longrightarrow \mathcal{O}_L \longrightarrow \mathcal{O}_L(1)^5 \longrightarrow T_{\mathbb{P}^4}|_L \longrightarrow 0 \tag{44}$$

Define $F \subset \mathcal{O}_L(1)^5$ to be the preimage of $T \subset T_{\mathbb{P}^4}|_L$. We have an exact sequence:

$$0 \longrightarrow \mathcal{O}_L \longrightarrow F \longrightarrow T \cong \mathcal{O}_L(-1) \longrightarrow 0 \tag{45}$$

As $\mathrm{Ext}^1_{\mathcal{O}_L}(\mathcal{O}_L(-1), \mathcal{O}_L) = H^1(L, \mathcal{O}_L(1)) = 0$, we see that $F \cong \mathcal{O}_L \oplus \mathcal{O}_L(-1)$. Therefore $E := F(-1)$ is isomorphic to $\mathcal{O}_L(-1) \oplus \mathcal{O}_L(-2)$. By construction E is a subbundle $E \hookrightarrow \mathcal{O}_L^5$. Defining $\Sigma = \mathbb{P}(E)$, we have a map $\Sigma \to \mathbb{P}(\mathcal{O}_L^5) \cong \mathbb{P}^1 \times \mathbb{P}^4$. Projecting onto the second factor, we get an induced map $f : \Sigma \to \mathbb{P}^4$.

The directrix D is the section of Σ corresponding to $\mathcal{O}_L(-1) \subset E$. The composite map $\mathcal{O}_L(-1) \hookrightarrow E \hookrightarrow \mathcal{O}_L^5$ is simply obtained from the first map of the Euler sequence by twisting by $\mathcal{O}_L(-1)$, and by construction of the Euler sequence, this is the same as the tautological map $\mathcal{O}_L(-1) \to \mathcal{O}_L^5$ induced by the inclusion $L \subset \mathbb{P}^4$. Therefore $f(D)$ is just our original embedding of L in $\mathbb{P}^4$. Moreover, the vertical tangent bundle of Σ on D is identified with

$$\mathrm{Hom}(\mathcal{O}_L(-1), E/\mathcal{O}_L(-1)) \subset \mathrm{Hom}(\mathcal{O}_L(-1), \mathcal{O}_L^5/\mathcal{O}_L(-1)) = f^*T_{\mathbb{P}^4}|_L. \tag{46}$$

By construction of E as the preimage of T, this is precisely $F/\mathcal{O}_L = f^*T \subset f^*T_{\mathbb{P}^4}|_D$. As well, $f^*\mathcal{O}_{\mathbb{P}^4}(1) = \mathcal{O}_E(1) = \mathcal{O}_\Sigma(D + 2F)$, and therefore $f : \Sigma \to \mathbb{P}^4$ is the necessary scroll.

To see that Σ is unique, simply observe that the lines through the points of L are determined by the direction of T in $T_{\mathbb{P}^4}$. Since the scroll is the union of its lines through L, we see that T uniquely determines the scroll. ☐

Lemma 6.7. *Let $C \subset \mathbb{P}^4$ be a smooth conic curve and let $T \subset T_{\mathbb{P}^4}|_C$ be a sub-line bundle isomorphic to $\mathcal{O}_C(1)$ (a degree 1 line bundle on C, not $\mathcal{O}_{\mathbb{P}^4}(1)|_C$). Then there is a unique scroll $f : \Sigma \to \mathbb{P}^4$ and a factorization $i : C \to \Sigma$ of $C \to \mathbb{P}^4$ such that the differential $df : T_\Sigma \to f^*T_{\mathbb{P}^4}$ maps the vertical tangent bundle $i^*T_{\Sigma/C}$ to $i^*f|_C^*T$ on C.*

Proof. As in the proof of lemma 6.6, define F to be the subbundle of $\mathcal{O}_C(2)^5$ which is the preimage of $T \subset T_{\mathbb{P}^4}|_L$. We have a short exact sequence:

$$0 \longrightarrow \mathcal{O}_C \longrightarrow F \longrightarrow T \cong \mathcal{O}_C(1) \longrightarrow 0 \tag{47}$$

Since $H^1(C, \mathcal{O}_C(-1)) = 0$, we have that $F \cong \mathcal{O}_C \oplus \mathcal{O}_C(1)$. Therefore $E := F(-2) \subset \mathcal{O}_C^5$ is isomorphic to $\mathcal{O}_C(-2) \oplus \mathcal{O}_C(-1)$. Define $\Sigma = \mathbb{P}(E)$. The injective map $E \to \mathcal{O}_C^5$ induces a morphism $f : \Sigma \to \mathbb{P}^4$. Define $i : C \to \Sigma$ to be the section associated to the twist by $\mathcal{O}_C(-2)$ of the injection from equation (47): $\mathcal{O}_C \to F$.

The composite map $\mathcal{O}_C(-2) \to E \to \mathcal{O}_C^5$ is just the twist of the map in the Euler sequence $\mathcal{O}_C \to \mathcal{O}_C(2)^5$. By construction of the Euler sequence, this is the map $\mathcal{O}_C(-2) \to \mathcal{O}_C^5$ corresponding to $\mathcal{O}_{\mathbb{P}^4}(-1) \to \mathcal{O}_{\mathbb{P}^4}^5$ induced by the inclusion $C \to \mathbb{P}^4$. Therefore $f(i(C))$ is just our original embedding of C in $\mathbb{P}^4$. Finally, notice that the restriction to $i(C)$ of the vertical tangent bundle is simply

$$\mathrm{Hom}(\mathcal{O}_C(-2), E/\mathcal{O}_C(-2)) \longrightarrow \mathrm{Hom}(\mathcal{O}_C(-2), \mathcal{O}_C^5/\mathcal{O}_C(-2)) = f^*T_{\mathbb{P}^4}|_C \tag{48}$$

By construction of E, this is precisely $F/\mathcal{O}_C = f^*T \subset f^*T_{\mathbb{P}^4}|_C$. Finally, $f^*\mathcal{O}_{\mathbb{P}^4}(1) = \mathcal{O}_E(1) = \mathcal{O}_\Sigma(D+2F)$, and therefore $f : \Sigma \to \mathbb{P}^4$ is the necessary scroll.

To see that Σ is uniquely determined, observe that $T \subset T_{\mathbb{P}^4}|_C$ determines the lines in Σ which pass through C. Since Σ is the union of the lines which pass through C, this shows that Σ is unique. ☐

6.3. **Cubic Scrolls and Quartic Rational Curves.**

Recall that $\mathrm{Pic}(\Sigma) = \mathbb{Z}\{D, F\}$ where D is the directrix and F is the class of a line of ruling. The intersection product is given by $D^2 = -1$, $D.F = 1$, $F^2 = 0$. The canonical class is given by $K_\Sigma = -2D - 3F$. Our definition of a cubic scroll is that the finite map $f : \Sigma \to \mathbb{P}^4$ should come from the line bundle $f^*\mathcal{O}_{\mathbb{P}^n}(1) = \mathcal{O}_\Sigma(D + 2F)$.

The linear system $|F|$ is nef because it is the pullback of $\mathcal{O}_{\mathbb{P}^1}(1)$ under the projection $\pi : \Sigma \to \mathbb{P}^1$. Similarly, $|D+F|$ is nef because it is the pullback of $\mathcal{O}_{\mathbb{P}^2}(1)$ in the realization of Σ as $\mathbb{P}^2$ blown up at a point. Thus for any effective curve class $aD+bF$ we have the two inequalities $a = (aD+bF).F \geq 0$, $b = (aD+bF).(D+F) \geq 0$.

Suppose that $C \subset \Sigma$ is an effective divisor of degree 4 and arithmetic genus 0. By the adjunction formula

$$K_\Sigma.[C] + [C].[C] = 2p_a - 2 = -2. \tag{49}$$

So if $[C] = aD + bF$, then we have the conditions

$$a \geq 0,\ b \geq 0,\ a+b = 4,\ a^2 - 2ab + a + 2b = 2. \tag{50}$$

It is easy to check that there are precisely two solutions $[C] = 2D + 2F$, and $[C] = D + 3F$. We will see that both possibilities occur and describe some constructions related to each possibility.

We start with the case $[C] = 2D + 2F$.

Lemma 6.8. *Let $C \subset \mathbb{P}^4$ be a smooth quartic rational curve and let $V \subset |\mathcal{O}_C(2)|$ be a pencil of degree 2-divisors on C without basepoints. There exists a unique cubic scroll $f : \Sigma \to \mathbb{P}^4$ and a factorization $i : C \to \Sigma$ of $C \to \mathbb{P}^4$ such that $[i(C)] = 2D + 2F$ and such that the pencil of degree 2 divisors $\pi^{-1}(t) \cap C$ (for $t \in \mathbb{P}^1$) is the pencil V.*

Proof. Let $g : C \to \mathbb{P}^1$ be a degree 2-morphism defining V. Define $E^\vee := g_*(\mathcal{O}_{\mathbb{P}^4}(1)|_C)$. Since g is finite and flat of degree 2, $E^\vee$ is locally free of rank 2. Since $g^*\mathcal{O}_{\mathbb{P}^1}(1) \cong \mathcal{O}_C(2)$, the projection formula shows that $E^\vee \cong \mathcal{O}_{\mathbb{P}^1}(2) \otimes g_*\mathcal{O}_C$. But $g : C \to \mathbb{P}^1$ is a cyclic cover of degree 2 branched over a divisor of degree 2. The theory of cyclic covers [11, definition 2.50] shows that $g_*\mathcal{O}_C$ decomposes as a sum of $\mathbb{Z}/2\mathbb{Z}$-eigensheaves: $g_*\mathcal{O}_C \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. Thus $E \cong \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$.

On $\mathbb{P}^4$ we have the surjection of vector bundles $\mathcal{O}^5_{\mathbb{P}^4} \to \mathcal{O}_{\mathbb{P}^4}(1)$ given by global sections of $\mathcal{O}_{\mathbb{P}^4}(1)$. Restricting this to C gives $\mathcal{O}^5_C \to \mathcal{O}_{\mathbb{P}^4}(1)|_C$, which by adjunction induces a map $\mathcal{O}^5_{\mathbb{P}^1} \to g_*\mathcal{O}_C(2) = E^\vee$. Since $C \to \mathbb{P}^4$ is an embedding, for each pair of points $\{p, q\} \subset C$ (possibly infinitely near), we have that $\mathcal{O}^5_C \to \mathcal{O}_{\mathbb{P}^4}(1)|_{\{p,q\}}$ is surjective. In particular, taking $\{p, q\} = g^{-1}(t)$ for $t \in \mathbb{P}^1$, we conclude that $\mathcal{O}^5_{\mathbb{P}^1} \to E^\vee|_t$ is surjective. Define $\Sigma = \mathbb{P}(E)$, then the surjective map $\mathcal{O}^5_{\mathbb{P}^1} \to E^\vee$ induces a morphism $f : \Sigma \to \mathbb{P}^4$. We have $E \cong \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ so that $\Sigma \cong \mathbb{F}_1$, and $f^*\mathcal{O}_{\mathbb{P}^4}(1) = \mathcal{O}_E(1) = \mathcal{O}_\Sigma(D + 2F)$.

