LOW DEGREE COMPLETE INTERSECTIONS ARE RATIONALLY SIMPLY CONNECTED

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Abstract. We prove smooth, low degree complete intersections in projective space are rationally simply connected in a strong sense. Using a result of Hassett, we deduce weak approximation for smooth, low degree complete intersections defined over the function field of a curve.

Contents

1. Main results

sec-intro

There is a well-developed algebro-geometric analogue of path connectedness: A nonempty, projective, complex variety is *rationally connected* if each pair of closed points is contained in a rational curve, cf. [Kol96], [Deb01]. To give a flavor of this notion, we mention that a smooth complete intersection of c hypersurfaces in $\mathbb{P}^n$ of degrees $d_1, \dots, d_c$ is rationally connected if and only if

$$n \geq \sum_{i=1}^{c} d_i.$$

A path connected space is simply connected if the space of based paths is path connected. Barry Mazur suggested an algebro-geometric analogue of simple connectedness: a rationally connected variety should be *rationally simply connected* if the space of based, 2-pointed rational curves of suitably positive class is rationally connected.

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We must assume the curve class is suitably positive or else the space may be empty or atypical in some other way. Similarly we must assume the two basepoints are general. Finally, since the space of 2-pointed rational curves is typically not compact, it should be compactified. We compactify with the Kontsevich moduli space of stable maps. For a projective scheme X and nonnegative integers g, m and e, the *Kontsevich space* $\overline{\mathcal{M}}_{g,m}(X,e)$ parametrizes data $(C, p_1, \dots, p_m, h)$ of

(i) a proper, connected, at-worst-nodal, arithmetic genus g curve C,
(ii) an ordered collection $p_1, \dots, p_m$ of distinct smooth points of C,
(iii) and a morphism $h : C \to X$ whose image has degree e

such that $(C, p_1, \dots, p_m, h)$ has only finitely many automorphisms, cf. [FP97]. Oftentimes this is refined by using a *curve class* β in X in place of the degree e. There is an evaluation morphism

$$\mathrm{ev} : \overline{\mathcal{M}}_{g,m}(X,e) \to X^m, \quad (C, p_1, \dots, p_m, h) \mapsto (h(p_1), \dots, h(p_m)).$$

In particular, a general fiber of

$$\mathrm{ev} : \overline{\mathcal{M}}_{0,2}(X,e) \to X \times X$$

is an algebraic analogue of the space of based paths in topology.

thm-main

Theorem 1.1. *Let X be a smooth complete intersection in $\mathbb{P}^n$ of type $\underline{d} = (d_1, \dots, d_c)$. For simplicity, assume all $d_i \geq 2$. The variety X is rationally simply connected if*

$$n + 1 \geq \sum_{i=1}^{c} d_i^2$$

with the one exception $n = 3$ and $\underline{d} = (2)$, i.e., a quadric surface. Also $\mathbb{P}^n$ is rationally simply connected for $n \geq 2$.

To be precise, for every $e \geq 2$ there is a canonically defined irreducible component $M_{e,2} \subset \overline{\mathcal{M}}_{0,2}(X,e)$ such that the restriction

$$ev : M_{e,2} \to X \times X$$

is dominant with rationally connected general fiber.

A general point of $M_{e,2}$ parametrizes embedded, smooth rational curves. Moreover, if X is general then $\overline{\mathcal{M}}_{0,2}(X,e)$ is irreducible, i.e., $M_{e,2} = \overline{\mathcal{M}}_{0,2}(X,e)$, cf. [HRS04].

What of m-pointed curves with $m > 2$? The fiber of

$$\mathrm{ev} : \overline{\mathcal{M}}_{0,m}(X,e) \to X^m$$

over a general point $(p_1, \dots, p_m)$ is an algebraic analogue of the space of based m-pointed paths to a path connected topological space X

$$\mathcal{P}_{p_1,\dots,p_m} = \{(\gamma, t_1, \dots, t_m) | \gamma : [0,1] \to X \text{ continuous },$$
$$0 = t_1 < t_2 < \dots < t_m = 1, \ \gamma(t_1) = p_1, \dots, \gamma(t_m) = p_m\}.$$

Of course this is homotopy equivalent to the $(m-1)$-fold product of the space of based paths in X. So if X is simply connected, then $\mathcal{P}_{p_1,\dots,p_m}$ is path connected.

Unfortunately the analogue in algebraic geometry is unclear (and quite possibly false): rational connectedness of the space of based m-pointed rational curves does not obviously follow from rational connectedness of the space of based 2-pointed rational curves. However, the methods for proving rational connectedness of the space of based 2-pointed rational curves often do prove rational connectedness of the space of based m-pointed rational curves.

thm-main2

Theorem 1.2. *Let X be a smooth complete intersection in $\mathbb{P}^n$ of type $\underline{d} = (d_1, \dots, d_c)$. For simplicity, assume all $d_i \geq 2$. The variety X is strongly rationally simply connected if*

$$n + 1 \geq \sum_{i=1}^{d} (2d_i^2 - d_i).$$

Also linear varieties of dimension ≥ 2 and smooth quadric hypersurfaces of dimension ≥ 3 are rationally simply connected.

To be precise, for every integer $m \geq 2$ and every integer $e \geq 4m - 6$ there is a canonically defined irreducible component $M_{e,m} \subset \overline{\mathcal{M}}_{0,m}(X, e)$ such that the restriction

$$ev : M_{e,m} \to X^m$$

is dominant with rationally connected general fiber.

The irreducible component $M_{e,m}$ above is the unique one dominating the irreducible component $M_{e,1}$ from Theorem 1.1, in particular a general point parametrizes a smooth, embedded curve. The inequality in Theorem 1.2 is worse than the inequality in Theorem 1.1. We expect Theorem 1.2 holds whenever

$$n \geq \sum_{i=1}^{c} d_i^2.$$

This would be the analogue of the theorem above that a smooth complete intersection is rationally connected when

$$n \geq \sum_{i=1}^{c} d_i.$$

The methods of this paper combined with [Sta04] prove the following.

thm-main3

Theorem 1.3. *A* general *degree d hypersurface in $\mathbb{P}^n$ is strongly rationally simply connected if*

$$n \geq d^2,$$

i.e., the locus parametrizing strongly rationally simply connected hypersurfaces is a dense Zariski open subset of the parameter space of all degree d hypersurfaces.

Weak approximation. There are applications of these theorems to *weak approximation* for varieties defined over the function field of a curve. There are equivalent algebraic and geometric formulations of weak approximation.

Weak approximation: Algebraic formulation. First let R be a semilocal Dedekind domain with maximal ideals $\mathfrak{m}_1, \dots, \mathfrak{m}_n$ and fraction field K. A smooth K-algebraic space (or K-scheme if you prefer) X is said to satisfy *weak approximation with respect to R, and $\mathfrak{m}_1, \dots, \mathfrak{m}_n$* if for every smooth R-algebraic space X_R with generic fiber $X_K \cong X$, the image of the induced map of rational points

$$X_R(R) \to \prod_{i=1}^{n} X_R(\widehat{R}_{\mathfrak{m}_i})$$

is dense for the adic topology. In other words, for every sequence $(\widehat{\sigma}_1, \dots, \widehat{\sigma}_n)$ of elements $\widehat{\sigma}_i \in X_R(\widehat{R}_{\mathfrak{m}_i})$, and for every integer N, there exists an element $\sigma \in X_R(R)$ such that for every $i = 1, \dots, n$,

$$\sigma \cong \widehat{\sigma}_i(\text{ modulo } \mathfrak{m}_i^N).$$

More generally, let R be a Dedekind domain with fraction field K, not necessarily semilocal. A smooth K-algebraic space X is said to satisfy *weak approximation with respect to* R if it satisfies weak approximation with respect to every semilocal localization of R.

Weak approximation: Geometric formulation. Let B be a connected, smooth curve over an algebraically closed field, let Z be a proper closed subset of B, and let K be the function field of B. A smooth algebraic space X over K is said to satisfy *weak approximation with respect to* B *and* Z if for every smooth morphism

$$\pi : \mathcal{X} \to B$$

with generic fiber $\mathcal{X} \otimes_{\mathcal{O}_B} K \cong X$ and for every morphism of B-schemes $\sigma_Z : Z \to \mathcal{X}$, there exists an open subset U of B containing Z and a commutative diagram of B-schemes

$$\begin{array}{ccc} Z & \xrightarrow{\sigma_Z} & \mathcal{X} \\ \subset\big\downarrow & & \big\downarrow= \\ U & \xrightarrow{\sigma_U} & \mathcal{X} \end{array}$$

In other words, there exists a B-morphism $\sigma_U : U \to \mathcal{X}$ such that $\sigma_U|_Z = \sigma_Z$.

More generally, let B be a connected, smooth curve over an algebraically closed field and let K be the function field of B. A smooth algebraic space X over K is said to satisfy *weak approximation with respect to* B if for every proper closed subset Z of B, it satisfies weak approximation with respect to B and Z.

The connection between the algebraic formulation and geometric formulation comes by setting R to be the coordinate ring of any open affine subset of B containing Z. Although nontrivial, the geometric formulation holds for arbitrary proper closed subschemes Z if and only if it holds for *reduced* proper closed subschemes, i.e., finite sets of closed points of B, cf. [HT06, Proposition 11].

Weak approximation and rational simple connectedness. The strong version of rational simple connectedness is related to weak approximation by the following beautiful theorem of Hassett.

thm-Hassett **Theorem 1.4.** [Has04] *Let K be the function field of a curve over an algebraically closed field of characteristic 0. A smooth, projective K-scheme X satisfies weak approximation if the geometric generic fiber $X \otimes_K \overline{K}$ is strongly rationally connected.*

More precisely, X satisfies weak approximation if for every sufficiently positive and divisible integer m there exists a Galois-invariant curve class β in $X \otimes_K \overline{K}$ and a Galois-invariant, irreducible, closed subset $M_{\beta,m}$ of $\overline{\mathcal{M}}_{0,m}(X \otimes_K \overline{K}, \beta)$ such that some point of $M_{\beta,m}$ parametrizes an irreducible curve and the restriction to $M_{\beta,m}$ of the evaluation map

$$ev|_M : M_{\beta,m} \subset \overline{\mathcal{M}}_{0,m}(X \otimes_K \overline{K}, \beta) \to (X \otimes_K \overline{K})^m$$

is dominant with rationally connected geometric generic fiber.

cor-main3

Corollary 1.5. *Let K be the function field of a curve over an algebraically closed field of characteristic 0. Let $X \subset \mathbb{P}^n_K$ be a smooth complete intersection of type $\underline{d} = (d_1, \dots, d_c)$. If*

$$n + 1 \geq \sum_{i=1}^{c} (2d_i^2 - d_i)$$

then X satisfies weak approximation.

If $n \geq d^2$ and if $X \subset \mathbb{P}^n_K$ is a "general" hypersurface of degree d, then X satisfies weak approximation. To be precise, there exists a Zariski dense open subset U in the parameter space $\mathbb{P}^N_{\overline{\mathbb{Q}}}$, $N = \binom{n+d}{d} - 1$, of degree d hypersurfaces in $\mathbb{P}^n$ defined over $\overline{\mathbb{Q}}$. If the K-valued point of $\mathbb{P}^N_{\overline{\mathbb{Q}}}$ associated to X lies in U, then X satisfies weak approximation.

At least conjecturally, rational simple connectedness should imply existence of rational points for varieties defined over the function field of a surface.

prin-deJong

Principle 1.6 (de Jong)**.** A proper, smooth variety defined over the function field of a surface over an algebraically closed field of characteristic 0 has a rational point if the base-change to the algebraic closure of the function field satisfies a strong version of rational connectedness and a certain Brauer obstruction vanishes.

In a forthcoming paper, we will present de Jong's strategy for proving Principle 1.6 and prove Principle 1.6 under some additional hypotheses.