By adjunction we have a map of sheaves $g^*E^\vee \to \mathcal{O}_{\mathbb{P}^4}(1)|_C$. This map is surjective since g is finite. Moreover $g^*E^\vee \to \mathcal{O}_{\mathbb{P}^4}(1)|_C$ even separates points: for points in distinct fibers this is clear. For points $\{p, q\} = g^{-1}(t)$ it follows because $E^\vee|_t$ is precisely $\mathcal{O}_{\mathbb{P}^4}(1)|_{\{p,q\}}$. So the induced morphism $i : C \to \Sigma$ is even an embedding. Moreover, the pullback $i^*\mathcal{O}^5_\Sigma \to i^*\mathcal{O}_E(1)$ is precisely our original map $\mathcal{O}^5_C \to \mathcal{O}_{\mathbb{P}^4}(1)|_C$ so that $f \circ i : C \to \mathbb{P}^4$ is our original embedding of C in $\mathbb{P}^4$.

By construction of E, we have $i^*|F| = V$, and $g^*\mathcal{O}_{\mathbb{P}^1}(-1) = \mathcal{O}_C(-2)$ so we see $i^*\mathcal{O}_\Sigma(D) \cong \mathcal{O}_C(2)$. Thus we have $i(C) \sim 2D + 2F$. Therefore $f : \Sigma \to \mathbb{P}^4$, $i : C \to \Sigma$ are the necessary maps.

The map f is only an embedding if C is nondegenerate, otherwise f is the normalization map for its image, which is a singular cubic surface.

To see that this is unique, notice that the lines $f(\pi^{-1}(t))$ are simply the lines obtained by taking the joins of the degree 2 divisors on C which lie in in V. Since $f(\Sigma)$ is the union of this system of lines, this proves that $f(\Sigma)$ is uniquely determined. But $f : \Sigma \to f(\Sigma)$ is simply the normalization map so that f is also uniquely determined. ☐

Remark While we are at it, let's mention a specialization of the construction above, namely what happens when V is *not* basepoint free. Then $V = p + |\mathcal{O}_C(1)|$, where $p \in C$ is some basepoint. Consider the projection morphism $f : \mathbb{P}^4 \dashrightarrow \mathbb{P}^3$ obtained by projection from p (this is a rational map undefined at p). The image of C is a rational cubic curve B (possibly a singular plane cubic). Consider the cone S' in $\mathbb{P}^4$ over B with vertex p. This surface contains C. If we blowup $\mathbb{P}^4$ at p, then the proper transform of S' in $\widetilde{\mathbb{P}^4}$ is a surface whose normalization S is a Hirzebruch surface $\mathbb{F}_3$ (normalization is only necessary if B is a plane curve). The pullback of the exceptional divisor of $\widetilde{\mathbb{P}^4}$ plays the role of the directrix D of S. The inclusion $C \subset S'$ induces a factorization $i : C \to S$ of $C \to \mathbb{P}^4$, with $[i(C)] = D + 4F$. The intersection of D and $i(C)$ is precisely the point p. Finally, the linear system $i^*|F|$ is exactly $|\mathcal{O}_C(1)|$.

Next we consider the case of a rational curve $C \subset \Sigma$ such that $[C] = D + 3F$.

Lemma 6.9. *Let $C \subset \mathbb{P}^4$ be a smooth quartic rational curve and let $L \subset \mathbb{P}^4$ be a line such that $L \cap C = Z$ is a degree 2 divisor. Let $\phi : C \to L$ be an isomorphism such that $\phi(Z) = Z$ and $\phi|_Z$ is the identity map. Then there exists a unique triple (f, i, j) where $f : \Sigma \to \mathbb{P}^4$ is a cubic scroll, $i : C \to \Sigma$ and $j : L \to \Sigma$ are factorizations of $C \to \mathbb{P}^4$ and $L \to \mathbb{P}^4$, and such that $j(L) = D$ is the directrix, $[i(C)] = D + 3F$, and the lines of ruling induce the original isomorphism $\phi : C \to L$.*

Proof. Choose isomorphisms $g : \mathbb{P}^1 \to C$ and $h : \mathbb{P}^1 \to L$ such that $\phi \circ g = h$. Consider the rank 2 vector bundle

$$G = g^*\mathcal{O}_{\mathbb{P}^4}(1) \oplus h^*\mathcal{O}(1) \cong \mathcal{O}_{\mathbb{P}^1}(4) \oplus \mathcal{O}_{\mathbb{P}^1}(1). \tag{51}$$

Let $Z = g^*Z = h^*Z$. Since $g(Z) = h(Z)$ as subschemes of $\mathbb{P}^4$, we have an identification of $g^*\mathcal{O}_{\mathbb{P}^4}(1)|_Z$ with $h^*\mathcal{O}_{\mathbb{P}^4}(1)|_Z$. Define $E^\vee \subset G$ to be the subsheaf of E of sections (s_C, s_L) such that $s_C|_Z = s_L|_Z$ under our identification.

Since the map $g^*\mathcal{O}_{\mathbb{P}^4}(1) \to g^*\mathcal{O}_{\mathbb{P}^4}(1)|_Z$ is surjective, we conclude that $E^\vee \cong \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$. Moreover the linear series

$$H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(1)) \to H^0(\mathbb{P}^1, g^*\mathcal{O}_{\mathbb{P}^4}(1) \oplus h^*\mathcal{O}_{\mathbb{P}^4}(1)) \tag{52}$$

clearly factors through $H^0(\mathbb{P}^1, E^\vee)$. The question arises whether

$$H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(1)) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^1} \to E^\vee = \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \tag{53}$$

is surjective. Certainly the corresponding maps to $g^*\mathcal{O}_{\mathbb{P}^4}(1)$ and $h^*\mathcal{O}_{\mathbb{P}^4}(1)$ are surjective. The condition that $C \cap L = Z$ is precisely the condition that the image of

$$H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(1)) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^1} \to f^*\mathcal{O}_{\mathbb{P}^4}(1) \oplus g^*\mathcal{O}_{\mathbb{P}^4}(1) \tag{54}$$

is $E^\vee$. Thus the morphism is surjective. Denoting $\Sigma = \mathbb{P}(E)$, we conclude that there is a well-defined morphism $f : \Sigma \to \mathbb{P}^4$ such that $f^*\mathcal{O}_{\mathbb{P}^4}(1) = \mathcal{O}_E(1)$ and the pullback map

$$H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(1)) \to H^0(\mathbb{P}(E), \mathcal{O}_E(1)) = H^0(\mathbb{P}^1, E^\vee) \tag{55}$$

is the map above.

The composition of $E^\vee \to g^*\mathcal{O}_{\mathbb{P}^4}(1) \oplus h^*\mathcal{O}_{\mathbb{P}^4}(1)$ with the two projections define two surjective maps which yield sections $i : \mathbb{P}^1 \to \Sigma$, $j : \mathbb{P}^1 \to \Sigma$. Clearly $\pi \circ i = g^{-1}$, and $\pi \circ j = h^{-1}$. From this it follows that the isomorphism $C \cong L$ induced by the ruling of Σ is the same as the isomorphism ϕ. Thus (f, i, j) is a triple as in the statement of the lemma.

To see that this is unique, notice that the lines $f(\pi^{-1}(t))$ are simply the lines obtained by $\text{span}(p, \phi(p))$. Since $f(\Sigma)$ is the union of this system of lines, we conclude that $f(\Sigma)$ is unique. But $f : \Sigma \to f(\Sigma)$ is just the normalization map so that f is also uniquely determined. □

6.4. **Cubic Scrolls and Quintic Elliptics.**

Suppose that $E \subset \Sigma$ is an effective Cartier divisor of degree 5 and arithmetic genus 1. Writing $E = aD + bF$ we see (a, b) satisfies the relations $a, b \geq 0$, $a + b = 5$ and $a(b-3) + b(a-2) - a(a-2) = 0$. These relations give the unique integer solution $E = 2D + 3F = -K$. In particular, if E is smooth then $\pi : E \to \mathbb{P}^1$ is a finite morphism of degree 2, i.e., a g^1_2 on E. Thus a pair $(f : \Sigma \to \mathbb{P}^n, E \subset \Sigma)$ of a cubic scroll and a quintic elliptic determines a pair $(g : E \to \mathbb{P}^n, \pi : E \to \mathbb{P}^1)$ where $g : E \to \mathbb{P}^n$ is a quintic elliptic and $\pi : E \to \mathbb{P}^1$ is a degree 2 morphism.

Suppose we start with a pair $(h : E \to \mathbb{P}^n, \pi : E \to \mathbb{P}^1)$ where $h : E \to \mathbb{P}^n$ is an embedding of a quintic elliptic curve and $\pi : E \to \mathbb{P}^1$ is a degree 2 morphism. Consider the rank 2 vector bundle $\pi_* h^*\mathcal{O}_{\mathbb{P}^n}(1)$.

Lemma 6.10. *Suppose E is an elliptic curve and $\pi : E \to \mathbb{P}^1$ is a degree 2 morphism. Suppose L is an invertible sheaf on E of degree d. Then we have*

$$\pi_* L \cong \begin{cases} \mathcal{O}_{\mathbb{P}^1}(e) \oplus \mathcal{O}_{\mathbb{P}^1}(e-1) & d = 2e+1, \\ \mathcal{O}_{\mathbb{P}^1}(e) \oplus \mathcal{O}_{\mathbb{P}^1}(e-2) & d = 2e, L \cong \pi^*\mathcal{O}_{\mathbb{P}^1}(e), \\ \mathcal{O}_{\mathbb{P}^1}(e-1) \oplus \mathcal{O}_{\mathbb{P}^1}(e-1) & d = 2e, L \ncong \pi^*\mathcal{O}_{\mathbb{P}^1}(e) \end{cases} \tag{56}$$

Proof. Using the projection formula, we see that the lemma for L is equivalent to the lemma for $L \otimes \pi^* \mathcal{O}_{\mathbb{P}^1}(m)$. For any L there is an m such that $L \otimes \pi^* \mathcal{O}_{\mathbb{P}^1}(m)$ has degree 0 or degree 1. Thus we are reduced to the two cases $d = 0$ and $d = 1$. Notice also that in all cases we have $\chi(\pi_* L) = \chi(L)$, so that by Riemann-Roch for E and $\mathbb{P}^1$ we have

$$\deg(\pi_* L) + \operatorname{rank}(\pi_* L) \;=\; \deg(L) \;=\; d, \tag{57}$$

i.e., $\deg(\pi_* L) = d - 2$. By Grothendieck's lemma about vector bundles on $\mathbb{P}^1$ we know $\pi_* L = \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(d-2-a)$.

Suppose now that $d = 0$. Then $\pi_* L = \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(-2-a)$. We also have $h^0(\pi_* L) = h^0(L)$. If $L \cong \mathcal{O}_E$ then $h^0(L) = 1$ so that we have $\pi_* L = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$. If $L \not\cong \mathcal{O}_E$ then $h^0(L) = 0$ so that we have $\pi_* L = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. Thus the lemma is proved when $d = 0$.

Next suppose that $d = 1$. Then $\pi_* L = \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(-1-a)$, and $h^0(\pi_* L) = h^0(L) = 1$. Thus we have $\pi_* L = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. So the lemma is proved when $d = 1$. Thus the lemma is proved in all cases. $\square$

By the lemma we see that the vector bundle $G^\vee := \pi_* h^* \mathcal{O}_{\mathbb{P}^n}(1)$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2)$. Associated to the linear series $\mathcal{O}_E^{n+1} \to h^* \mathcal{O}_{\mathbb{P}^n}(1)$ defining the embedding h, we have the adjoint map $\mathcal{O}_{\mathbb{P}^1}^{n+1} \to \pi_* h^* \mathcal{O}_{\mathbb{P}^4}(1) = G^\vee$. Since h is an embedding, for each pair of points $\{p, q\} \subset E$ (possibly infinitely near), we have that $\mathcal{O}_E^{n+1} \to h^* \mathcal{O}_{\mathbb{P}^n}(1)|_{\{p,q\}}$ is surjective. In particular taking $\{p, q\} = \pi^{-1}(t)$ for $t \in \mathbb{P}^1$, we conclude that $\mathcal{O}_{\mathbb{P}^1}^{n+1} \to G^\vee|_t$ is surjective. Thus we have an induced morphism $\mathbb{P}(G) \to \mathbb{P}^n$ which pulls back $\mathcal{O}_{\mathbb{P}^n}(1)$ to $\mathcal{O}_G(1)$. Let us denote $\Sigma := \mathbb{P}(G)$ and let us denote the morphism by $f : \Sigma \to \mathbb{P}^n$. Abstractly Σ is isomorphic to $\mathbb{F}_1$ and $f : \Sigma \to \mathbb{P}^n$ is a cubic scroll.

The tautological map $\pi^* \pi_* h^* \mathcal{O}_{\mathbb{P}^n}(1) \to h^* \mathcal{O}_{\mathbb{P}^n}(1)$ is clearly surjective. Thus there is an induced morphism $i : E \to \Sigma$. Chasing definitions, we see that $h = f \circ i$. So we conclude that given a pair $(h : E \to \mathbb{P}^n, \pi : E \to \mathbb{P}^1)$ as above, we obtain a pair $(f : \Sigma \to \mathbb{P}^n, i : E \to \Sigma)$. Thus we have proved the following:

Lemma 6.11. *There is an equivalence between the collection of pairs $(f : \Sigma \to \mathbb{P}^n, i : E \to \Sigma)$ with $f : \Sigma \to \mathbb{P}^n$ a cubic scroll and $f \circ i : E \to \mathbb{P}^n$ an embedded quintic elliptic curve and the collection of pairs $(h : E \to \mathbb{P}^n, \pi : E \to \mathbb{P}^1)$ where $h : E \to \mathbb{P}^n$ is an embedded quintic elliptic curve and $\pi : E \to \mathbb{P}^1$ is a degree 2 morphism.*

Stated more precisely, this gives an isomorphism of the parameter schemes of such pairs, but we won't need the result in this form.

6.5. **Cubic Scrolls and Quintic Rational Curves.**

If one carries out the analogous computations as at the beginning of subsection 6.3 one sees that the only effective divisor classes $aD + bF$ on a cubic scroll Σ with degree 5 and arithmetic genus 0 are $D + 4F$ and $3D + 2F$. But the divisor class $3D + 2F$ cannot be the divisor of an irreducible curve because $(3D + 2F).D = -1$. Thus if $C \subset \Sigma$ is an irreducible curve of degree 5 and arithmetic genus 0, then $[C] = D + 4F$.

Lemma 6.12. *Let $C \subset \mathbb{P}^4$ be a smooth quintic rational curve and let $L \subset \mathbb{P}^4$ be a line such that $L \cap C$ is a degree 3 divisor Z. Let $\phi : C \to L$ be an isomorphism such that $\phi(Z) = Z$ and $\phi|_Z$ is the identity map. Then there exists a unique triple (f, i, j) such that $f : \Sigma \to \mathbb{P}^4$ is a cubic scroll, $i : C \to \Sigma$, and $j : L \to \Sigma$ are factorizations of $C \to \mathbb{P}^4$, and $L \to \mathbb{P}^4$, and such that $j(L) = D$ is the directrix, $[i(C)] = D + 4F$, and the lines of ruling induce the original isomorphism $\phi : C \to L$.*

Proof. The proof is very similar to the proof of lemma 6.9. $\square$

7. Quartic Rational Curves

In this section we will prove that the space $\mathcal{H}^{4,0}(X)$ of smooth quartic rational curves $C \subset X$ is irreducible of dimension 8. Recall from section 2.1 that every irreducible component of $\mathcal{H}^{4,0}(X)$ has dimension at least $-K_X.C = 2 \times 4 = 8$. First we prove that the open subset $\mathcal{U} \subset \mathcal{H}^{4,0}(X)$ parametrizing curves C with $\operatorname{span}(C) = \mathbb{P}^4$ is Zariski dense. To prove this it suffices to prove that the complement $\mathcal{D} \subset \mathcal{H}^{4,0}(X)$ has dimension at most 7.

Lemma 7.1. *Every irreducible component of the closed subset $\mathcal{D} \subset \mathcal{H}^{4,0}(X)$ parametrizing degenerate curves C (i.e., span$(C) \neq \mathbb{P}^4$) has dimension at most* 7.

Proof. By Riemann-Roch we see that a smooth quartic rational curve $C \subset \mathbb{P}^3$ lies on at least one quadric surface S. It cannot lie on two distinct quadric surfaces, for then it would have arithmetic genus 1 which is a contradiction. Thus to each point $[C] \in \mathcal{D}$, there is an associated quadric surface $S \subset \text{span}(C)$. Moreover the residual to $C \subset S \cap X$ is a pair of lines L_1, L_2 (possibly a single non-reduced line). Thus there is a morphism $\mathcal{D} \rightarrow \text{Hilb}_2(F)$. Since F is a smooth surface, $\text{Hilb}_2(F)$ has dimension 4.

Now there are two types of behaviors depending on whether or not $\text{span}(L_1, L_2)$ is a 2-plane or a 3-plane. The set of pairs $\{L_1, L_2\}$ such that $\text{span}(L_1, L_2)$ is a 2-plane corresponds to a point in the 3-dimensional divisor $\mathcal{I} \subset \text{Sym}^2(F)$ parameterizing incident lines. For each pair $\{L_1, L_2\}$ on this 3-fold, there is a 1-parameter family of hyperplanes containing $\text{span}(L_1, L_2)$. For each such hyperplane, there is a $\mathbb{P}^3$ of quadric surfaces Q in this hyperplane which contain $L_1 \cup L_2$. Thus the locus of all curves $[C] \in \mathcal{D}$ whose associated pair $\{L_1, L_2\}$ lies in $\mathcal{I}$ has dimension at most $3 + 1 + 3 = 7$.

Next consider the case that $\text{span}(L_1, L_2)$ is a 3-plane. Then every quadric surface containing these lines lies in this 3-plane. The set of quadric surfaces in this 3-plane which contain L_1 and L_2 is itself a $\mathbb{P}^3$. Thus the set of curves $[C] \in \mathcal{D}$ whose associated pair $\{L_1, L_2\}$ spans a 3-plane has dimension at most $4 + 3 = 7$. So the lemma is proved. ☐

Recall from lemma 4.2 that given any smooth quartic rational curve $C \subset X$, the subscheme $A_C \subset \text{Grass}(2,5)$ parametrizing the 2-secant lines to C is either 1-dimensional or else is 0-dimensional of length 16. In either case we conclude that there exists a 0-dimensional, length 2 subscheme $Z \subset A_C$ (in fact many such subschemes). Suppose given a 0-dimensional, length 2 subscheme $Z \subset A_C$. Then Z either consists of two reduced points $[L_1], [L_2] \in A_C$ or else Z corresponds to a non-reduced point of A_C. In the case that $Z = \{[L_1], [L_2]\}$, there are again two behaviors depending on whether Z is *planar*, i.e., $\text{span}(L_1, L_2)$ is a 2-plane, or whether Z is *non-planar*, i.e., $\text{span}(L_1, L_2)$ is a 3-plane. In the case that Z is planar, notice that we have the *distinguished point* $p \in X$ corresponding to the intersection of L_1 and L_2. In order to explain the analogues of planar and non-planar in the case that Z is non-reduced, we make a brief digression on ribbons.

A *ribbon* (for our purposes) is a degree two subscheme R of $\mathbb{P}^4$ supported along a line L, such that the ideal sheaf $\mathcal{I}_L$ of L in R satisfies $\mathcal{I}_L^2 = 0$, and such that the conormal sheaf $\mathcal{I}_L/\mathcal{I}_L^2 = \mathcal{I}_L$ is a line bundle on L. A ribbon is therefore a kind of doubled line in $\mathbb{P}^4$, such that at each point p of L the doubling occurs in a specified direction (given by the normal bundle $N_{L/R} = \mathcal{I}_L^\vee$), and such that this doubling direction varies reasonably along the line. Starting with a fixed line L in $\mathbb{P}^4$, and a sub-line-bundle $N \subset N_{L/\mathbb{P}^4}$, there is a unique ribbon R supported on L with $N_{L/R} = N$.