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subsec-ov

1.1. **Overview.** In fact, the methods of this paper are not particular to complete intersections, although this is the application we have in mind. Theorems 1.1 and 1.2 are special cases of the following theorems about Fano manifolds whose Chern classes satisfy certain inequalities, together with linear sections of these Fano manifolds.

thm-trc1

Theorem 1.7. *Let $(X, \mathcal{O}(1))$ be a smooth, projective variety. Let α be an $\mathcal{O}(1)$-degree 1 curve class. Assume $c_1(T_X) = \langle c_1(T_X), \alpha \rangle c_1(\mathcal{O}(1))$, i.e., $c_1(T_X)$ is a multiple of $c_1(\mathcal{O}(1))$.*

Assume $(X, \mathcal{O}(1))$ is neither a linear variety nor a quadric hypersurface. Assume that

$$ch_2(T_X) = \frac{\langle 2ch_2(T_X), \Pi \rangle}{2} c_1(\mathcal{O}(1))^2$$

for some integer $\langle 2ch_2(T_X), \Pi \rangle$, i.e., $2ch_2(T_X)$ is a multiple of $c_1(\mathcal{O}(1))^2$. Assume that

$$\langle c_1(T_X), \alpha \rangle > 0,$$
$$\langle 2ch_2(T_X), \Pi \rangle \geq 0$$

and

$$dim(X) \geq 2(dim(X) - \langle c_1(T_X), \alpha \rangle - 1) + 4.$$

Assume there exists a smooth, projective variety Y such that X is a codimension c linear section of Y. Assume

$$c \geq 2(\mathit{dim}(X) - \langle c_1(T_X), \alpha\rangle - 1) - \mathit{dim}(X) + 5,$$

$$c > \mathit{dim}(X) - 2(\mathit{dim}(X) - \langle c_1(T_X), \alpha\rangle - 1) - 2\langle 2\mathit{ch}_2(T_X), \Pi\rangle + 2,$$

and

$$\mathit{CH}^p(Y) = \mathbb{Z}\{c_1(\mathcal{O}(1))^p\}, \text{ for } 0 \leq p \leq 2(\mathit{dim}(X) - \langle c_1(T_X), \alpha - 1) + 4.$$

By Barth's theorems, [Bar70]*, this last condition holds if*

$$c \geq 4(\mathit{dim}(X) - \langle c_1(T_X), \alpha\rangle - 1) - \mathit{dim}(X) + (h^0(X, \mathcal{O}(1)) - \mathit{dim}(X) - 2) + 9.$$

Then, first, there exists a unique irreducible component $M_{\alpha,1}$ of $\overline{\mathcal{M}}_{0,1}(X,\alpha)$ dominating X, and a general fiber of $\overline{\mathcal{M}}_{0,1}(X,\alpha)$ over X is geometrically irreducible and geometrically rationally connected. Second, for every integer $e \geq 2$ there is a canonically defined irreducible component $M_{e\cdot\alpha,2}$ of $\overline{\mathcal{M}}_{0,2}(X, e\alpha)$, constructed in Notation 3.7, for which the fiber of the restriction

$$\mathit{ev}|_M : M_{e\cdot\alpha,2} \subset \overline{\mathcal{M}}_{0,2}(X, e\alpha) \to X^2$$

over a general point of X^2 is geometrically irreducible and geometrically rationally connected.

The same conclusion holds for every linear variety of dimension ≥ 2 and every smooth quadric variety of dimension ≥ 3.

thm-trc2

Theorem 1.8. *Let $(X, \mathcal{O}(1))$ be a smooth, projective variety. Let α be an $\mathcal{O}(1)$-degree 1 curve class. Assume $c_1(T_X) = \langle c_1(T_X), \alpha\rangle c_1(\mathcal{O}(1))$, i.e., $c_1(T_X)$ is a multiple of $c_1(\mathcal{O}(1))$.*

Assume $(X, \mathcal{O}(1))$ is neither a linear variety nor a quadric hypersurface. Assume that

$$\mathit{ch}_2(T_X) = \frac{\langle 2\mathit{ch}_2(T_X), \Pi\rangle}{2} c_1(\mathcal{O}(1))^2$$

for some integer $\langle 2\mathit{ch}_2(T_X), \Pi\rangle$, i.e., $2\mathit{ch}_2(T_X)$ is a multiple of $c_1(\mathcal{O}(1))^2$. Assume that

$$\langle c_1(T_X), \alpha\rangle > 0,$$

$$2\langle 2\mathit{ch}_2(T_X), \Pi\rangle \geq \langle c_1(T_X), \alpha\rangle$$

and

$$\mathit{dim}(X) \geq 4(\mathit{dim}(X) - \langle c_1(T_X), \alpha\rangle - 1) + 4.$$

Assume there exists a smooth, projective variety Y such that X is a codimension c linear section of Y. Assume

$$c \geq 4(\mathit{dim}(X) - \langle c_1(T_X), \alpha\rangle - 1) - \mathit{dim}(X) + 4,$$

$$c > \mathit{dim}(X) - 2(2\langle 2\mathit{ch}_2(T_X), \alpha\rangle - \langle c_1(T_X), \alpha\rangle) - 4(\mathit{dim}(X) - \langle c_1(T_X), \alpha\rangle - 1) + 6,$$

and

$$\mathit{CH}^p(Y) = \mathbb{Z}\{c_1(\mathcal{O}(1))^p\}, \text{ for } 0 \leq p \leq 2(\mathit{dim}(X) - \langle c_1(T_X), \alpha - 1) + 4.$$

By Barth's theorems, [Bar70]*, this last condition holds if*

$$c \geq 4(\mathit{dim}(X) - \langle c_1(T_X), \alpha\rangle - 1) - \mathit{dim}(X) + (h^0(X, \mathcal{O}(1)) - \mathit{dim}(X) - 2) + 9.$$

Then, first, there exists a unique irreducible component $M_{\alpha,1}$ of $\overline{\mathcal{M}}_{0,1}(X,\alpha)$ dominating X, and a general fiber of $\overline{\mathcal{M}}_{0,1}(X,\alpha)$ over X is geometrically irreducible and geometrically rationally connected. Next, for every integer $m \geq 2$, for every integer $e \geq 4m-6$, there is a canonically defined irreducible component $M_{e\cdot\alpha,m}$ of $\overline{\mathcal{M}}_{0,m}(X,e\alpha)$, constructed in Notation 3.7, for which the fiber of the restriction

$$ev|_M : M_{e\cdot\alpha,m} \subset \overline{\mathcal{M}}_{0,m}(X,e\alpha) \to X^m$$

over a general point of X^m is geometrically irreducible and geometrically rationally connected.

The same conclusion holds for every linear variety of dimension ≥ 2 and every smooth quadric variety of dimension ≥ 3.

2. Elementary results about complete intersections

sec-eci

We gather here some elementary, well-known results about complete intersections which we use. The first result we use is the computation of the Chern character of a complete intersection.

lem-Chern

Lemma 2.1. *For a smooth, complete intersection X of type $\underline{d} = (d_1,\dots,d_c)$ in $\mathbb{P}^n$, the Chern character equals*

$$ch(T_X) = n - c + \sum_{k=1}^{n-c}\left(n+1-\sum_{i=1}^{c} d_i^k\right)\frac{c_1(\mathcal{O}(1))^k}{k!}.$$

Proof. There is a short exact sequence

$$0 \longrightarrow T_X \longrightarrow T_{\mathbb{P}^n}|_X \longrightarrow \oplus_{i=1}^{c}\mathcal{O}_{\mathbb{P}^n}(d_i)|_X \longrightarrow 0.$$

Therefore, by the axioms of the Chern character,

$$\mathrm{ch}(T_X) = \mathrm{ch}(T_{\mathbb{P}^n})_X - \sum_{i=1}^{c} e^{d_i c_1(\mathcal{O}(1))} = \mathrm{ch}(T_{\mathbb{P}^n}|_X) - c - \sum_{k=1}^{n-c}\sum_{i=1}^{c} d_i^k \frac{c_1(\mathcal{O}(1))^k}{k!}.$$

Also, by the Euler sequence,

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n+1)} \longrightarrow T_{\mathbb{P}^n} \longrightarrow 0$$

and the axioms of the Chern character,

$$\mathrm{ch}(T_{\mathbb{P}^n}) = (n+1)e^{c_1(\mathcal{O}(1))} - 1 = n + \sum_{k=1}^{n}(n+1)\frac{c_1(\mathcal{O}(1))^k}{k!}.$$

Combined, these formulas give the formula for $\mathrm{ch}(T_X)$. □

The second result we use is that every complete intersection is a linear section of a smooth complete intersection of arbitrarily large dimension.

lem-ext1

Lemma 2.2. *Let $Y \subset \mathbb{P}^N$ be a smooth, quasi-projective variety, let Π be a linear subspace of $\mathbb{P}^N$ intersecting Y transversally in a smooth subvariety Y_Π, let d be a positive integer, and let F be a global section of $\mathcal{O}_Y(d)$ whose zero locus $Z = \mathbb{V}(F)$ does not contain Y_Π. Denote the ideal sheaf of Π in $\mathbb{P}^N$ by $\mathcal{I}_\Pi$.*

If

$$codim(\Pi \subset \mathbb{P}^N) \geq dim(Sing(\Pi \cap Z)) + 1,$$

then there exists a global section G of $\mathcal{I}_\Pi \cdot \mathcal{O}_{\mathbb{P}^N}(d)$ such that the zero locus $Z' = \mathbb{V}(F+G|_Y)$ is smooth. Thus Z' is a smooth Cartier divisor on Y such that $Z' \cap \Pi = Z \cap \Pi$.

Proof. The base locus of the linear subsystem L of $|\mathcal{O}_Y(d)|$ spanned by F and the image of $|\mathcal{I}_\Pi \cdot \mathcal{O}_{\mathbb{P}^N}(d)|$ is contained in $Z \cap \Pi$. By Bertini's theorem, cf. [Jou83, Théorème I.6.3.2], a general member Z' of L is smooth away from $Z \cap \Pi$. Thus to prove a general member is everywhere smooth it suffices to prove there exists at least one member that is smooth at every point of $Z \cap \Pi$.

Denote the "bad locus" $\mathrm{Sing}(Z \cap \Pi)$ by B. Denote by a the codimension $\mathrm{codim}(Y_\Pi \subset Y)$. Denote by b the dimension $\dim(B)$. Choose homogeneous coordinates $[X_0, \ldots, X_N]$ on $\mathbb{P}^N$ so that

$$\Pi = \mathbb{V}(X_{N+1-a}, \ldots, X_N).$$

Because $a > b$, there exist elements $G_{N+1-a}, \ldots, G_N \in H^0(\mathbb{P}^N, \mathcal{O}(d-1))$ so that

$$B \cap \mathbb{V}(G_{N+1-a}, \ldots, G_N) = \emptyset.$$

For each scalar λ, denote

$$F_\lambda := F + \lambda \sum_{i=1}^{a} X_{N-a+i} G_{N-a+i}$$

and denote the zero locus of F_λ by Z_λ. The claim is that Z_λ is smooth at every point of $Z \cap \Pi$ for a general choice of the scalar λ.

Associated to F_λ there is a derivative map

$$dF_\lambda : T_Y|_{Z_\lambda} \to \mathcal{O}_Y(d)|_{Z_\lambda}.$$

The singular locus of Z_λ is the support of the cokernel of dF_λ. Thus Z_λ is smooth at every point of $Z \cap \Pi$ if and only if the induced map

$$dF_\lambda : T_Y|_{Z\cap\Pi} \to \mathcal{O}_Y(d)|_{Z\cap\Pi}$$

is surjective. The restriction to the subsheaf $T_{Y_\Pi}|_{Z\cap\Pi}$ is precisely

$$d(F|_{Y_\Pi}) : T_{Y_\Pi}|_{Z\cap\Pi} \to \mathcal{O}_{Y_\Pi}(d)|_{Z\cap\Pi}.$$

Denote the cokernel by

$$C := \mathrm{Coker}(d(F|_{Y_\Pi})).$$

This is a locally principal coherent sheaf supported on B. There is an induced map

$$\widetilde{dF}_\lambda : N_{Y_\Pi/Y}|_{Z\cap\Pi} \to C.$$

The map $\widetilde{dF}_\lambda$ is *linear* in λ. To be precise there exists a map

$$\widetilde{G} : N_{Y_\Pi/Y}|_{Z_\Pi} \to C$$

such that

$$\widetilde{dF}_\lambda = \widetilde{dF}_0 + \lambda \widetilde{G}.$$

The sequence $(X_{N+1-a}, \ldots, X_N)$ is an ordered basis for $\mathcal{I}_\Pi(1)$, i.e., the induced map $\mathcal{O}^{\oplus a} \to \mathcal{I}_\Pi(1)$ is an isomorphism. Transposing, twisting by $\mathcal{O}(1)$ and restricting to Y gives an isomorphism $N_{Y_\Pi/Y} \cong \mathcal{O}_{Y_\Pi}(1)^{\oplus a}$. Inverting the isomorphism and composing with $\widetilde{G}$ gives the map

$$\widehat{G} : \mathcal{O}_{Y_\Pi}(1)^{\oplus a} \to \mathcal{O}_{Y_\Pi}(d)$$

whose restriction to the i^{th} summand $\mathcal{O}_{Y_\Pi}(1)$ of $\mathcal{O}_{Y_\Pi}(1)^{\oplus a}$ is the restriction to Y_Π of the multiplication map

$$G_{N-a+i} : \mathcal{O}(1) \to \mathcal{O}(d).$$

For scalars λ, μ, consider the map

$$\mu \widetilde{dF} + \lambda \widetilde{G} : N_{Y_\Pi/Y}|_{Z_\Pi} \to C.$$

For $\mu = 0$ and $\lambda = 1$, the map is surjective because

$$B \cap \mathbb{V}(G_{N+1-a}, \ldots, G_N) = \emptyset.$$

Therefore, by upper-semicontinuity, the map is surjective for general choice of (λ, μ). In particular, there exists $\mu \neq 0$ such that the map is surjective. But then the map is also surjective for $(\lambda/\mu, 1)$. Thus $Z_{\lambda/\mu}$ is smooth at every point of Z_Π. □

notat-ext1

Definition 2.3. Let P be a pure-dimensional variety. A sequence $Y_1, \ldots, Y_r$ of locally closed, pure-dimensional subvarieties intersect *dimensionally transversally* if either $Y_1 \cap \cdots \cap Y_r$ is empty or else it is pure-dimensional of dimension

$$\dim(Y_1 \cap dots \cap Y_r) = \dim(Y_1) + \cdots + \dim(Y_r) - (r-1)\dim(P).$$

lem-ext2

Lemma 2.4. *Let $(Y, \Pi, r, \underline{d}, F_1, \ldots, F_r)$ be a datum of a smooth, quasi-projective variety $Y \subset \mathbb{P}^N$, a linear subspace Π of $\mathbb{P}^N$ intersecting Y transversally in a smooth subvariety Y_Π, a positive integer r, a sequence of positive integers $\underline{d} = (d_1, \ldots, d_r)$, and for $i = 1, \ldots, r$, a global section F_i of $\mathcal{O}_Y(d_i)$ with zero locus $Y_i = \mathbb{V}(F_i)$ such that $Y_1, \ldots, Y_r$ and Π intersect dimensionally transversally. Denote the ideal sheaf of Π in $\mathbb{P}^N$ by $\mathcal{I}_\Pi$.*

If

$$\mathit{codim}(\Pi \subset \mathbb{P}^N) \geq (2^r - 1)\mathit{dim}(Y_0 \cap \Pi) - 2^r + r + 1$$

then for $i = 1, \ldots, r$ there exist global sections G_i of $\mathcal{I}_\Pi \cdot \mathcal{O}_{\mathbb{P}^N}(d_i)$ giving Cartier divisors $Y_i' = \mathbb{V}(F_i + G_i|_Y)$ such that for every $k = 1, \ldots, r$ the intersection $Y_1' \cap \cdots \cap Y_k'$ is smooth.

Proof. This is proved by induction on r. The base case, $r = 1$ is Lemma 2.2. Thus, by way of induction, assume $r > 1$ and assume the result is known for $r - 1$. By Bertini's theorem, cf. [Jou83, Théorème I.6.3], for a general linear space Λ of dimension

$$\dim(\Lambda) = \dim(\Pi) + \dim(Y \cap \Pi)$$

containing Π, the intersection $Y \cap \Lambda$ is transverse and smooth. The singular locus of $Y_1 \cap \Pi$ has dimension

$$\dim(\mathrm{Sing}(Y_1 \cap \Pi)) \leq \dim(Y_1 \cap \Pi) = \dim(Y \cap \Pi) - 1.$$

Thus

$$\mathrm{codim}(\Pi \subset \Lambda) > \dim(\mathrm{Sing}(\Pi \cap Y_1)).$$

Thus Lemma 2.2 implies there exists a global section G_1 of $\mathcal{I}_\Pi \cdot \mathcal{O}_{\mathbb{P}^N}(d)$ such that the zero locus $Y_{1,\Lambda}' = \mathbb{V}(F_1|_\Lambda + G_1|_\Lambda)$ is smooth. Since $Y_{1,\Lambda}'$ is smooth, a second application of Lemma 2.2 implies there exists G as above such that also $Y_1' = \mathbb{V}(F_1 + G_1|_Y)$ is smooth.

For each of $i = 2, \ldots, r$, after replacing F_i by $F_i + G_i|_Y$ for a general member G_i of $\mathcal{I}_\Pi \cdot \mathcal{O}_{\mathbb{P}^N}(d_i)$, $Y_1, \ldots, Y_r$ and Λ intersect dimensionally transversally.

Now form the new datum $(Z, \Lambda, r-1, \underline{e}, H_1, \dots, H_{r-1})$ where $Z = Y_1'$, $\underline{e} = (d_2, \dots, d_r)$, and for $i = 1, \dots, r-1$, $H_i = F_{i+1}|_Z$. For each $i = 1, \dots, r-1$, denote $Z_i = \mathbb{V}(H_i) = Z_1' \cap Y_{i+1}$. The claim is that this datum also satisfies the hypotheses of the lemma. One hypothesis that requires verification is:

$$\dim(\mathbb{P}^N) - \dim(\Lambda) \geq (2^{r-1} - 1)\dim(Z \cap \Lambda) - 2^{r-1} + r.$$

First observe that

$$\dim(Y_1 \cap \Lambda) = \dim(Y \cap \Lambda) - 1 =$$

$$(\dim(Y \cap \Pi) + \dim(\Lambda) - \dim(\Pi)) - 1 = 2\dim(Y \cap \Pi) - 1.$$

By construction of Λ and the hypothesis on $\dim(\Pi)$,

$$\dim(\mathbb{P}^N) - \dim(\Lambda) = \dim(\mathbb{P}^N) - \dim(\Pi) - \dim(Y \cap \Pi) \geq$$

$$[(2^r - 1)\dim(Y \cap \Pi) - 2^r + r + 1] - \dim(Y \cap \Pi) =$$

$$(2^r - 2)\dim(Y \cap \Pi) - 2^r + r + 1 = (2^{r-1} - 1)(2\dim(Y \cap \Pi)) - 2^r + r + 1.$$

Using the identity above, this is

$$(2^{r-1} - 1)\dim(Y_1 \cap \Lambda) + 2^{r-1} - 1 - 2^r + r + 1.$$

Using the elementary identity that

$$2^{r-1} - 1 - 2^r + r + 1 = -(2^r - 2^{r-1}) + r = -2^{r-1} + r,$$

it follows that

$$\dim(\mathbb{P}^N) - \dim(\Lambda) \geq (2^{r-1} - 1)\dim(Z \cap \Lambda) - 2^{r-1} + r.$$

So the datum satisfies the hypotheses of the lemma.

By the induction assumption, for each $i = 1, \dots, r-1$, there exist elements J_i of $\mathcal{I}_\Lambda \cdot \mathcal{O}_{\mathbb{P}^N}(d_{i+1})$ giving Cartier divisors $Z_i' = \mathbb{V}(H_i + J_i|_Z)$ such that for every $k = 1, \dots, r-1$, the intersection $Z_1' \cap \dots \cap Z_k'$ is smooth. In other words, setting $G_{i+1} = J_i$, for the Cartier divisors $Y_{i+1}' = \mathbb{V}(F_{i+1} + G_{i+1}|_Y)$,

$$Y_1' \cap Y_2' \cap \dots \cap Y_{k+1}' = Z_1' \cap \dots \cap Z_k'$$

is smooth. Thus $(G_1, G_2, \dots, G_r)$ satisfy the conditions of the lemma. Therefore the lemma is proved by induction on r. $\square$

lem-ext3

Lemma 2.5. *Let X be a complete intersection, not necessarily smooth, in $\mathbb{P}^n$ of type $\underline{d} = (d_1, \dots, d_r)$. For every integer*

$$c \geq (2^r - 1)n - 2^r + r + 1,$$

for every linear embedding $\mathbb{P}^n \subset \mathbb{P}^{n+c}$, there exists a smooth complete intersection X' in $\mathbb{P}^{n+c}$ of type $\underline{d}$ such that $X = \mathbb{P}^n \cap X'$.

Proof. This follows from Lemma 2.4 by setting $\mathbb{P}^N = \mathbb{P}^{n+c}$, $\Pi = \mathbb{P}^n$, $Y = \mathbb{P}^N$, and, for $i = 1, \dots, r$, letting F_i be a general section of $\mathcal{I}_X \cdot \mathcal{O}_{\mathbb{P}^N}(d_i)$. $\square$

rmk-ext3

Remark 2.6. The inequality in Lemma 2.5 is certainly not the best possible, but it suffices for our purposes.

3. Canonical irreducible components

sec-cic

The space of rational curves of a given class in a rationally connected variety may be reducible. However, often there is a distinguished irreducible component. In this section we present a construction associating to a distinguished component for one curve class α and a positive integer e, a distinguished component for the curve class $e\alpha$. In fact, there are two constructions: the first uses multiple covers and the second uses reducible curves. The main result of this section, Lemma 3.5, proves these constructions agree under a suitable hypothesis.

Lemma 3.1. [Kol96, Theorem II.7.6] *Let $f : C \to X$ be a stable map such that every component of C is contracted or free. Then $\overline{\mathcal{M}}_{0,0}(X)$ is smooth at $[(C, f)]$, and there exist deformations of (C, f) smoothing all the nodes of C. A general such deformation is free.* lem-cic0.1

The following criterion from [Gro67, §4] will be useful.

Lemma 3.2. *Let*

$$i : N \to M, \quad e : M \to X$$

be morphisms of irreducible schemes. If i maps the generic point of N to a normal point of M, and if $e \circ i$ is dominant with irreducible geometric generic fiber, then also e is dominant with irreducible geometric generic fiber.