A non-reduced 0-dimensional subscheme Z of $\text{Grass}(2,5)$ of length 2 determines a ribbon R in $\mathbb{P}^4$. The line L of the ribbon is given by the point of support of Z in $\text{Grass}(2,5)$, while the tangent direction of Z corresponds to a global section of $N_{L/\mathbb{P}^4}$, and there is a unique sub-line-bundle N of $N_{L/\mathbb{P}^4}$ containing this section, which gives the ribbon.

There are two possibilities.

First of all we could have $N_{L/R} \cong \mathcal{O}_L(1)$. In this case we say that R is *planar ribbon* since there is a unique 2-plane $P \subset \mathbb{P}^4$ such that $R \subset P$ – in fact P is the unique 2-plane such that $N_{L/P} = N_{L/R}$ as subbundles of $N_{L/\mathbb{P}^4}$. Notice that in this case the global section $\mathcal{O}_L \rightarrow N_{L/R}$ is not determined by the ribbon R – in fact the data of this section is equivalent to a point $p \in L$ such that the length 2 scheme Z is simply the tangent direction at $[L]$ to the pencil of lines in P which pass through p. We refer to the point $p \in L$ as the *distinguished point* of L determined by Z.

The second possibility for the ribbon is that $N_{L/R} \cong \mathcal{O}_L$. First of all notice that in this case Z is uniquely determined by the ribbon. Second, given any subbundle $\mathcal{O}_L \cong N \subset N_{L/\mathbb{P}^4} \cong \mathcal{O}_L(1)^3$, there is a unique 3-plane $H \subset \mathbb{P}^4$ such that the map $\mathcal{O}_L \rightarrow N_{L/\mathbb{P}^4}$ factors through $N_{L/H} \subset N_{L/\mathbb{P}^4}$. Moreover in H there is a $\mathbb{P}^3$ of quadric surfaces $Q \subset H$ such that $R \subset Q$. The general surface Q in this $\mathbb{P}^3$ will be smooth and we will have $N_{L/R} = N_{L/Q}$ as subbundles of $N_{L/H}$. We will call a ribbon of this type a *non-planar ribbon.*

Define $I = I_{4,0} \subset \mathcal{U} \times \mathrm{Hilb}_2(F)$ to be the incidence correspondence of pairs $([C],[Z])$ such that $Z \subset A_C$ is a 0-dimensional length 2 subscheme. The idea of the proof of irreducibility of $\mathcal{H}^{4,0}(X)$ is to consider for such a pair $([C],[Z])$ a certain cubic surface Σ which contains the curve which is the union of C and the scheme parametrized by Z. The residual of this curve in $\Sigma \cap X$ will be a cubic curve, and for general $([C],[Z])$ this will be a smooth cubic rational curve. Moreover, if we associate to this cubic rational curve its image in Θ under the Abel-Jacobi map, then we obtain a rational transformation $I \to \Theta \times \mathrm{Hilb}_2(F)$ as the product of this map with projection $I \subset \mathcal{H}^{4,0}(X) \times \mathrm{Hilb}_2(F) \to \mathrm{Hilb}_2(F)$. The main fact is that this rational transformation is birational.

In order to prove the claims made in the last paragraph, we must first dispense with some degenerate possibilities. Let $I_P \subset I$ denote the closed subset parametrizing pairs $([C],[Z])$ such that Z is planar.

Lemma 7.2. *Every irreducible component of I_P has dimension at most 7.*

Proof. In the reduced case $Z = \{[L_1],[L_2]\}$, we have that $C \cap (L_1 \cup L_2)$ is a 0-dimensional subscheme of length 4 unless the distinguished point $p \in C$, in which case $C \cap (L_1 \cup L_2)$ has length 3. Similarly in the case that Z is non-reduced and gives rise to a planar ribbon R, we have $C \cap R$ is length 4 unless the distinguished point p is on C. But since $\mathrm{span}(C) = \mathbb{P}^4$, there is no 2-plane P such that $P \cap C$ has length 4; if such a 2-plane exists, then for any point $q \in C - P \cap C$ we have the hyperplane $H = \mathrm{span}(P,q)$ intersects C in a scheme of length 5 which contradicts Bézout's theorem since $C \not\subset H$. Therefore we conclude that if $Z \subset A_C$ is planar, then the distinguished point p lies on C.

Now define S to be the cone over C with vertex p. The projection of $\mathbb{P}^4$ away from p (to $\mathbb{P}^4/p \cong \mathbb{P}^3$) maps C birationally to a smooth cubic rational curve C', and S is simply the cone over this cubic curve. Moreover, S contains the curve E which is the union of C and the degree 2 subscheme parametrized by Z (either $L_1 \cup L_2$ or else the ribbon R determined by Z). By the Lefschetz hyperplane theorem, X contains no cubic surfaces other than the (degenerate) hyperplane sections of X. Now S is non-degenerate since it contains C and C is non-degenerate. Therefore S is not contained in X. So $S \cap X$ is a divisor on S of degree $\deg(S) \times \deg(X) = 3 \times 3 = 9$. But E has degree 6. Thus the residual curve D to E is a curve of degree 3. The only curves of degree 3 on S are hyperplane sections. There are two possible cases depending on whether or not $p \in D$.

Suppose that $p \in D$. In this case D is a union of three lines in S through p (or some degeneration thereof). Let $H \subset \mathbb{P}^4$ be the tangent hyperplane to X at p. Then every line $L \subset X$ containing p is contained in H. In particular, the residual subscheme to C in $S \cap X$ is contained in H. But an easy divisor class calculation on the blowup of S at p shows that the residual to C intersects C in a divisor of degree 5 (not counting p where the residual isn't well-defined). So $C \cap H$ is a divisor on C of degree at least 5. This contradicts Bézout's theorem unless $C \subset H$. But by assumption $\mathrm{span}(C) = \mathbb{P}^4$. So we conclude that $p \notin D$.

Every hyperplane section of S which does not contain p is a smooth cubic rational curve $D \subset X$. Thus we have a well-defined morphism $I_P \to \mathcal{H}^{3,0}(X)$. Let us define $\Pi \subset \mathrm{Hilb}_2(F)$ to be the divisor parametrizing planar subschemes $Z \subset F$. Then we can define a morphism

$$f_P : I_P \to \Pi \times \Theta \tag{58}$$

as the product of the projection map $I_P \to \mathrm{Hilb}_2(F)$ (which factors through Π by construction) and the composition of $I_P \to \mathcal{H}^{3,0}(X)$ with the Abel-Jacobi map $\mathcal{H}^{3,0}(X) \to \Theta$.

The claim is that the morphism f_P is injective. Recall that the fiber of the Abel-Jacobi map $\mathcal{H}^{3,0}(X) \to \Theta$ containing some curve $[D] \in \mathcal{H}^{3,0}(X)$ is an open set of the 2-dimensional linear series determined by D on the cubic surface $X \cap \mathrm{span}(D)$. The scheme determined by Z will intersect this cubic surface in a 0-dimensional scheme of length 2. Such a scheme imposes 2 linearly independent conditions on divisors in the linear series $|D|$. Therefore there is a unique curve D in this linear system which contains this 0-dimensional scheme of length 2. Given the curve D and the distinguished point p (which is determined by Z), we can reconstruct the scroll Σ as the cone over D with vertex p. We can then reconstruct C as the curve residual to the scheme determined by Z and D in the intersection $\Sigma \cap X$. Thus we can uniquely recover $[C]$ from $f_P([C])$ which shows that f_P is injective. Therefore $\dim I_P \leq \dim \Pi + \dim \Theta = 3 + 4 = 7$. This proves the lemma. ∎

Now define $I_U \subset I$ to be the Zariski dense open subset parametrizing pairs $([C],[Z])$ with $Z \subset A_C$ a 0-dimensional scheme of length 2 such that $\text{span}(C) = \mathbb{P}^4$ and Z is non-planar. If we consider A_C as a subscheme of $\text{Sym}^2(C) \cong \mathbb{P}^2$, then the length 2 subscheme $Z \subset \text{Sym}^2(C)$ determines a line in $\text{Sym}^2(C)$, i.e., a linear series of degree 2 divisors on C. One consequence of the assumption that Z is non-planar is that this linear series has no base-points. By lemma 6.8, there is a unique cubic scroll $\Sigma \subset \mathbb{P}^4$ which contains C and such that the linear series of degree 2 divisors is the linear series of intersections of C with the lines of the ruling of Σ. Let D denote the directrix of Σ and let F denote the divisor class of a line of the ruling. Then the hyperplane class on Σ is $H \sim D + 2F$ so the intersection $X \cap \Sigma$ has divisor class $3D + 6F$. Now $C.F = 2$ and $C.H = 4$, thus $C \sim 2D + 2F$. On the other hand, the scheme determined by Z (either $L_1 \cup L_2$ if Z is reduced, or the ribbon R if Z is non-reduced) has divisor class $2F$. Thus the residual to C and the subscheme determined by Z is a divisor $D_2 \subset \Sigma$ linearly equivalent to $D + 2F$.

Theorem 7.3. *The space $\mathcal{H}^{4,0}(X)$ is irreducible of dimension* 8.

Proof. We continue to use the notation introduced in this section. Because of lemma 7.1 and lemma 7.2, it is equivalent to prove the I_U is irreducible of dimension 8. We stratify I_U according to the type of the residual curve D_2 defined above. If D_2 is a smooth curve, we say it is the first type. If D_2 is the union of a conic and a line of the ruling, we say it is the second type. If D_3 is the union of the directrix D and two lines of the ruling (possibly one non-reduced line), we say it is the third type. Define the corresponding loci in I_U to be I_1, I_2, and I_3.

Third type: First we deal with the third type because it is the most involved. We will prove that every irreducible component of I_3 has dimension at most 7. We can associate to each pair $([C],[Z]) \in I_3$ the configuration $([D],[W])$ where $[D] \in F$ is the directrix line and $[W] \in \text{Hilb}_4(F)$ is the length 4 scheme parametrizing the residual to D and C in Σ. Because of lemma 7.2, we may disregard the subvariety of I_3 such that any length 2 subscheme of W is planar. Let us define $I_{3,U} \subset I_3$ to be the open subset such that no length 2 subscheme of W is planar. Then we are reduced to proving that every irreducible component of $I_{3,U}$ has dimension at most 7.

Notice that W is a subscheme of the divisor $Z_D \subset F$ which parametrizes lines in X which intersect D. Let $M \subset F \times \text{Hilb}_4(F)$ be the closed subset parametrizing configurations $([D],[W])$ such that $W \subset Z_D$. Then we have a morphism $f_3 : I_{3,U} \rightarrow M$.