Proof. If $e \circ i$ is dominant, clearly e is dominant. By [Gro67, Proposition 4.5.9], to prove the geometric generic fiber of e is irreducible, it is equivalent to prove the field extension $K(M)/K(X)$ is a *primary* extension, i.e., $K(X)$ is separably closed in $K(M)$. Denote by η_i the image under i of the generic point of N. Because the $K(X)$-algebra $\mathcal{O}_{M,\eta_i}$ is normal, the separable closure of $K(X)$ in $K(M)$ equals the separable closure of $K(X)$ in $\mathcal{O}_{M,\eta_i}$. Because the separable closure is a field, it maps isomorphically to its image in the residue field $\kappa(\eta_i)$ of $\mathcal{O}_{M,\eta_i}$. Thus the separable closure of $K(X)$ in $K(M)$ is contained in the separable closure of $K(X)$ in $\kappa(\eta_i)$. Of course $\kappa(\eta_i)/K(X)$ is a subextension of $K(N)/K(X)$. Because the geometric generic fiber of $e \circ i$ is irreducible, $K(X)$ is separably closed in $K(N)$, thus separably closed in $\kappa(\eta_i)$. Therefore also $K(X)$ is separably closed in $K(M)$. $\square$

The first construction of distinguished irreducible components uses multiple covers.

lem-cic0.2

Lemma 3.3. *Let $M_{\alpha,0}$ be an irreducible component of $\overline{\mathcal{M}}_{0,0}(X, \alpha)$ whose general point parametrizes a smooth, free curve. For every positive integer e there exists a unique irreducible component $M_{e\cdot\alpha,0}$ of $\overline{\mathcal{M}}_{0,0}(X, e\alpha)$ parametrizing a degree e cover of a smooth, free curve parametrized by $M_{\alpha,0}$.*

For every positive integer e denote by $M_{e\cdot\alpha,1}$ the unique irreducible component of $\overline{\mathcal{M}}_{0,1}(X, e\alpha)$ dominating $M_{e\cdot\alpha,0}$. If the geometric generic fiber of

$$ev|_M : M_{\alpha,1} \to X$$

is irreducible, then for every e the geometric generic fiber of

$$ev|_M : M_{e\cdot\alpha,1} \to X$$

is irreducible.

Proof. The subvariety S of $\overline{\mathcal{M}}_{0,0}(X, e\alpha)$ parametrizing degree e covers of smooth, free curves parametrized by $M_{\alpha,0}$ fibers over $M_{\alpha,0}$ with fibers isomorphic to $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^1, e)$. Because $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^1, e)$ is smooth and geometrically connected, S is irreducible. Moreover, because a multiple cover of a free curve satisfies the hypothesis of Lemma 3.1, $\overline{\mathcal{M}}_{0,0}(X, e\alpha)$ is smooth at every point of S. Therefore there exists a unique irreducible component $M_{e\cdot\alpha,0}$ that intersects S.

The proof of the second part uses Lemma 3.2. The role of e is played by

$$\mathrm{ev}|_M : M_{e\cdot\alpha,1} \to X.$$

The role of $i : N \to M$ is played by the inverse image S_1 of S in $M_{e\cdot\alpha,1}$. By the argument above, S_1 is in the normal locus of $M_{e\cdot\alpha,1}$. By Lemma 3.2, to prove the geometric generic fiber of e is irreducible, it suffices to prove that $i \circ e$ is dominant with irreducible geometric generic fiber.

The morphism $e \circ i$ factors as

$$S_1 \to M_{\alpha,1} \to X$$

where the first morphism "forgets" about the multiple covering. The first morphism is dominant and its geometric generic fiber is isomorphic to a base change of the geometric generic fiber of

$$\mathrm{ev} : \overline{\mathcal{M}}_{0,1}(\mathbb{P}^1, e) \to \mathbb{P}^1,$$

which is irreducible. Thus both

$$S \to M_{\alpha,1} \text{ and } M_{\alpha,1} \to X$$

are dominant morphisms whose geometric generic fiber is irreducible. Therefore the composite $e \circ i$ is also dominant with irreducible geometric generic fiber. □

The second construction of distinguished irreducible components uses reducible curves.

lem-cic0.3

Lemma 3.4. *For $i = 1, 2$, let M_i be an irreducible component of $\overline{\mathcal{M}}_{0,1}(X, \beta_i)$ whose general point parametrizes a smooth, free curve. Assume that for at least one of $i = 1, 2$, the geometric generic fiber of*

$$ev|_{M_i} : M_i \subset \overline{\mathcal{M}}_{0,1}(X, \beta_i) \to X$$

is irreducible. Then there exists a unique irreducible component M of $\overline{\mathcal{M}}_{0,0}(X, \beta_1 + \beta_2)$ parametrizing a reducible curve $C = C_1 \cup C_2$ whose component C_i is a smooth, free curve parametrized by M_i. Moreover a general point of M parametrizes a smooth, free curve. And, if ev_{M_i} has geometrically irreducible generic fiber for both $i = 1$ and 2, then for the unique irreducible component M' of $\overline{\mathcal{M}}_{0,1}(X, \beta_1 + \beta_2)$ dominating M, the restriction

$$ev_{M'} : M' \subset \overline{\mathcal{M}}_{0,1}(X, \beta_1 + \beta_2) \to X$$

has geometrically irreducible generic fiber.

Proof. Denote by M_i° the open subset of M_i parametrizing smooth, free curves. Then, by [Kol96, Corollary II.3.5.4.2], the evaluation morphism

$$\mathrm{ev}|_{M_i^\circ} : M_i^\circ \subset \overline{\mathcal{M}}_{0,1}(X, \beta_i) \to X$$

is smooth. Therefore the fiber product

$$F := M_1^\circ \times_X M_2^\circ$$

is smooth, and projection onto each factor is smooth. Moreover, the geometric generic fiber of projection onto M_2°, resp. onto M_1°, equals the geometric generic fiber of

$$\mathrm{ev}|_{M_i^\circ} : M_i^\circ \to X.$$

By hypothesis, at lease one of the geometric generic fibers is irreducible. Therefore the geometric generic fiber of one of the two projections of F is irreducible. Since the base of each projection is irreducible, it follows that F is irreducible.

There is a boundary morphism

$$\Delta : \overline{\mathcal{M}}_{0,1}(X,\beta_1) \times_X \overline{\mathcal{M}}_{0,1}(X,\beta_2) \to \overline{\mathcal{M}}_{0,0}(X,\beta_1+\beta_2)$$

associating to a pair $((C_1,p_1,f_1),(C_2,p_2,f_2))$ with $f_1(p_1) = f_2(p_2)$ the reducible curve $C = C_1 \cup C_2$ obtained by gluing p_1 to p_2, and the unique morphism $f : C \to X$ whose restriction to C_i equals f_i for each i. As F is irreducible, also the image $\Delta(F)$ is also irreducible. By Lemma 3.1, $\overline{\mathcal{M}}_{0,0}(X,\beta_1+\beta_2)$ is smooth at every point of $\Delta(F)$. Therefore there exists a unique irreducible component M of $\overline{\mathcal{M}}_{0,0}(X,\beta_1+\beta_2)$ intersecting $\Delta(F)$. By Lemma 3.1, the generic point of M parametrizes a smooth, free curve.

Next assume that $\mathrm{ev}|_{M_i}$ has geometrically irreducible generic fiber for each of $i = 1,2$. Denote by $(M_1^o)'$ the open subset of $\overline{\mathcal{M}}_{0,2}(X,\beta_1)$ parametrizing smooth, 2-pointed curves whose associated 1-pointed curve (forgetting the first point) is a free curve parametrized by M_1. The projection

$$(M_1^o)' \to M_1^\circ, \quad [(C_1,p_0,p_1,f_1)] \mapsto [(C_1,p_1,f_1)]$$

is smooth with geometrically irreducible fibers. Therefore also the fiber product

$$F' := (M_1^o)' \times_{M_1^\circ} F =$$

$$\{([(C_1,p_0,p_1,f_1)],[(C_2,p_2,f_2)])|[(C_i,p_i,f_i)] \in M_i^\circ \text{ for } i=1,2, \text{ and } f_1(p_1)=f_2(p_2)\}$$

is smooth and irreducible. There is a boundary map

$$\Delta : \overline{\mathcal{M}}_{0,2}(X,\beta_1) \times_X \overline{\mathcal{M}}_{0,1}(X,\beta_2) \to \overline{\mathcal{M}}_{0,1}(X,\beta_1+\beta_2),$$

$$([(C_1,p_0,p_1,f_1)],[(C_2,p_2,f_2)]) \mapsto [(C,p_0,f)]$$

where C and f are as above. Again by Lemma 3.1, $\overline{\mathcal{M}}_{0,1}(X,\beta_1+\beta_2)$ is smooth at every point of $\Delta(F')$, so there is a unique irreducible component M' intersecting $\Delta(F')$. By Lemma 3.1, the generic point of M' parametrizes a smooth curve. Thus the image of M' in $\overline{\mathcal{M}}_{0,0}(X,\beta_1+\beta_2)$ is an irreducible component intersecting $\Delta(F)$, i.e., it is precisely M.

To prove the restriction

$$\mathrm{ev}_{M'} : M' \subset \overline{\mathcal{M}}_{0,1}(X,\beta_1+\beta_2) \to X$$

has geometrically irreducible generic fiber, we use Lemma 3.2 with $\mathrm{ev}_{M'}$ in the role of e, and with

$$\Delta : F' \to M'$$

in the role of i. Thus it suffices to prove that the geometric generic fiber of

$$\mathrm{ev} \circ \Delta : F' \to X$$

is irreducible. By construction, this is a composition of the dominant morphisms

$$F' \to (M_1^o)' \to M_1^o \to X.$$

Each of these has irreducible geometric generic fibers by the hypotheses that $\mathrm{ev}|_{M_i}$ have geometrically irreducible fibers. Thus the composition also has irreducible geometric generic fibers. $\square$

The following lemma shows that in good cases the constructions from Lemmas 3.3 and 3.4 agree.

lem-cic1

Lemma 3.5. *Let $M_{\alpha,0}$ be an irreducible component of $\overline{\mathcal{M}}_{0,0}(X,\alpha)$ whose general point parametrizes a smooth, free curve. Denote by $M_{\alpha,1}$ the unique irreducible component of $\overline{\mathcal{M}}_{0,1}(X,\alpha)$ dominating $M_{\alpha,0}$. Assume that the generic fiber of the restriction*

$$ev|_M : M_{\alpha,1} \to X$$

is geometrically irreducible.

For every positive integer e there is a unique irreducible component $M_{e\cdot\alpha,0}$ of $\overline{\mathcal{M}}_{0,0}(X,e\alpha)$ parametrizing (among others) a reducible curve whose (non-contracted) components are all multiple covers of free curves parametrized by $M_{\alpha,0}$.

A general point of $M_{e\cdot\alpha,0}$ parametrizes a smooth, free curve. Denoting by $M_{e\cdot\alpha,1}$ the unique irreducible component of $\overline{\mathcal{M}}_{0,1}(X,e\alpha)$ dominating $M_{e\cdot\alpha,0}$, the restriction

$$ev|_M : M_{e\cdot\alpha,1} \to X$$

is dominant with irreducible geometric generic fiber.

Proof. The very last part is just the restatement of the second half of Lemma 3.3.

Let (D,g) be a stable map all of whose non-contracted components D_i are multiple covers of free curves C_i parametrized by $M_{\alpha,0}$. By Lemma 3.1, $\overline{\mathcal{M}}_{0,0}(X)$ is smooth at $[(D,g)]$. In particular, there is a unique irreducible component M containing $[(D,g)]$. Let e be the degree of D_i over C_i. Since the space of multiple covers $\overline{\mathcal{M}}_{0,r}(\mathbb{P}^1,e)$ is irreducible, we can deform D_i to become a reducible cover of C_i all of whose non-contracted components map isomorphically to C_i. So there is a deformation of (D,g) in M which specializes to a stable map (C,f) all of whose non-contracted components are equal to free curves C_i parametrized by $M_{\alpha,0}$. Therefore it suffices to prove there exists a unique irreducible component M parametrizing a stable map (C,f) all of whose non-contracted components are free curves parametrized by $M_{\alpha,0}$.