Now we consider the dimension of M. By [1, lemma 10.5], for a general $[D] \in F$, the divisor Z_D is smooth. Therefore the fiber of projection onto the first factor $M \to F$ over the point $[D]$ is simply the 4-dimensional scheme $\text{Sym}^4(Z_D)$. It also follows by [1, lemma 10.5] that when Z_D is singular, the singular set consists of a single ordinary node. The dimension of the space of length d subschemes of a nodal curve whose reduced scheme is the node is 0 if $d = 1$ and 1 if $d > 0$. Therefore, even when Z_D is singular, the dimension of $\text{Hilb}_4(Z_D)$ is still only 4. So we conclude that $M \rightarrow F$ has fiber dimension 4 so that $\dim(M) = 6$.

Now we consider the fiber dimension of $f_3 : I_{3,U} \rightarrow M$. We will prove that every fiber of f_3 has dimension at most 1. Suppose given a configuration $([D],[W])$. We want to consider the quotient $\mathbb{P}^4/D \cong \mathbb{P}^2$. Because no length 2 subscheme of W is planar, the image of W in $\mathbb{P}^4/D$ is a 0-dimensional subscheme of length 4. Given any scroll Σ which contains D as the directrix, the image of Σ in $\mathbb{P}^4/D$ is a smooth plane conic. Given a 0-dimensional subscheme of $\mathbb{P}^2$ of length 4, there is a pencil of conics containing this subscheme. So if Σ is a scroll which contains both D and W, then the image of Σ in $\mathbb{P}^4/D$ must be a conic B in the pencil determined by the image of W. Moreover, the scroll Σ determines an isomorphism $\phi : D \rightarrow B$ which associates to each point in D, the image in $\mathbb{P}^4/D$ of the line of the ruling through that point. Notice that given a conic B in $\mathbb{P}^4/D$ which contains the image of W, there is at most one isomorphism $\phi : D \to B$ such that ϕ identifies the projection of W onto D with the projection of W onto B (because the only automorphism of $\mathbb{P}^1$ which fixes a divisor of degree 4 is the identity map).

By the last paragraph, associated to a configuration $([D],[W])$ there is a pencil of smooth conics B in $\mathbb{P}^4/D$ which contain the image of W, and to each B there is (at most) one automorphism $\phi : D \rightarrow B$ which identifies the two projections of W. What extra information is needed to determine a scroll Σ such that Σ contains D and W and such that the image of Σ is B? To answer this question we recall lemma 6.6 which says that to determine a scroll $\Sigma \subset \mathbb{P}^4$ which contains a line D as the directrix, it is equivalent to determine the rank 1 subbundle $T \subset T_{\mathbb{P}^4}|_D$, where $T \cong \mathcal{O}_D(-1)$ is the bundle of tangent spaces to line of the ruling of Σ. Also we have that T is everywhere distinct (as a rank 1 subspace of $T_{\mathbb{P}^4}$) from T_D. So first we consider the image of T in $N_{D/\mathbb{P}^4}$. But of course this is just $N_{D/\Sigma} \subset N_{D/\mathbb{P}^4}$. Also $N_{D/\mathbb{P}^4} = \mathcal{O}_D(1) \otimes_{\mathbb{C}} N$ where N is the rank 3 vector space whose associated projective space is canonically $\mathbb{P}^4/D$. The map $\phi : D \to \mathbb{P}^4/D$ is equivalent to a subbundle $N' \subset \mathcal{O}_D \otimes_{\mathbb{C}} N$ where N' is isomorphic to $\mathcal{O}_D(-2)$, and the subbundle $N_{D/\Sigma} \subset \mathcal{O}_D(1) \otimes_{\mathbb{C}} N$ is simply $N' \otimes \mathcal{O}(1)$. So $N_{D/\Sigma} \subset N_{D/\mathbb{P}^4}$ is uniquely determined by the map $\phi : D \to \mathbb{P}^4/D$.

Finally, to determine the scroll Σ, we have to determine a subbundle $T \subset T_{\mathbb{P}^4}|_D$ which projects isomorphically to $N_{D/\Sigma}$. The set of such subbundles is equivalent to the set of global sections of the bundle $\mathrm{Hom}(\mathcal{O}_D(-1), T_D \oplus N_{D/\Sigma}) \cong \mathcal{O}_D \oplus \mathcal{O}_D(3)$ (modulo non-zero scaling) whose composition with $T_D \oplus N_{D/\Sigma} \to N_{D/\Sigma}$ is an isomorphism. In other words, the set of such subbundles is simply $\mathrm{Hom}(\mathcal{O}_D(-1), T_D) \cong \mathcal{O}_D(3)$. But this section is determined along the projection of W since W must be a subscheme of the scheme of lines of the ruling of Σ. Since a length 4 subscheme of $\mathbb{P}^1$ imposes 4 linear conditions on $\mathcal{O}_D(3)$, we see that there is a unique section which restricts to W in the appropriate way. So finally we conclude that the scroll Σ is determined by the configuration $([D], [W])$ together with the conic B. Since M has dimension 6, and since for each $([D], [W])$ there is at most a 1-dimensional family of possible B's, we conclude that $I_{3,U}$ has dimension at most 7.

Second Type: Next we consider the second type. We will prove that every irreducible component of I_2 has dimension at most 7. Let B denote the smooth conic. Again let $W \subset Z_B \subset F$ denote the length 3 subscheme parametrizing the lines which make up the residual to $C \cup B \subset X$. By lemma 7.2, we may suppose that every length 2 subscheme of W is non-planar. The claim is that the subscheme parametrized by W spans $\mathbb{P}^4$. By way of contradiction, suppose that it is contained in a hyperplane H. By Bézout's theorem, B is also contained in H. But then the intersection of H and the scroll Σ contains the degree 5 curve which is the union of B and the subscheme parametrized by W. This contradicts Bézout's theorem unless $\Sigma \subset H$. But then C is also contained in H, and this contradicts the hypothesis on C. Therefore the scheme parametrized by W spans $\mathbb{P}^4$.

Let $M_2 \subset \mathcal{H}^{2,0}(X) \times \mathrm{Hilb}_3(F)$ be the locally closed subset parametrizing configurations $([B], [W])$ such that B is smooth, such that every length 2 subscheme of W is planar, such that the subscheme of X parametrized by W spans $\mathbb{P}^4$, and such that $W \subset Z_B$, where $Z_B \subset F$ is the locally closed set which parametrizes lines which intersect B exactly once (there is exactly one line which intersects B twice). By the same type of argument at in the first case, we conclude that $\dim M_2 = \dim \mathcal{H}^{2,0}(X) + 3 = 4 + 3 = 7$.

There is an obvious morphism $f_2 : I_2 \to M_2$, and we are reduced to showing that this map is injective. Now given a subscheme $[W] \in \mathrm{Hilb}_3(\mathrm{Grass}(2, 5))$ such that no length 2 subscheme of W is planar, and such that the scheme parametrized by W (in $\mathbb{P}^4$) spans $\mathbb{P}^4$, then there is precisely one line L whose intersection with this scheme is of length 3. We will only give the proof when W is reduced – the non-reduced case is only slightly more technical. Suppose that $W = \{[L_1], [L_2], [L_3]\}$. Then L_1, L_2, L_3 are all disjoint. If a line L intersects L_1 and L_2, then it lies in the hyperplane $H = \mathrm{span}(L_1, L_2)$. Since $\mathrm{span}(L_1, L_2, L_3) = \mathbb{P}^4$, the line L_3 is not contained in H. Therefore $H \cap L_3$ is a point p which does not lie on L_1 or L_2. We conclude that the lines L which intersect L_1, L_2, and L_3 are exactly the lines $L \subset H$ which intersect L_1, L_2 and which pass through p. If we consider projection away from p, then the set of such lines corresponds to the intersection points in $H/p \cong \mathbb{P}^2$ of the images of L_1 and L_2. Since these lines are skew and don't contain p, their images in H/p consist of two distinct lines, and two distinct lines in $\mathbb{P}^2$ intersect in precisely one point.

But given a scroll Σ, the directrix line D is a line which intersects L_1, L_2 and L_3. Thus we conclude that the directrix line D is uniquely determined by the configuration $([B], [W])$ (in fact just by $[W]$). Moreover, the lines of the ruling induce an isomorphism $\phi : D \to B$ which carries each intersection $L_i \cap D$ to the intersection $L_i \cap B$. There is a unique isomorphism $\phi : D \to B$ with this property (because the only automorphism of $\mathbb{P}^1$ which fixes a length 3 divisor is the identity). Thus ϕ is also determined by the configuration $([B], [W])$. From ϕ we recover the scroll Σ. From the scroll Σ we recover C as the residual to $B \cup L_1 \cup L_2 \cup L_3$ in $\Sigma \cap X$. Thus we conclude that f_2 is injective, which proves that I_2 has dimension at most 7.

First type: Finally we consider I_1. This analysis will also be very important for describing the Abel-Jacobi map $\alpha_{4,0} : \mathcal{H}^{4,0}(X) \to J(X)$. Denote the residual curve by A (this is the curve we were calling D_2). Let $N \subset \mathcal{H}^{3,0}(X) \times F \times F$ denote the locally closed subscheme parametrizing triples $([A], [L_1], [L_2])$ such that L_1, L_2 are skew lines each of which intersects A transversely in 1 point and such that $\mathrm{span}(A, L_1, L_2) = \mathbb{P}^4$.

Of course N fibers over $\mathcal{H}^{3,0}(X)$ and the fiber is an open subset of $D_A \times D_A$. Thus we have

$$\dim(N) \; = \; \dim(\mathcal{H}^{3,0}) + 2\dim(D_A) \; = \; 6 + 2 \; = \; 8. \tag{59}$$

There is an obvious map $I_1 \; \rightarrow \; N$, and the only nontrivial condition to verify is that $\mathrm{span}(A, L_1, L_2) \; = \; \mathbb{P}^4$, but this follows by applying Bézout's theorem to Σ.

What are the fibers of $I_1 \; \rightarrow \; N$? Consider the 3-plane $P \; = \; \mathrm{span}(L_1, L_2)$. This intersects A in a degree 3 divisor. Two of the points of this divisor are the points of intersection of A and L_1, L_2. The third point p lies on neither L_1 nor L_2 since $A.L_i$ is a degree 1 divisor. Now there is a unique line M which contains p and which intersects both L_1 and L_2: if we project P away from p, then the line M corresponds to the unique point of intersection of the images of L_1 and L_2 in $\mathbb{P}^2$. Now suppose that Σ is a scroll which contains L_1 and L_2 and A. Let D denote the directrix. Since D intersects L_1 and L_2, it must lie in P. If D does not contain p, then there is a line of the ruling F of Σ which passes through p. But then $L_1 \cup L_2 \cup D \cup F$ is a divisor of degree 4 in the hyperplane section $P \cap \Sigma$. This contradicts Bézout's theorem. What we conclude is that D must equal M. Moreover the isomorphism $\phi : A \; \rightarrow \; M$ corresponding to projection of $\Sigma \; \rightarrow \; M$ must be the unique isomorphism such that $\phi(p) \; = \; p$ and such that $\phi(A \cap L_i) \; = \; \phi(M \cap L_i)$. Of course the scroll Σ is determined by M and the isomorphism ϕ. Thus there is a unique scroll Σ which contains $A \cup L_1 \cup L_2$, i.e., $I_1 \; \rightarrow \; N$ maps 1-to-1 to its image. Since there is some irreducible component of I of dimension 8, in fact we must have that $I_1 \; \rightarrow \; N$ is dominant, i.e., $I_1 \; \rightarrow \; N$ is an open immersion. Finally notice that N fibers over the irreducible space $\mathcal{H}^{3,0}$ and the general fiber $D_A \times D_A$ is irreducible. Thus N is irreducible. So I_1 is irreducible of dimension 8.

Since $I \rightarrow \mathcal{H}^{4,0}(X)$ is surjective, every component of $\mathcal{H}^{4,0}(X)$ is dominated by a component of I, which must be at least eight dimensional, since every component of $\mathcal{H}^{4,0}$ is. The only component of I with this property is $\overline{I_1}$, which is precisely eight dimensional, so we conclude that $\mathcal{H}^{4,0}$ is irreducible of dimension eight. □

Remark: Let I_1 be as in the proof above and let J_1 be the quotient of I_1 by the involution $([C], [L_1], [L_2]) \; \mapsto \; ([C], [L_2], [L_1])$ Notice that $J_1 \; \rightarrow \; \mathcal{H}^{4,0}(X)$ is still dominant.

8. Quintic Elliptic Curves

In section 7 we proved that $\mathcal{H}^{4,0}(X)$ is irreducible by residuating the union of a quartic curve and a pair of 2-secant lines in the intersection of X with a suitable cubic scroll Σ. In this section we will prove that $\mathcal{H}^{5,1}(X)$ by residuating a quintic genus 1 curve in the intersection of X with a suitable cubic scroll. The idea of the proof is very similar to the proof of theorem 7.3. As in that proof, there are several degenerate behaviors which we need to rule out as generic.

Theorem 8.1. *The space $\mathcal{H}^{5,1}(X)$ is irreducible of dimension* 10.

Proof. By lemma 6.11 the irreducibility of $\mathcal{H}^{5,0}$ is equivalent to showing that $\widetilde{H}^{5,1}$ is irreducible, where $\widetilde{H}^{5,1}$ is the parameter space for pairs $(f : \Sigma \; \rightarrow \; \mathbb{P}^4, i : E \; \rightarrow \; \Sigma)$ such that $f \circ i : E \; \rightarrow \; \mathbb{P}^4$ is an embedding of E as a quintic elliptic curve. Indeed, we have seen that the fiber of projection $\widetilde{H}^{5,1} \; \rightarrow \; \mathcal{H}^{5,1}(X)$ over a point $[E]$ is simply the set of $g^1_2 s$ on E, i.e., $\mathrm{Pic}^2(E) \cong E$. Since the fibers are irreducible of constant fiber dimension 1, we conclude that $\mathcal{H}^{5,1}(X)$ is irreducible of dimension 10 iff $\widetilde{H}^{5,1}$ is irreducible of dimension 11.

On the other hand each pair $(f : \Sigma \; \rightarrow \; \mathbb{P}^4, i_E : E \; \rightarrow \; \Sigma)$ is equivalent to a pair $(f : \Sigma \; \rightarrow \; \mathbb{P}^4, i_C : C \; \rightarrow \; \Sigma)$ where C is the residual quartic curve, $[C] = D + 3F$. We decompose $\widetilde{H}^{5,1}$ into a union of locally closed subsets $\widetilde{H}_1$, $\widetilde{H}_2$, $\widetilde{H}_3$, $\widetilde{H}_4$ parametrizing the set where $i_C : C \; \rightarrow \; \Sigma$ is in the first, second, third or fourth case (we say that $i_C : C \; \rightarrow \; \Sigma$ is in the i-th case if $C'.D \; = \; i - 2$ where C' is the unique irreducible component of C which projects isomorphically to $\mathbb{P}^1$ under π). We will show that for $i \neq 1$, $\widetilde{H}_i$ has dimension ≤ 10, and we will show that $\widetilde{H}_1$ is irreducible of dimension 11.

First Case: Now $\widetilde{H}_1$ parameterizes pairs $(f : \Sigma \rightarrow \mathbb{P}^4,\; i_C : C \rightarrow \Sigma)$ where Σ is in the first case, and $f(i_C(C)) \subset X$. There is a projection $\widetilde{H}_1 \rightarrow \mathcal{H}^{4,0}$ which assigns to (f, i_C) the curve $C \subset X$, the embedding being given by $f \circ i_C$. We have seen in lemma 6.9 that the fibre of this projection over a particular curve $C \subset \mathbb{P}^4$ consists of the data of a line L in $\mathbb{P}^4$ intersecting C in a subscheme Z of length 2, and an isomorphism ϕ between C and L which is the identity map on Z. The length 2 subscheme Z uniquely determines L, and given a fixed Z, there is a $\mathbb{C}^*$ worth of choices of such isomorphisms ϕ. Therefore each fibre of $\widetilde{H_1} \to \mathcal{H}^{4,0}$ is itself a $\mathbb{C}^*$ bundle over the space $\mathrm{Sym}^2(C) = \mathbb{P}^2$ parameterizing the Z's. We see that $\widetilde{H_1} \to \mathcal{H}^{4,0}$ has irreducible fibres of dimension 3. By theorem 7.3, $\mathcal{H}^{4,0}(X)$ is irreducible of dimension 8, and therefore $\widetilde{H_1}$ is irreducible of dimension 11.

Second Case: Now $\widetilde{H}_2$ parametrizes pairs $(f : \Sigma \to \mathbb{P}^4,\ i_C : C \to \Sigma)$ where $C = C' \cup F$ the union of a smooth rational cubic curve C' and a line of ruling F, and $f(i_C(C)) \subset X$. Consider the morphism $\widetilde{H}_2 \to H^{3,0}$ which associates to (f, i_C) the curve $C' \subset X$, Recall that $\dim(H^{3,0}) = 6$. We analyze the fibre of this map by looking for the data necessary to reconstruct Σ. The irreducible component $F \subset C$ is mapped to a line in X which intersects C' in a single point. The directrix D of Σ is mapped to a line in $\mathbb{P}^4$ which intersects F, and also intersects C' in a single point. Given a fixed $C' \subset X$, there is a one parameter family of lines in X to serve as an F. Given a fixed F, we recover the directrix as follows: pick any point p on C', then there is a $\mathbb{P}^1$ of lines D passing through p and intersecting F (in case $p \in F \cap C'$, the limiting condition is that D lie in the $\mathbb{P}^2$ spanned by F and the tangent line to C' at p). Finally, we need to specify the isomorphism $\phi : C' \to D$ induced by the lines of ruling. Since this must be the identity on p and on $F \cap C'$, this is parameterized by $\mathbb{C}^*$. As in the other lemmas on reconstructing cubic scrolls in section 6, this data is sufficient to specify Σ. Altogether we see that the dimension of $\widetilde{H}_2$ is the sum of 6 for $\dim(\mathcal{H}^{3,0})$, 1 for the choice of the line F, 1 for the choice of point $p \in C'$, 1 for the choice of D going through p and intersecting F, and 1 for the $\mathbb{C}^*$ of isomorphisms between C' and D satisfying our conditions, i.e., $\dim(\widetilde{H}_2) = 10$.

Third Case: This time the curve C' is a smooth conic, and C consists of C' and two lines F_1, F_2 of ruling (possibly a double line). The inclusion i_C takes the lines of ruling to two lines (or possibly a nonplanar ribbon) in X which intersect C'. The directrix D of Σ maps to a line in $\mathbb{P}^4$ which intersects C' once and the union of the lines in a subscheme of length two. We have a projection $\widetilde{H}_3 \to H^{2,0}$, given by forgetting all of the data except the conic C'. Reversing this procedure, if we start with a smooth conic $C' \subset X$, the choices of two lines F_1, F_2, in X meeting C' form a two dimensional family. Given the two lines, the directrix D must meet each of them, and so is also parameterized by a two dimensional family, namely the choices of the intersection points on the two lines. Finally, given this data, we have to specify the isomorphism $\phi : C' \to D$ corresponding to the lines of ruling. This isomorphism must take $F_i \cap C'$ to $F_i \cap D$ for $i = 1, 2$, and so we see that there is a $\mathbb{C}^*$ of choices. Altogether the dimension of $\widetilde{H}_3$ is the sum of $4 = \dim(H^{2,0})$, 2 for the union of two lines intersecting C', 2 for the 2-parameter family of possibilities for the directrix D, and 1 for the $\mathbb{C}^*$ of isomorphisms $\pi : C' \to D$, i.e., $\dim(\widetilde{H}_3) = 9$.

Fourth Case: Finally we consider the fourth case. This time C' is the directrix of Σ, and $C' \subset X$ is a line in X. The Fano scheme of lines in X has dimension 2. The remaining components of C are mapped to a union of three lines intersecting C' (or some degeneration thereof). For fixed C', the dimension of such triples of lines is 3. By lemma 6.6, in order to construct a scroll Σ containing C' as the directrix, we need to provide a sub-line-bundle $T \subset T_{\mathbb{P}^4}|C'$, with $T \cong \mathcal{O}_{C'}(-1)$. The set of such bundles is a $\mathbb{P}^{12}$, since $\hom(\mathcal{O}_{C'}(-1), T_{\mathbb{P}^4}|_{C'}) = 13$. In order for the scroll to contain the three lines touching C', this sub-bundle must agree with the direction of each line at the point of contact with C'. For each line, this is a three dimensional linear condition. Therefore the space of scrolls containing C' as the directrix, as well as the three lines as lines of ruling is a $\mathbb{P}^3$. Thus altogether $\widetilde{H}_4$ has dimension $(2 + 3 + 3) - 1 = 7$.