For each integer e, if $M_{e\cdot\alpha,1}$ is an irreducible component satisfying the hypotheses, then the image $M_{e\cdot\alpha,0}$ of $M_{e\cdot\alpha,1}$ under the forgetful morphism

$$\overline{\mathcal{M}}_{0,1}(X,e\alpha) \to \overline{\mathcal{M}}_{0,0}(X,e\alpha)$$

is an irreducible component of $\overline{\mathcal{M}}_{0,0}(X,e\alpha)$, and $M_{e\cdot\alpha,1}$ is the unique component of $\overline{\mathcal{M}}_{0,1}(X,e\alpha)$ dominating $M_{e\cdot\alpha,0}$.

The lemma is established by induction on e. The base case $e=1$ is tautological. Thus, by way of induction, assume $e>1$ and assume the lemma is true for $e-1$. The irreducible components $M_{\alpha,1}$ and $M_{(e-1)\cdot\alpha,1}$ satisfy the hypotheses for M_1 and M_2 in Lemma 3.4. Therefore there exists a unique irreducible component $M_{e\cdot\alpha,0}$ of $\overline{\mathcal{M}}_{0,0}(X,e\alpha)$ parametrizing a reducible curve $C=C_1\cup C_2$ whose component C_1, resp. C_2, is a smooth, free curve parametrized by $M_{\alpha,1}$, resp. $M_{(e-1)\cdot\alpha,1}$. Moreover a general point of $M_{e\cdot\alpha,0}$ parametrizes a smooth,free curve and the generic fiber of $M_{e\cdot\alpha,1}$ over X is geometrically irreducible. By the induction hypothesis, there is a reducible curve C_2 parametrized by $M_{(e-1)\cdot\alpha,1}$, all of whose non-contracted components are free curves parametrized by $M_{\alpha,0}$. Because they are free curves, for a general deformation of C_2 as above, the marked point is a general point of X. Because $M_{\alpha,1}$ dominates X, there exists a smooth, free curve C_1 parametrized by $M_{\alpha,1}$ whose marked point agrees with the marked point of C_2. Thus, by the construction of $M_{e\cdot\alpha,0}$, $C = C_1 \cup C_2$ is a point of $M_{e\cdot\alpha,0}$, and all of its non-contracted components are free curves parametrized by $M_{\alpha,0}$.

It remains to prove that $M_{e\cdot\alpha,0}$ is the *only* irreducible component of $\overline{\mathcal{M}}_{0,0}(X, e\alpha)$ parametrizing a curve all of whose non-contracted components are free curves parametrized by $M_{\alpha,0}$. Let C be such a curve. There exists an irreducible component C_1 of C meeting the remainder C_2 of C in a single node. Necessarily C_1 is not contracted, since it does not contain 3 special points. Therefore C_1, marked by the node, is a free curve in $M_{\alpha,1}$. Moreover, C_2 is a curve parametrized by $\overline{\mathcal{M}}_{0,0}(X,(e-1)\alpha)$ all of whose non-contracted components are free curves parametrized by $M_{\alpha,0}$. By the induction hypothesis, C_2, marked by the node, is parametrized by $M_{(e-1)\alpha,1}$. Thus C is in the image $\Delta(M_{\alpha,1} \times_X M_{(e-1)\alpha,1})$. By the construction in the proof of Lemma 3.4, C is contained in $M_{e\cdot\alpha,0}$. By Lemma 3.1, $\overline{\mathcal{M}}_{0,0}(X, e\alpha)$ is smooth at $[C]$, therefore $M_{e\cdot\alpha,0}$ is the unique irreducible component containing C. □

Here is a useful "recognition criterion" for the irreducible component $M_{e\cdot\alpha,0}$.

Lemma 3.6. *Let C be a smooth point of $\overline{\mathcal{M}}_{0,0}(X, e\alpha)$. Assume that every non-contracted component C_i of C is in one of the sets $M_{e_i\cdot\alpha,0}$. Moreover, assume that at most one of the C_i is not a free curve. Then $M_{e\cdot\alpha,0}$ is the unique irreducible component of $\overline{\mathcal{M}}_{0,0}(X, e\alpha)$ containing C.* lem-cic1.5

Proof. Let C_i be an irreducible component of C such that for every other component $C_j \neq C_i$, C_j is free. Then, for every deformation of C_i, there exists a deformation of the remaining irreducible components C_j giving rise to a deformation of C. Because C is a smooth point of $\overline{\mathcal{M}}_{0,0}(X, e\alpha)$, the unique irreducible component containing C also contains this deformation. Because C_i is in one of the sets $M_{e_i\cdot\alpha,0}$, a general deformation of C_i is a free curve. Thus the unique irreducible component containing C also contains a curve C^{new} as above all of whose components are free. Thus, without loss of generality, assume C has been deformed so that every irreducible component of C is a free curve parametrized by some $M_{e'\cdot\alpha}$.

The proof is by induction on the number of components of C. If C is irreducible, then the result is tautological. Thus assume the number of components c is positive, and the result is known for all smaller values of c. Let C_i be a leaf of C and let C' denote the union of all components $C_j \neq C_i$. Mark each of C_i and C' by their intersection point p. By the induction hypothesis, the unique irreducible component of $\overline{\mathcal{M}}_{0,0}(X,(e-e_i)\cdot\alpha)$ containing C' is $M_{(e-e_i)\cdot\alpha,0}$. Thus the curve C is in the image of the boundary map

$$\Delta : M_{e_i\cdot\alpha,1} \times_X M_{(e-e_i)\cdot\alpha,1} \to \overline{\mathcal{M}}_{0,0}(X, e\cdot\alpha).$$

Moreover, C is in the unique component M' dominating $M_{(e-e_i)\cdot\alpha,1}$. By the same argument as in the previous proof, the unique component of $\overline{\mathcal{M}}_{0,0}(X, e\cdot\alpha)$ containing the image of M' contains a reducible curve, every (non-contracted) component of which is a smooth, free curve in $M_{\alpha,0}$. Therefore, by Lemma 3.5, the unique component containing the image of M' is $M_{e\cdot\alpha,0}$. $\square$

It is useful to extend this to m-pointed maps.

notat-cic2

Notation 3.7. For every integer $e \geq 1$ and every integer $m \geq 0$, denote by $M_{e\cdot\alpha,m}$ the unique irreducible component of $\overline{\mathcal{M}}_{0,m}(X, e\cdot\alpha)$ whose image under the forgetful morphism

$$\overline{\mathcal{M}}_{0,m}(X, e\cdot\alpha) \to \overline{\mathcal{M}}_{0,0}(X, e\cdot\alpha)$$

equals $M_{e\cdot\alpha,0}$.

Finally, one can consider Behrend-Manin stable maps whose components are free curves parametrized by some $M_{e\cdot\alpha,m}$.

notat-cic3

Notation 3.8. Let τ be a genus 0, stable A-graph for X in the sense of [BM96], all of whose curve classes are multiples of α. Denote by M_τ the closure in the Behrend-Manin stack $\overline{\mathcal{M}}(X,\tau)$ of the locus parametrizing stable maps each of whose (non-contracted) irreducible components is a smooth, free curve parametrized by some $M_{e\cdot\alpha,m}$.

sec-pl

4. Pointed lines

In order to apply Lemma 3.5, we must first have an irreducible component $M_{\alpha,1}$ dominating X whose geometric generic fiber over X is irreducible. When α is the class of a line, i.e., the $\mathcal{O}(1)$-degree of α is 1, there are conditions on the dimension of X and the intersection number $\langle c_1(T_X), \alpha\rangle$ guaranteeing the existence of such a component $M_{\alpha,1}$, which is moreover the unique component dominating X. The conditions are stronger than might be expected (by doing a parameter count, for instance). However, if we assume that X is a linear section of a smooth, projective variety of sufficiently high dimension (which is true for every smooth complete intersection), then the strong conditions may be replaced by weaker conditions.

Let X be a smooth, projective variety with a very ample invertible sheaf $\mathcal{O}(1)$. All degrees are relative to $\mathcal{O}(1)$. Let α be a curve class on X. Denote by

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & X \\ {\scriptstyle\pi}\downarrow & & \\ \overline{\mathcal{M}}_{0,1}(X,\alpha) & & \end{array}$$

the universal family of stable maps over $\overline{\mathcal{M}}_{0,1}(X,\alpha)$. If α is "indecomposable", then we know quite a bit about $\overline{\mathcal{M}}_{0,1}(X,\alpha)$ and the evaluation morphism ev.

lem-pl1

Lemma 4.1. *Assume there is no nontrivial decomposition $\alpha = \alpha' + \alpha''$ of α as a sum of effective rational curve classes, e.g., the $\mathcal{O}(1)$-degree of α equals* 1.

(i) *A general fiber of*
$$ev : \overline{\mathcal{M}}_{0,1}(X,\alpha) \to X$$
is smooth and projective.

(ii) *Every (nonempty) connected component M of a general fiber of ev has dimension*
$$dim(M) = \langle c_1(T_X), \alpha\rangle - 2.$$
In particular, if ev is dominant then $\langle c_1(T_X), \alpha\rangle \geq 2$.

(iii) *The first Chern class of M is*

$$c_1(T_M) = \pi_* f^*[ch_2(T_X) + \frac{1}{2\langle c_1(T_X), \alpha\rangle} c_1(T_X)^2] - \psi,$$

where $-\psi$ is the first Chern class of the normal bundle of the marked section in the universal curve over M. In particular, if the $\mathcal{O}(1)$-degree of α equals 1, this reduces to

$$c_1(T_M) = \pi_* f^*[ch_2(T_X) + \frac{1}{2\langle c_1(T_X), \alpha\rangle} c_1(T_X)^2 - c_1(\mathcal{O}(1))^2].$$

Proof. The first part follows from [KMM92, 1.1]. The second part follows from [Kol96, Theorem II.1.7.2]. The third part follows from [dJS05]. □

Because of Lemma 4.1(ii), ev is dominant only if

$$\langle c_1(T_X), \alpha\rangle \geq 2.$$

Moreover, if there is equality, a general fiber is finite. If the degree is not 1, then the fiber is disconnected. However, if the inequality is strict, i.e., if

$$\langle c_1(T_X)\alpha\rangle \geq 3,$$

we may well expect a general fiber of ev to be irreducible. The next lemmas prove dominance of ev and irreducibility of a general fiber of ev under inequalities stronger than this naive one. After that, we prove the naive inequality suffices if X is a linear section of a smooth projective variety of large dimension.

lem-pl1.5

Lemma 4.2. *Let α be an $\mathcal{O}(1)$-degree 1 rational curve class. Assume that*

$$2\langle c_1(T_X), \alpha\rangle \geq dim(X) + 3.$$

If a general fiber of

$$ev : \overline{\mathcal{M}}_{0,1}(X, \alpha) \to X$$

is nonempty, then it is irreducible.