□

Remark Of course the proof shows more than just that $\mathcal{H}^{5,1}(X)$ is irreducible. We see that for a general quintic elliptic $E \subset X$ and a general cubic scroll containing E, the residual curve is a smooth quartic rational curve.

9. Quintic Rational Curves

In this section we will prove that the space $\mathcal{H}^{5,0}(X)$ is irreducible.

Lemma 9.1. *Let $C \subset \mathbb{P}^n$ be a rational normal curve and let $P \subset \mathbb{P}^n$ be a linear r-plane. If $(r+2)k \geq (r+1)(n+1)$, then P is contained in a k-secant $(k-1)$-plane of C, i.e., there exists a divisor $D = q_1 + \dots q_k$ on C such that $P \subset span(D)$.*

Proof. We identify C with $\mathbb{P}^1$ so that $\mathcal{O}_C(1)$ is a degree 1 line bundle, and $\mathcal{O}_{\mathbb{P}^n}(1)|_C$ is a degree n line bundle. Up to a choice of basis of $\mathbb{P}^n$, we can identify the inclusion $C \hookrightarrow \mathbb{P}^n$ with the morphism associated to the complete linear series $|\mathcal{O}_C(n)|$.

Let $\mathbb{P}^k$ be identified with the complete linear series $|\mathcal{O}_C(k)|$. Then on $\mathbb{P}^k$ we have the tautological injection of vector bundles $\mathcal{O}_{\mathbb{P}^k}(-1) \rightarrow H^0(C, \mathcal{O}_C(k)) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^k}$. If we take the tensor product of this map with $H^0(C, \mathcal{O}_C(n-k))$ and then use the product map

$$H^0(C, \mathcal{O}_C(n-k)) \otimes_{\mathbb{C}} H^0(C, \mathcal{O}_C(k)) \longrightarrow H^0(C, \mathcal{O}_C(n)), \tag{60}$$

we have the composite map

$$H^0(C, \mathcal{O}_C(n-k)) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^k}(-1) \longrightarrow H^0(C, \mathcal{O}_C(n)) \otimes \mathcal{O}_{\mathbb{P}^k}. \tag{61}$$

If we think of $\mathbb{P}^k$ as the parameter space for degree k divisors $D = q_1 + \cdots + q_k$ on C, i.e., as $\mathrm{Sym}^k(C)$, then the fiber of this map of vector bundles at a point $[D]$ is just $H^0(C, \mathcal{O}_C(n)(-D)) \rightarrow H^0(C, \mathcal{O}_C(n))$.

Under the identification $H^0(C, \mathcal{O}_C(n)) = H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^1}(1))$, we have a restriction map $H^0(C, \mathcal{O}_C(n)) \rightarrow H^0(P, \mathcal{O}_{\mathbb{P}^n}(1)|_P)$. Thus we have an induced map of vector bundles on $\mathbb{P}^k$ obtained as the composite map

$$H^0(C, \mathcal{O}_C(n-k)) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^k}(-1) \longrightarrow H^0(C, \mathcal{O}_C(n)) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^k} \longrightarrow H^0(P, \mathcal{O}_{\mathbb{P}^n}(1)|_P) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^k}. \tag{62}$$

Suppose the fiber of this map is the zero map at a point $[D]$. Then every linear polynomial of $\mathbb{P}^n$ which vanishes on D also vanishes on P. Since $\mathrm{span}(D) \subset \mathbb{P}^n$ is cut out by the linear polynomials which vanish on D, we conclude that the ideal of $\mathrm{span}(D)$ is contained in the ideal of P, i.e., $P \subset \mathrm{span}(D)$. So we are reduced to showing that some fiber of this map is zero, i.e., this map of vector bundles has nonempty zero locus.

We may think of the map above as a global section of the bundle

$$\mathrm{Hom}_{\mathbb{C}}(H^0(C, \mathcal{O}_C(n-k)), H^0(P, \mathcal{O}_{\mathbb{P}^k}(1)|_P)) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^k}(1). \tag{63}$$

The rank of this vector bundle is

$$\dim H^0(C, \mathcal{O}_C(n-k)) \times \dim H^0(P, \mathcal{O}_{\mathbb{P}^k}(1)|_P) = (n+1-k)(r+1). \tag{64}$$

Thus the map is a global section of $\mathcal{O}_{\mathbb{P}^k}(1)^{(n+1-k)(r+1)}$. The zero locus is just defined by the vanishing of $(n+1-k)(r+1)$ linear polynomials. So long as $(n+1-k)(r+1) \leq k$, these linear polynomials always have a solution. Thus if $(r+2)k \geq (n+1)(r+1)$, then the zero locus is nonempty. □

Remark 9.2. Notice that the proof also shows that the set of k-secant $(k-1)$-planes which contain P is a linear subspace of $\mathbb{P}^k$. In particular, when this set is finite, there is a unique solution.

Corollary 9.3. *If $C \subset \mathbb{P}^4$ is a smooth, nondegenerate quintic rational curve, then C has a unique 3-secant line $L \subset \mathbb{P}^4$, and L is not a 4-secant line. If $C \subset \mathbb{P}^3$ is a smooth quintic rational curve, the C has a 1-parameter family of 3-secant lines $L \subset \mathbb{P}^3$. If every 3-secant line to C is a 4-secant line, then C lies on a smooth quadric surface as a divisor of type $(1,4)$.*

Proof. First consider the case where C is nondegenerate. Then we can think of $C \subset \mathbb{P}^4$ as the projection of a rational normal curve $C' \subset \mathbb{P}^5$ from a point p not on C'. By lemma 9.1, we see that there is a 3-secant 2-plane $\mathrm{span}(D)$ which contains p. The projection of P is a 3-secant line L to C. On the other hand, suppose that C has a 4-secant line L. The preimage of L is a 4-secant 2-plane to C'. But since any 4 points on C are linearly independent (or more generally any degree 4 divisor on C imposes 4 conditions on linear forms), we see that C' does not have a 4-secant 2-plane. Thus C has a 3-secant line, but does not have a 4-secant line.

Suppose that C has two distinct 3-secant lines L, M. Consider $H = \mathrm{span}(L, M)$. If this is a hyperplane in $\mathbb{P}^4$, then $H \cap C$ has degree 6. This contradicts Bézout's theorem unless $C \subset H$, i.e., C is degenerate. If H is a 2-plane, choose any point $p \in C$ not contained in H and let $H' = \mathrm{span}(H, p)$. Then H' is a hyperplane, and again $H' \cap C \supset \{p\} \cup (H \cap C)$ has degree at least 6. Again by Bézout's theorem we conclude that $C \subset H'$ so that C is degenerate.

Suppose now that C is degenerate. Since C is smooth, $\mathrm{span}(C)$ is a hyperplane in $\mathbb{P}^4$. Thus we may think of C as the projection of a rational normal curve $C' \subset \mathbb{P}^5$ from a line $N \subset \mathbb{P}^5$. Now the 3-secant lines to C correspond to 3-secant 3-planes to C in $\mathbb{P}^5$ which contain N. It is a bit simpler to think of this as the set of 3-secant 2-planes which intersect the line N. Since there is one such 2-plane for each point of N, we see that C has a pencil of 3-secant lines. Suppose moreover that each of these 3-secant lines is actually a 4-secant line. If two of these lines, L, M intersect nontrivially, then $P = \text{span}(L, M)$ is a 2-plane and $P \cap C$ has degree at least 7. This contradicts Bézout's theorem unless $C \subset P$, which itself contradicts that C is smooth. Thus all of the 4-secant lines are skew. Now let S be the surface swept out by the 4-secant lines. Then S contains C. Choose any 2-secant line M to C. For each 4-secant line L to C which intersects M, consider the 2-plane span(L, M). If L does not pass through one of the 2 points of intersection of $M \cap C$ then span(L, M) intersects C in at least 6 points, which contradicts Bézout's theorem. Therefore the only lines L which intersect M are the lines through the 2 points of intersection of $M \cap C$. Thus S intersects M in exactly 2 points, i.e., S is a quadric surface. Since S contains a 1-parameter family of skew lines, we conclude that S is a smooth quadric surface. Finally, every smooth quintic rational curve on a smooth quadric surface has divisor class $(1, 4)$ (with respect to some ordering of the two rulings). $\square$

Now suppose that $[C] \in \mathcal{H}^{5,0}(X)$. If C has a 1-parameter family of 4-secant lines, then we see by corollary 9.3 that C is a divisor of type $(1, 4)$ on a smooth quadric Q. But $Q \cap X$ is a divisor of type $(3, 3)$ on Q, it cannot contain a divisor of type $(1, 4)$ as an irreducible component. This contradiction shows there are no such curves.

Define $I = I_{5,0} \subset \mathcal{H}^{5,0}(X) \times \mathbb{G}(1, 4)$ to be the locally closed subvariety parametrizing pairs (C, L) where L is a 3-secant line to C which is not a 4-secant line. Given such a pair, let $Z = L \cap C$. This is a degree 3 divisor on both L and C, so there is a unique isomorphism $\phi : L \to C$ such that $\phi(Z) = Z$. By lemma 6.12 associated to the data C, L, and ϕ there is a unique triple (f, i, j) with $f : \Sigma \to \mathbb{P}^4$ a cubic scroll, $i : C \to \Sigma$ and $j : L \to \Sigma$ factorizations of $C \to \mathbb{P}^4$, and $L \to \mathbb{P}^4$, and such that L is the directrix of Σ.

Conversely, given a cubic scroll $f : \Sigma \to \mathbb{P}^4$ and a factorization $i : C \to \Sigma$ of the inclusion with $i(C) \sim D + 4F$, we see that $f(D)$ is a 3-secant line which is not a 4-secant line. Therefore I also parametrizes triples $(C, f : \Sigma \to \mathbb{P}^4, \phi)$.

Now for each cubic scroll $f : \Sigma \to \mathbb{P}^4$ and $j : C \to \Sigma$ as above, the residual C_2 to $j(C)$ in $f^{-1}(X)$ is a divisor of type $2D + 2F$. We know from subsection 6.3 that such a divisor is a quartic curve of arithmetic genus 0, e.g. a quartic rational curve.

Theorem 9.4. *$\mathcal{H}^{5,0}(X)$ is irreducible of dimension 10. For a general $[C] \in \mathcal{H}^{5,0}(X)$, if $f : \Sigma \to \mathbb{P}^4$ is the unique cubic scroll containing C, the residual curve C_2 to $C \subset f^{-1}(X)$ is a smooth quartic rational curve.*

Proof. Decompose I depending on the *type* of C_2. We say C_2 is the first type if it is a smooth quartic rational curve. We say C_2 is the second type if C_2 is a union of two smooth conics $A \cup B$. We say that C_2 is the third type if it is a union of the directrix and a twisted cubic $D \cup A$. We say that C_2 is the fourth type if it is the union of a conic, the directrix, and a line of the ruling $A \cup D \cup F$. We say that C_2 is the fifth type if C_2 is the union of the double of the directrix and two lines of the ruling $2D \cup F_1 \cup F_2$. Finally, we say that C_2 is of the sixth type if C_2 is the double of a conic. We will label the corresponding locally closed subsets of I by $I_1, \dots, I_6$.

First we show that for each $i > 1$, $\dim I_i \leq 9$.

Second type: Suppose that C_2 is the second type. The scroll $f : \Sigma \to \mathbb{P}^4$ is determined by giving the union of the two conics $A \cup B$ meeting at a point p, and by giving the isomorphism $\phi : A \to B, \phi(p) = p$ induced by the lines of the ruling of Σ. Thus we see that I_2 fibers over the Hilbert scheme of intersecting conics with fibers of dimension 2: the set of isomorphisms is a principal homogeneous space for the 2-dimensional subgroup of automorphisms in $PGL(2)$ which fix a point of $\mathbb{P}^1$. To specify a conic in X it is equivalent to specify a line in X and a 2-plane containing this line (the conic is the residual of the line). Thus to specify two conics intersecting in a point $p \in X$, it is equivalent to specify a pair of lines L, M and then let the 2-planes be span(L, p) and span(M, p). So we see that the Hilbert scheme of intersecting conics is birational to $X \times \text{Sym}^2(F)$, and so has dimension $3 + 2 + 2 = 7$. So I_2 has dimension $2 + 7 = 9$.

Third type: Suppose that C_2 is the third type. To specify the scroll it is equivalent to specify the twisted cubic A, the directrix D which intersects A in a point p, and an isomorphism $\phi : A \to D$ such that $\phi(p) = p$. Thus I_3 fibers over the Hilbert scheme of unions $A \cup D$ with fibers which are 2-dimensional. We have seen that $\mathcal{H}^{3,0}$ has dimension 6, and that the set of lines intersecting a twisted cubic A has dimension 1. Thus the Hilbert scheme of unions $A \cup B$ has dimension 7. So I_3 has dimension $2 + 7 = 9$.

Fourth type: Suppose that C_2 is the fourth type. To specify the scroll it is equivalent to specify the directrix line D, the conic A, a line of ruling F intersecting both D and A (in distinct points), and an isomorphism $\phi : D \rightarrow A$ such that $\phi(F \cap D) = F \cap A$. Thus I_4 fibers over the Hilbert scheme of curves $A \cup D \cup F$ with fibers which are 2-dimensional. To specify $A \cup D \cup F$, it is equivalent to specify $D \cup F$, a point $p \in F$ and the residual line L to A. The dimension of the space of intersecting lines is 3. The dimension of choices for p is 1, and the dimension of choices for L is 2. Thus the dimension of the space of curves $A \cup D \cup F$ is $3 + 1 + 2 = 6$. So I_4 has dimension 8.

Fifth type: Suppose that C_2 is the fifth type. By lemma 6.6, we know that the scroll Σ is determined by the vertical tangent bundle $T \subset T_{\mathbb{P}^4}|_D$. The condition that the intersection of Σ contain the double of D is exactly that the normal bundle $N_{D/\Sigma} \subset N_{D/\mathbb{P}^4}$ is contained in $N_{D/X}$. But this normal bundle is simply the image of T in the quotient $N_{D/\mathbb{P}^4}$ of $T_{\mathbb{P}^4}|_D$. Thus the scrolls Σ such that $\Sigma \cap X$ contains $2D$ are the same as sub-line-bundles $T \subset T_X|_D$ of degree -1. In both of the cases $T_X|_D \cong \mathcal{O}_D(2) \oplus \mathcal{O}_D \oplus \mathcal{O}_D$ and $T_X|_D \cong \mathcal{O}_D(2) \oplus \mathcal{O}_D(1) \oplus \mathcal{O}_D(-1)$ we have that $H^0(C, T_X|_D(1))$ is an 8-dimensional vector space. Moreover, each of the two lines F_1, F_2 of the ruling contained in X imposes two linear conditions on the sections. Since the set of scrolls is the projective space associated to the possible sections, we see that there is at most a 3-dimensional family of scrolls which contain the double of D and F_1, F_2. Therefore the dimension of the space of pairs $([D], \{F_1, F_2\})$ is just the sum of 2 for the line in X, and 1 each for the F_i's. Altogether, we see that $\dim(I_5) \leq 2 + 1 + 1 + 3 = 7$.

Sixth type: Finally we consider the sixth type. By lemma 6.7, to specify a scroll containing a conic $A \subset X$ is the same as giving a sub-line bundle $T \subset T_{\mathbb{P}^4}|_A$ of degree 1. As in the last case, the condition that $\Sigma \cap X$ contain $2A$ is exactly that $T \subset T_X|_A$. The two possibilities for $T_X|_A$ are $\mathcal{O}_A(2) \oplus \mathcal{O}_A(1) \oplus \mathcal{O}_A(1)$ and $\mathcal{O}_A(2) \oplus \mathcal{O}_A(2) \oplus \mathcal{O}_A$. In both cases we see that $H^0(A, T_X|_A(-1))$ is a vector space of dimension 4. Thus there is a $\mathbb{P}^3$ of scrolls Σ such that $\Sigma \cap X$ contains $2A$. Since $\dim(\mathcal{H}^{2,0}(X)) = 4$, we conclude that $\dim(I_6) = 4 + 3 = 7$.

In each case the dimension is at most 9. By proposition 2.1 every irreducible component of $\mathcal{H}^{5,0}$ has dimension at least 10. By corollary 9.3 we know $I \rightarrow \mathcal{H}^{5,0}$ is surjective. So we conclude that the image of $I_1 \rightarrow \mathcal{H}^{5,0}$ is Zariski dense and has dimension at least 10.

Fixing a quartic rational curve $C_2 \subset X$, by lemma 6.8 the set of cubic scrolls $f : \Sigma \rightarrow \mathbb{P}^4$ containing C_2 is equivalent to the set of (basepoint free) g^1_2's on C_2. The set of g^1_2's on C_2 is simply $\mathrm{Sym}^2(C_2) \cong \mathbb{P}^2$. We see that I_1 fibers over $\mathcal{H}^{4,0}$ as a $\mathbb{P}^2$-fibration. By theorem 7.3, $\mathcal{H}^{4,0}$ is irreducible of dimension 8. Thus I_1 is irreducible of dimension 10. So the image of $I_1 \rightarrow \mathcal{H}^{5,0}$ is irreducible of dimension at most 10. On the other hand we know the image has dimension at least 10. So $\mathcal{H}^{5,0}$ is irreducible of dimension 10. $\square$

10. Quintic Curves of Genus 2

By Bézout's theorem, X cannot contain a plane curve of degree $d > 3$. Thus the next case after quintic elliptic curves is quintic curves of genus 2.

Suppose $C \subset X$ is a quintic curve of genus 2. Let H denote the hyperplane class on C. Since $\deg(H) = 5 > 2 = \deg(K_C)$, we conclude that $H^1(C, \mathcal{O}_C(H)) = 0$. Thus by Riemann-Roch we have

$$\dim H^0(C, \mathcal{O}_C(H)) = \deg(H) + 1 - g = 5 + 1 - 2 = 4. \tag{65}$$

Thus the complete linear system $|H|$ is a $\mathbb{P}^3$, i.e., C is contained in a $\mathbb{P}^3$ inside $\mathbb{P}^4$. Moreover, by Riemann-Roch we also have that

$$H^0(C, \mathcal{O}_C(2H)) = 10 + 1 - 2 = 9 < 10 = H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)). \tag{66}$$

Therefore C is contained in a quadric surface $C \subset S$. Now $S \cap X$ is a Cartier divisor of degree 6 on S. Since C is degree 5, the residual of $C \subset S \cap X$ is a divisor of degree 1, i.e., a line. Therefore every quintic genus 2 curve is residual to a line $L \subset X$ in a quadric surface.

Let $\mathbb{P}(Q^\vee) \rightarrow F$ denote the $\mathbb{P}^2$-bundle over F parametrizing pairs $([L], [H])$ where $L \subset H \subset \mathbb{P}^4$ is a line contained in a hyperplane contained in $\mathbb{P}^4$ such that $L \subset X$. Let $U \rightarrow \mathbb{P}(Q^\vee)$ denote the $\mathbb{P}^6$-bundle parametrizing triples $([L], [H], [S])$ where $S \subset H$ is a quadric surface containing L. The universal quadric surface $\widetilde{S} \subset U \times \mathbb{P}^4$ is a Cartier divisor inside the pullback of the universal hypersurface $\widetilde{H} \subset U \times \mathbb{P}^4$. Since U is smooth so is $\widetilde{H}$, therefore $\widetilde{S}$ is a local complete intersection. Next, $U \times X \subset U \times \mathbb{P}^4$ is a Cartier divisor.

Since $\widetilde{S}$ and $U \times X$ have no irreducible component in common, we see that $D := \widetilde{S} \cap U \times X \subset \widetilde{S}$ is a Cartier divisor locally cut out by a regular element, so D is also a local complete intersection. In particular, D is Gorenstein.

Let $D_1 \subset D$ denote the pullback from F of the universal line in X. Then $D_1 \to U$ is smooth, therefore D_1 is smooth. In particular D_1 is Cohen-Macaulay. Therefore by corollary 2.7, the residual D_2 to D_1 in D is a flat family of Cohen-Macaulay schemes. By specializing to a point $([L],[H],[S])$ with S smooth, we see that the general fiber of D_2 is a smooth quintic genus 2 curve in X. Thus there is an induced map $f : U \to \mathrm{Hilb}_{5t-1}(X)$. We have seen that this map is a bijection over $\mathcal{H}^{5,2}(X)$. Therefore the preimage $\overline{f^{-1}(\mathcal{H}^{5,2}(X))}$ is precisely the normalization $\widetilde{\mathcal{H}^{5,2}}(X)$. Since U is irreducible, we also conclude that $f(U) = \overline{\mathcal{H}^{5,2}}(X)$. Thus we have the following result.

Theorem 10.1. *The normalization $\widetilde{\mathcal{H}^{5,2}}(X)$ of $\mathcal{H}^{5,2}(X)$ is a smooth, connected variety of dimension* 10.

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