Proof. Let M' and M'' be irreducible components of the fiber of ev over a general point p of X. Denote by Π', resp. Π'', the subvarieties of X swept out by curves parametrized by M', resp. M''. By Lemma 4.1(ii),

$$\dim(\Pi') = \dim(\Pi'') = \langle c_1(T_X), \alpha\rangle - 1.$$

Moreover they intersect at p. Because X is smooth at p,

$$\dim_p(\Pi' \cap \Pi'') \geq \dim_p(\Pi') + \dim_p(\Pi'') - \dim_p(X) = 2\langle c_1(T_X), \alpha\rangle - 2 - \dim(X).$$

By hypothesis, this is ≥ 1. In particular, there exists $q \neq p$ such that Π' and Π'' intersect at q. There is a unique line L containing p and q. Thus

$$[L] \in M' \cap M''.$$

By Lemma 4.3(i), the fiber of ev over p is smooth. In particular, every connected component is irreducible. Therefore

$$M' = M'',$$

i.e., the fiber is irreducible. □

Because of Lemma 4.2, irreducibility follows from dominance of ev, if the index $\langle c_1(T_X), \alpha\rangle$ is sufficiently large. The next lemma shows that if the index is sufficiently large, then ev is dominant.

lem-pl2

Lemma 4.3. *Assume that for every effective rational curve class $\alpha' \neq \alpha$,*

$$\langle c_1(T_X), \alpha' \rangle \geq dim(X) + 2.$$

Then a general fiber of

$$ev : \overline{\mathcal{M}}_{0,1}(X, \alpha) \to X$$

is nonempty. In particular, if $c_1(T_X) = \langle c_1(T_X), \alpha \rangle c_1(\mathcal{O}(1))$ for an $\mathcal{O}(1)$-degree 1 class α, i.e., if $c_1(T_X)$ is a multiple of $c_1(\mathcal{O}(1))$, the inequality holds if

$$2\langle c_1(T_X), \alpha \rangle \geq dim(X) + 2.$$

Proof. This follows from [Kol96, Theorem V.1.6]. □

Together, Lemmas 4.2 and 4.3 imply that ev is dominant, resp. ev is dominant and a general fiber is irreducible, if

$$2\langle c_1(T_X), \alpha \rangle \geq \dim(X) + 2, \text{ resp. } \geq \dim(X) + 3.$$

However, as observed above, we expect this to hold under the weaker inequality

$$\langle c_1(T_X), \alpha \rangle \geq 2, \text{ resp. } \geq 3.$$

The next lemma proves the weaker inequality suffices provided X is a linear section of sufficiently high codimension of a smooth, projective variety Y.

lem-pl2a

Lemma 4.4. *Let α be an $\mathcal{O}(1)$-degree 1 curve class. Assume that*

$$\langle c_1(T_X), \alpha \rangle \geq 2, \text{ resp. } \geq 3.$$

Assume there exists a smooth projective variety Y such that X is a codimension c linear section of Y and

$$ev : \overline{\mathcal{M}}_{0,1}(Y, \alpha) \to Y$$

is dominant, resp. dominant with connected generic fiber. Then

$$ev : \overline{\mathcal{M}}_{0,1}(X, \alpha) \to X$$

is dominant, resp. dominant with connected generic fiber. In the latter case, the unique irreducible component $M_{\alpha,1}$ dominating X satisfies the hypotheses of Lemma 3.5.

In particular, if $c_1(T_X) = \langle c_1(T_X), \alpha \rangle c_1(\mathcal{O}(1))$, i.e., if $c_1(T_X)$ is a multiple of $c_1(\mathcal{O}(1))$, and if X is a codimension c linear section of a smooth projective variety Y, then

$$ev : \overline{\mathcal{M}}_{0,1}(X, 1) \to X$$

is surjective, resp. surjective with connected generic fiber, if

$$\langle c_1(T_X), \alpha \rangle \geq 2, \text{ resp. } \geq 3$$

and

$$c \geq dim(X) - 2\langle c_1(T_X), \alpha \rangle + 2, \text{ resp. } \geq dim(X) - 2\langle c_1(T_X), \alpha \rangle + 3.$$

In the latter case, the unique irreducible component $M_{\alpha,1}$ dominating X satisfies the hypotheses of Lemma 3.5

Proof. By hypothesis, the morphism

$$\text{ev} : \overline{\mathcal{M}}_{0,1}(Y, 1) \to Y$$

is dominant. Thus for some $\mathcal{O}(1)$-degree 1 curve class α on Y, the morphism

$$\text{ev} : \overline{\mathcal{M}}_{0,1}(Y, \alpha) \to Y$$

is dominant. By the Lefschetz hyperplane theorem, the curve class α on Y is the image of a curve class α on X. By Lemma 4.1, a general fiber of ev has dimension

$$\langle c_1(T_Y), \alpha\rangle - 2 = c + \langle c_1(T_X), \alpha\rangle - 2.$$

By the hypothesis that $\langle c_1(T_X), \alpha\rangle \geq 2$, this implies a general fiber has dimension $\geq c$. By upper semicontinuity of fiber dimension, every fiber has dimension $\geq c$.

Let p be a point of X. Of course Y is embedded in a projective space $\mathbb{P}^N$ and X is the intersection of Y with a codimension c linear subspace Λ containing p. The variety of lines in $\mathbb{P}^N$ containing p is isomorphic to $\mathbb{P}^{N-1} = \mathbb{P}^N/\{p\}$. By the previous paragraph, the variety of lines in Y containing p is a subvariety of $\mathbb{P}^{N-1}$ of dimension $\geq c$. The variety of lines in X containing p is the intersection of this subvariety with the codimension c linear subspace $\Lambda/\{p\}$. Because every c-dimensional variety in projective space intersects every codimension c linear subspace, there exists a line in X containing p, i.e.,

$$\text{ev} : \overline{\mathcal{M}}_{0,1}(X, 1) \to X$$

is surjective.

Next assume that

$$\langle c_1(T_X), \alpha\rangle \geq 3.$$

By [Kol96, Theorem II.1.7.2], every irreducible component of $\overline{\mathcal{M}}_{0,1}(Y, \alpha)$ has dimension

$$\geq \langle c_1(T_Y), \alpha\rangle + \dim(Y) - 2.$$

By the argument above, the fiber of

$$\text{ev}_X : \overline{\mathcal{M}}_{0,1}(X, \alpha) \to X$$

over a point p is an intersection of c ample divisors in the fiber of

$$\text{ev}_Y : \overline{\mathcal{M}}_{0,1}(Y, \alpha) \to Y$$

over p. If p is a general point of X, then by Lemma 4.1(ii),

$$\dim(\text{ev}_X^{-1}(p)) \leq \langle c_1(T_X), \alpha\rangle - 2.$$

Therefore

$$\dim(\text{ev}_Y^{-1}(p)) \leq \langle c_1(T_X), \alpha\rangle - 2 + c = \langle c_1(T_Y), \alpha\rangle - 2 = \dim(\overline{\mathcal{M}}_{0,1}(Y, \alpha) - \dim(Y).$$

This implies that ev_Y is a flat, local complete intersection morphism at every point of $\text{ev}_Y^{-1}(p)$, cf. [Kol96, Theorem II.1.7.3]. In particular, every irreducible component of $\text{ev}_Y^{-1}(p)$ is contained in an irreducible component M of $\overline{\mathcal{M}}_{0,1}(Y, \alpha)$ that dominates Y. By hypothesis, a general fiber of ev_Y is irreducible. Thus, in fact, there is only one irreducible component M. The restriction

$$\text{ev}|_M : M \to Y$$

is a projective morphism whose geometric generic fiber is connected. By [Har77, Corollary III.11.5], every fiber is connected. Therefore the fiber $\mathrm{ev}_Y^{-1}(p)$ is connected. Also, as mentioned above, it is a local complete intersection variety of dimension

$$\geq \langle c_1(T_X), \alpha\rangle - 2 + c \geq c + 1.$$

Thus an intersection with c ample divisors is still connected, cf. [Har77, Corollary III.7.9]. Therefore $\mathrm{ev}_X^{-1}(p)$ is connected.

The last part of the lemma follows from the first parts together with Lemmas 4.2 and 4.3. $\square$

The main application is to complete intersections.

cor-pl3 **Corollary 4.5.** *Let X be a smooth complete intersection in $\mathbb{P}^n$ of multidegree $\underline{d} = (d_1, \dots, d_c)$ and thus of index*

$$\langle c_1(T_X), \alpha\rangle = n + 1 - \sum d_i.$$

A general fiber of ev is nonempty, resp. nonempty and irreducible, if

$$\langle c_1(T_X), \alpha\rangle \geq 2, \ \textit{resp.} \ \geq 3.$$

In the latter case, the unique irreducible component $M_{\alpha,1}$ of $\overline{\mathcal{M}}_{0,1}(X, \alpha)$ dominating X satisfies the hypotheses of Lemma 3.5.

Proof. This follows from Lemma 4.4: for every sufficiently positive integer c there exists a smooth projective variety Y such that X is a codimension c linear section of Y by Lemma 2.5. $\square$

sec-mpc

5. Minimal pointed curves. Existence and connectedness

The proofs of Theorems 1.1 and 1.2 are essentially induction arguments. A key role is played by rational curves having minimal curve class among those that contain 1, resp. 2 and 4, general fixed points of X. More generally, a curve having minimal curve class among those containing m general fixed points of X is called a *minimal pointed curve*. The case of 1 point, i.e., pointed lines, was considered in Section 4. The higher cases are considered in this section and Section 6.

There are a few features of the space of minimal pointed curves that are important. First of all, in order to prove irreducibility and rational connectedness of a general fiber of

$$\mathrm{ev} : \overline{\mathcal{M}}_{0,m}(X, \beta) \to X^m$$

for large m and β, it is crucial to first prove irreducibility and rational connectedness of a general fiber of

$$\mathrm{ev} : \overline{\mathcal{M}}_{0,2}(X, 2\alpha) \to X^2.$$

Secondly, in order to make the induction argument work, we need to prove existence of some special rational surfaces in X. Although it is possible one can prove this directly, we find it useful to think of such rational surfaces as minimal degree free curves in a space of minimal pointed curves. For this reason, it is important to study the canonical bundle and uniruledness of the spaces of minimal pointed curves. This is done in Section 6. Finally, although we are primarily concerned with the cases $m = 2$ and $m = 4$, the methods apply to arbitrary m. Thus the results are proved for arbitrary m.

As above, X is a smooth, projective variety with a very ample invertible sheaf $\mathcal{O}(1)$. The goal in this section is to understand curve classes β such that the geometric generic fiber of

$$\mathrm{ev} : \overline{\mathcal{M}}_{0,m}(X, \beta) \to X^m$$

is smooth, resp. smooth and automorphism free. There is one simple criterion for smoothness of a general fiber of ev. lem-mpc1

Lemma 5.1. *If every point in a general fiber of ev parametrizes a curve whose irreducible components are all free, then a (non-empty) general fiber of ev is smooth of the expected dimension*

$$\langle c_1(T_X), \beta\rangle - (m-1)\,dim(X) + m - 3$$

and the intersection with the boundary Δ is a simple normal crossings divisor.

Proof. This is trivial if a general fiber is empty. Thus assume ev is dominant.

The open substack U of $\overline{\mathcal{M}}_{0,m}(X, \beta)$ parametrizing unions of free curves is smooth and $U \cap \Delta$ is a simple normal crossings divisor, cf. [Kol96, Theorem II.7.6]. By generic smoothness the intersection of U with a general fiber of ev is smooth and the intersection with $U \cap \Delta$ is a simple normal crossings divisor. If U contains a general fiber of ev, the fiber is smooth and the intersection with Δ is a simple normal crossings divisor. □

There is a simple criterion on the curve class β insuring the hypothesis of Lemma 5.1. The relevant definitions are the following, and the criterion is the next lemma. defn-mpc2

Definition 5.2. A curve class β is *m-dominant* if the evaluation morphism

$$\mathrm{ev} : \overline{\mathcal{M}}_{0,m}(X, \beta) \to X^m$$

is dominant.

An m-dominant curve class β is *m-minimal* if for every partition

$$m = m_1 + \cdots + m_r$$

and for every collection $(\beta_1, \dots, \beta_r)$ of m_i-dominant curve classes β_i satisfying

$$\sum_{i=1}^{r} \beta_i \leq \beta,$$

in fact

$$\sum_{i=1}^{r} \beta_i = \beta.$$

In particular, this implies that β is minimal among m-dominant curve classes. lem-mpc3

Lemma 5.3. *If β is m-minimal, then every point in a general fiber of*

$$ev : \overline{\mathcal{M}}_{0,m}(X, \beta) \to X^m$$

is a union of free curves. Therefore a general fiber of ev is smooth of the expected dimension

$$\langle c_1(T_X), \beta\rangle - (m-1)\,dim(X) + m - 3$$

and intersects Δ in a simple normal crossings divisor. Moreover, every point in a general fiber parametrizes an automorphism-free stable map.

Proof. Let $p = (p_1, \dots, p_m)$ be *the* geometric generic point of X^m. Then for every subset $I \subset \{1, \dots, m\}$, setting $l := \#I$, the l-tuple $(p_i | i \in I)$ maps to the geometric generic point of X^l.

By Lemma 5.1, to prove the lemma it suffices to prove that every point $(C, q_1, \dots, q_r, f)$ in the fiber of

$$\mathrm{ev} : \overline{\mathcal{M}}_{0,m}(X, \beta) \to X^m$$

over p is a union of free curves. Let C' be the maximal subcurve of C whose irreducible components are free curves. Let $C'_1, \dots, C'_r$ denote the connected components of C' containing at least one of the marked points. Denote by C'' the union of all irreducible components of C not among $C'_1, \dots, C'_r$.

There are two observations. First of all, for every $i = 1, \dots, m$, every irreducible curve in X containing p_i is free by [KMM92, 1.1]. Therefore every point q_i is contained in one of the components $C'_1, \dots, C'_r$. Secondly, let C_j be a contracted irreducible component of C whose image equals p_i. Every irreducible component of C intersecting C_j is either contracted – and thus trivially free – or else maps to an irreducible curve in X containing p_i – and thus free by the first observation. Therefore every connected component C'_i contains at least one irreducible component that is not contracted.

Let β_i denote the curve class of C'_i. Denote by m_i the number of elements of $\{q_1, \dots, q_r\}$ contained in C'_i. Because every subset of $\{p_1, \dots, p_m\}$ is general, β_i is m_i-dominant. And

$$\sum_{i=1}^{r} \beta_i \leq \beta.$$

Therefore, since β is m-minimal,

$$\sum_{i=1}^{r} \beta_i = \beta.$$

In particular, every irreducible component of C'' is contracted. Because contracted curves are free, every irreducible component of C is free.

Finally, to prove that C is automorphism-free, it suffices to prove that every non-contracted component C_i has degree 1 over its image. But if the degree is > 1, then C_i can be replaced by a curve with smaller curve class also meeting all marked points contained in C_i: namely the normalization of the image of C_i (if any "special" points are mapped to the same point in the image, one may also need to attach some contracted components). Thus C can be replaced by an m-dominating curve with smaller curve class. This contradicts that β is m-minimal. □

What is the $\mathcal{O}(1)$-degree of an m-minimal curve? Assuming

$$m \leq h^0(X, \mathcal{O}(1))$$

then m general points on X impose independent conditions on linear forms, i.e., they are in *linear general position*. The minimum $\mathcal{O}(1)$-degree of a rational curve in $\mathbb{P}^N$ containing m points in linearly general position is $m - 1$. However, there seem to be very few varieties X with an m-dominating class of $\mathcal{O}(1)$-degree $m - 1$.

ques-mpcCH

Question 5.4 (Coskun, Harris)**.** Let $(X, \mathcal{O}(1))$ be a smooth projective variety, and let m be an integer satisfying $2 \leq m \leq h^0(X, \mathcal{O}(1))$. Assume that every general m-tuple of points on X is contained in a curve in X of $\mathcal{O}(1)$-degree $m-1$. Is it true that $(X, \mathcal{O}(1))$ is a linear variety, a quadric hypersurface, or the projection of a rational normal scroll or a Veronese surface?

There is some evidence for a positive answer. This gives one criterion for a curve class to be m-minimal.

lem-mpcCH

Lemma 5.5. *Let $(X, \mathcal{O}(1))$ be a smooth, projective variety and let m be a positive integer. Assume that $c_1(T_X) = \langle c_1(T_X), \alpha\rangle c_1(\mathcal{O}(1))$ for some $\mathcal{O}(1)$-degree 1 curve class α, i.e., $c_1(T_X)$ is a multiple of $c_1(\mathcal{O}(1))$.*

If there exists an m-dominating class β of $\mathcal{O}(1)$-degree $< m$, then $(X, \mathcal{O}(1))$ is isomorphic to either $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ or a quadric hypersurface $(Q, \mathcal{O}_{\mathbb{P}^{n+1}}(1)|_Q)$.

If $(X, \mathcal{O}(1))$ is neither a linear variety nor a quadric hypersurface, then every $\mathcal{O}(1)$-degree m, m-dominating class is m-minimal.

If $(X, \mathcal{O}(1))$ is a linear variety, [line] is 1-minimal. If $(X, \mathcal{O}(1))$ is a quadric hypersurface, m[line] is m-minimal for $m = 1, 2$.

Proof. For a general m-tuple of points on X, every irreducible curve C containing the points satisfies

$$h^1(C, N_{C/X}(-m)) = 0$$

and therefore

$$\chi(C, N_{C/X}(-m)) \geq 0.$$

By Riemann-Roch, this implies

$$\langle c_1(T_X), C\rangle - 2 - (m-1)(\dim(X) - 1) = \deg(N_{C/X}(-m)) + \operatorname{rank}(N_{C/X}(-m)) \geq 0.$$

Substituting in

$$c_1(T_X) = \iota_X c_1(\mathcal{O}(1))$$

and

$$\langle c_1(\mathcal{O}(1)), C\rangle \leq m - 1$$

gives,

$$\iota_X \geq \dim(X)$$

As is well-known, this implies either $X \cong \mathbb{P}^n$ or X is a quadric hypersurface.

For linear varieties and quadric hypersurfaces, lines are obviously 1-minimal. For a general pair of points on a quadric hypersurface, the join of the pair is not contained in the hypersurface. Thus conics are 2-minimal on a quadric hypersurface.

Assume $(X, \mathcal{O}(1))$ is neither a linear variety nor a quadric hypersurface. Let β be an m-minimal curve class of $\mathcal{O}(1)$-degree m. Let

$$m = m_1 + \cdots + m_r$$

be a partition of m and let $(\beta_1, \ldots, \beta_r)$ be a sequence of m_i-dominant classes β_i such that

$$\sum_{i=1}^{r} \beta_i \leq \beta.$$

By the first part of the lemma, the $\mathcal{O}(1)$-degree of β_i is $\geq m_i$. Thus the sum of the $\mathcal{O}(1)$-degrees of the classes β_i equals the $\mathcal{O}(1)$-degree of β. Since $\mathcal{O}(1)$ is ample, this implies

$$\sum_{i=1}^{r} \beta_i = \beta.$$

Therefore β is m-minimal. $\square$

When is an $\mathcal{O}(1)$-degree m curve class m-dominant? By Lemma 5.1 if $c_1(T_X) = \langle c_1(T_X), \alpha \rangle c_1(\mathcal{O}(1))$ and if $m\alpha$ is m-dominant, then

$$\dim(X) \geq m(\dim(X) - \langle c_1(T_X), \alpha \rangle - 1) + 3.$$

If ev is not generically 1-to-1, then a general fiber of ev is irreducible only if

$$\dim(X) \geq m(\dim(X) - \langle c_1(T_X), \alpha \rangle - 1) + 4.$$

Assuming these inequalities hold, is $m\alpha$ an m-dominant class, resp. m-dominant with irreducible geometric generic fiber over X^m?

For $m = 1$, this is precisely the issue addressed by Lemma 4.4. For $m > 1$, the question can be reduced to the case of $m_i < m$.

Let m_1, m_2 be positive integers such that $m_1 + m_2 = m$. For $i = 1, 2$, let β_i be an $\mathcal{O}(1)$-degree m_i curve class. Denote

$$\beta = \beta_1 + \beta_2.$$

For $i = 1, 2$, let p_i be a general point of X^{m_i}. Denote

$$p = (p_1, p_2) \in X^m.$$

For $i = 1, 2$, let Π_i be the subvariety of X (possibly empty) swept out by curves in the fiber of

$$\text{ev} : \overline{\mathcal{M}}_{0,m_i}(X, \beta_i) \to X^{m_i}$$

over p_i.

lem-mpcdom

Lemma 5.6. *Let $(X, \mathcal{O}(1))$ be a smooth projective variety and let β be an $\mathcal{O}(1)$-degree m curve class on X. If for every general point $p \in X^m$ there exists a decomposition,*

$$m = m_1 + m_2, \quad \beta = \beta_1 + \beta_2$$

for which Π_1 intersects Π_2, then β is m-dominating.

Conversely, if β is m-dominating and if

$$dim(X) \geq m(dim(X) - 1) - \langle c_1(T_X), \beta \rangle + 4$$

then there exists a nontrivial decomposition as above such that Π_1 intersects Π_2.

Proof. The first part is obvious by taking a union of two intersecting curves. The second part follows from the bend-and-break lemma, cf. [Kol96, Corollary II.5.5]. $\square$

These results imply a consequence for the canonical irreducible components of Section 3.

lem-mpcM

Lemma 5.7. *Assume $(X, \mathcal{O}(1))$ is neither a linear variety nor a quadric hypersurface and $Pic(X) = \mathbb{Z}\{c_1(\mathcal{O}(1))\}$; in particular, $c_1(T_X) = \langle c_1(T_X), \alpha\rangle c_1(\mathcal{O}(1))$ for some $\mathcal{O}(1)$-degree 1 curve class α.*

Assume there exists a unique irreducible component $M_{\alpha,1}$ of $\overline{\mathcal{M}}_{0,1}(X, \alpha)$ such that the restriction

$$ev|_M : M_{\alpha,1} \to X$$

is dominant, and further assume the general fiber is irreducible. Assume

$$dim(X) \geq m(dim(X) - \langle c_1(T_X), \alpha\rangle - 1) + 4$$

If there exists an irreducible component M_m of $\overline{\mathcal{M}}_{0,m}(X, m\alpha)$ such that the restriction

$$ev|_M : M_m \to X^m$$

is dominant, then M_m equals the component $M_{m\cdot\alpha,m}$ from Notation 3.7.

Proof. The lemma will be proved by induction on m. For $m = 1$, it is tautological. Thus assume $m > 1$ and the lemma is true for all smaller values $m' < m$. First of all, by Lemmas 5.1, 5.3 and 5.5, $m\alpha$ is m-minimal and a general point of M_m parametrizes a smooth curve. Therefore the image of M_m in $\overline{\mathcal{M}}_{0,0}(X, m\alpha)$ is an irreducible component M_0, and M_m is the unique irreducible component of $\overline{\mathcal{M}}_{0,m}(X, m\alpha)$ dominating M_0.

Let $p = (p_1, \ldots, p_m)$ be a general point of X^m. By the second part of Lemma 5.6, there exists a decomposition

$$m = m_1 + m_2$$

and a reducible curve $C = C_1 \cup C_2$ parametrized by M_m such that

$$(C_1 \cap \{p_1, \ldots, p_m\}) \sqcup (C_1 \cap \{p_1, \ldots, p_m\})$$

is a partition of $\{p_1, \ldots, p_m\}$ into subsets of size m_1 and m_2 respectively. The sum of the $\mathcal{O}(1)$-degrees of C_1 and C_2 equals the $\mathcal{O}(1)$-degree of C, m. On the other hand, by Lemma 5.5, the $\mathcal{O}(1)$-degree of C_i is $\geq m_i$. Therefore the $\mathcal{O}(1)$-degree of C_i is precisely m_i.

Because p is general, also every subset of $\{p_1, \ldots, p_m\}$ is general. Therefore the component M'_{m_i} of $\overline{\mathcal{M}}_{0,m_i}(X, m_i\alpha)$ parametrizing $[C_i]$ dominates X^{m_i}. Moreover, because X is neither linear nor a quadric hypersurface,

$$\dim(X) - \langle c_1(T_X), \alpha\rangle - 1 \geq 0.$$

Therefore, since

$$\dim(X) - 4 \geq m(\dim(X) - \langle c_1(T_X), \alpha\rangle - 1),$$

also for $i = 1, 2$,

$$\dim(X) - 4 \geq m_i(\dim(X) - \langle c_1(T_X), \alpha\rangle - 1).$$

Thus m_i and M_{m_i} satisfy all the hypotheses of the Lemma. Because $m_i < m$, by the induction hypothesis, M'_{m_i} equals $M_{m_i\cdot\alpha,m_i}$.

By Lemmas 5.3 and 5.5, every irreducible component of C_i is a free curve. Therefore M_0 contains a point in the image of

$$\Delta : M_{m_1\cdot\alpha,1} \times_X M_{m_2\cdot\alpha,1} \to \overline{\mathcal{M}}_{0,0}(X, m\alpha)$$

which is a smooth point of $\overline{\mathcal{M}}_{0,0}(X, m\alpha)$. By Lemma 3.5, because both morphisms

$$\mathrm{ev} : M_{m_i \cdot \alpha, 1} \to X$$

are dominant with irreducible generic fiber, $M_{m_1\cdot\alpha,1} \times_X M_{m_2\cdot\alpha,1}$ is irreducible (by the same argument as in the proof of Lemma 3.4). Therefore M_0 is the unique irreducible component of $\overline{\mathcal{M}}_{0,0}(X, m\alpha)$ containing the image of $M_{m_1\cdot\alpha,1} \times_X M_{m_2\cdot\alpha,1}$.

By Lemma 3.5, each of $M_{m_i\cdot\alpha,1}$ parametrizes a reducible curve C_{m_i} whose (non-contracted) components are all smooth, free curves parametrized by $M_{\alpha,1}$. Because they are free, they deform so that the marked point p_i is a general point of X. In particular, there exists a pair of curves $((C_{m_1}, p_1), (C_{m_2}, p_2))$ as above so that p_1 and p_2 coincide in X, i.e., the pair is contained in the fiber product

$$M_{m_1\cdot\alpha,1} \times_X M_{m_2\cdot\alpha,1}.$$

The image under Δ parametrizes a reducible curve $C = C_1 \cup C_2$. Every (non-contracted) component of C is either a (non-contracted) component of C_1 or of C_2. Thus, by hypothesis, it is a smooth, free curve parametrized by $M_{\alpha,1}$. Since M_0 contains such a curve, by Lemma 3.5, M_0 equals $M_{m\cdot\alpha,0}$. Since M_m is the unique irreducible component dominating M_0, M_m equals $M_{m\cdot\alpha,m}$. The lemma is proved by induction on m. □

Lemma 5.6 provides the induction step in an inductive proof of the existence of an m-dominating curve class of $\mathcal{O}(1)$-degree m. For the induction step to apply, it is necessary that certain cycles in X have nonempty intersection. The simplest way to insure this is to require the relevant Chow groups of X to be generated by the appropriate power of $c_1(\mathcal{O}(1))$.

lem-mpcdom2

Lemma 5.8. *Assume that $c_1(T_X) = \langle c_1(T_X), \alpha\rangle c_1(\mathcal{O}(1))$ for an $\mathcal{O}(1)$-degree 1 curve class α, i.e., $c_1(T_X)$ is a multiple of $c_1(\mathcal{O}(1))$, resp. assume Pic(X) is generated by $c_1(\mathcal{O}(1))$. Assume that $(X, \mathcal{O}(1))$ is neither a linear space nor a quadric hypersurface. Let m be an integer $m \geq 2$. Assume that*

$$\mathit{dim}(X) \geq m(\mathit{dim}(X) - \langle c_1(T_X), \alpha\rangle - 1) + m + 2,$$

$$\textit{resp.} \ \geq 2(m-1)(\mathit{dim}(X) - \langle c_1(T_X), \alpha\rangle - 1) + 2m - 6.$$

Assume that for every pair of nonempty closed subsets Π_1, Π_2, resp. for every m-tuple of nonempty closed subsets $\Pi'_1, \dots, \Pi'_m$, of X with pure dimensions

$$\mathit{dim}(\Pi_1) = \langle c_1(T_X), \alpha\rangle - 1,$$

and

$$\mathit{dim}(\Pi_2) = \langle c_1(T_X), \alpha\rangle - (m-2)(\mathit{dim}(X) - \langle c_1(T_X), \alpha\rangle) + m - 3,$$

respectively

$$\mathit{dim}(\Pi'_i) = \langle c_1(T_X), \alpha\rangle - 1, \ \textit{for } i = 1, \dots, m,$$

the intersection $\Pi_1 \cap \Pi_2$ is nonempty, resp. $\Pi'_1 \cap \cdots \cap \Pi'_m$ is nonempty and connected. In particular this holds if

$$\mathit{CH}^p(X) = \mathbb{Z}\{c_1(\mathcal{O}(1))^p\}, \ \textit{for } 0 \leq p \leq \mathit{dim}(X) + 1 - \langle c_1(T_X), \alpha\rangle,$$

respectively

$$\textit{for } 0 \leq p \leq m(\mathit{dim}(X) + 1 - \langle c_1(T_X), \alpha\rangle).$$

Then $m\alpha$ is m-dominating, resp. $m\alpha$ is m-dominating and a general fiber of ev is connected.

Proof. First of all, because X is neither linear nor a quadric hypersurface,

$$\dim(X) - \langle c_1(T_X), \alpha\rangle - 1 \geq 0.$$

This means that the various inequalities above for m imply the analogous inequalities for $m' < m$. This is the basis for the induction argument.

There are two directions: nonemptiness and connectedness. Nonemptiness of a general fiber of

$$\mathrm{ev} : \overline{\mathcal{M}}_{0,m}(X, m\alpha) \to X^m$$

will be proved by induction on m. For $m = 1$, connectedness follows from Lemma 4.4. Thus, assume $m > 1$ and the result is known for $m-1$. By the induction hypothesis, the fiber of

$$\mathrm{ev} : \overline{\mathcal{M}}_{0,m-1}(X, (m-1)\alpha) \to X^{m-1}$$

over a general point $(p_1, \ldots, p_{m-1})$ is nonempty. By Lemmas 5.1, 5.3 and 5.5, the fiber has pure dimension

$$\langle c_1(T_X), \alpha\rangle - (m-2)(\dim(X) - \langle c_1(T_X), \alpha\rangle) + m - 4.$$

Therefore the subvariety Π'' swept out by the curves parametrized by the fiber has dimension one greater,

$$\dim(\Pi'') = \langle c_1(T_X), \alpha\rangle - (m-2)(\dim(X) - \langle c_1(T_X), \alpha\rangle) + m - 3.$$

Fix a general point p_m in X. By Lemma 4.4, a general fiber of

$$\mathrm{ev} : \overline{\mathcal{M}}_{0,1}(X, \alpha) \to X$$

is connected. By Lemma 4.1, the fiber has dimension $\langle c_1(T_X), \alpha\rangle - 2$. Therefore the subvariety Π' swept out by curves in the fiber has dimension one greater,

$$\dim(\Pi') = \langle c_1(T_X), \alpha\rangle - 1.$$

By hypothesis, Π' and Π'' intersect. Therefore there exists a curve C'' of class $(m-1)\alpha$ containing $p_1, \ldots, p_{m-1}$ and a curve C' of class α containing p_m such that C' intersects C''. The union $C = C' \cup C''$ is a curve of class $m\alpha$ containing $p_1, \ldots, p_{m-1}, p_m$. Therefore $m\alpha$ is m-dominating.

Next we address connectedness. Let M be an irreducible component of a the fiber of

$$\mathrm{ev} : \overline{\mathcal{M}}_{0,m}(X, m\alpha) \to X^m$$

over a general point $p = (p_1, \ldots, p_m)$ of X^m. By repeatedly applying the second part of Lemma 5.6, M parametrizes some curves C every irreducible component of which has class α and contains precisely one of the points $\{p_1, \ldots, p_m\}$. Moreover, the locus Λ of such curves has dimension

$$m\langle c_1(T_X), \alpha\rangle - (m-1)\dim(X) - 2.$$

Now consider the pointed lines parametrized by this locus. The total space of the family is a $\mathbb{P}^1$-bundle over the locus, together with a section contracted to a point p_i. For such a $\mathbb{P}^1$-bundle over a 1-dimensional base, any two effective curves that do not intersect the contracted section do intersect each other. From this it follows that there is a nonempty sublocus Λ' of dimension

$$m\langle c_1(T_X), \alpha\rangle - (m-1)\dim(X) - m$$

parametrizing curves such that every component is a curve of class α containing precisely one of the points $\{p_1, \ldots, p_m\}$, and such that all components intersect in a common point q. For $i = 1, \ldots, m$, denote by Π'_i the closed subvariety of X swept out by all α-curves containing p_i. Then clearly Λ' is a $\overline{\mathcal{M}}_{0,m}$-bundle over the intersection

$$\Pi' := \Pi'_1 \cap \cdots \cap \Pi'_m.$$

By Lemmas 4.1 and 4.3, each Π' is nonempty of dimension

$$\dim(\Pi'_i) = \langle c_1(T_X), \alpha \rangle - 1.$$

By hypothesis, the intersection Π' of an m-tuple of such subvarieties is nonempty and connected. Therefore Λ' is connected. Therefore $\mathrm{ev}^{-1}(p)$ is connected. □

The following two lemmas form the analogue for higher m of Lemma 4.4.

lem-mpcdom3

Lemma 5.9. *Let $(X, \mathcal{O}(1))$ be a smooth, projective variety that is neither a linear space nor a quadric hypersurface. Let m be a positive integer $\leq h^0(X, \mathcal{O}(1))$. Let β be an $\mathcal{O}(1)$-degree m curve class on X. Assume that*

$$dim(X) \geq m(dim(X) - 1) - \langle c_1(T_X), \beta \rangle + 3, \ \textit{resp.} \ \geq m(dim(X) - 1) - \langle c_1(T_X), \beta \rangle + 4.$$

In particular, if $\beta = m\alpha$, assume

$$dim(X) \geq m(dim(X) - \langle c_1(T_X), \alpha \rangle - 1) + 3, \ \textit{resp.} \ \geq m(dim(X) - \langle c_1(T_X), \alpha \rangle - 1) + 4.$$

Assume there exists a smooth projective variety Y such that X is a codimension c linear section of Y and

$$ev_Y : \overline{\mathcal{M}}_{0,m}(Y, \beta) \to Y^m$$

is dominant, resp. dominant with connected generic fiber. Then

$$ev_X : \overline{\mathcal{M}}_{0,m}(X, \beta) \to X^m$$

is dominant, resp. dominant with connected generic fiber.

Proof. The proof is very similar to the proof of Lemma 4.4. Embed Y in $\mathbb{P}^N$ and let $X = Y \cap \Lambda$ for a linear subspace Λ of codimension c. For an m-tuple of points $p_1, \ldots, p_m$, in X, a β-curve in Y containing $p_1, \ldots, p_m$ is contained in X if and only if its span is contained in $p_1, \ldots, p_m$. Using Lemmas 5.1, 5.3 and 5.5, together with the argument from the proof of Lemma 4.4, the morphism ev_Y is a flat, local complete intersection morphism at every point of the fiber $\mathrm{ev}_X^{-1}(p)$ over a general point p of X^m. The claim is that $\mathrm{ev}_X^{-1}(p)$ is an intersection of c big, base-point-free divisors in $\mathrm{ev}_Y^{-1}(p)$. This will immediately imply that every component of $\mathrm{ev}_Y^{-1}(p)$ meets $\mathrm{ev}_X^{-1}(p)$, in particular $\mathrm{ev}_X^{-1}(p)$ is not empty. Moreover, if a general fiber of ev_Y is connected, by the same argument from the proof of Lemma 4.4, $\mathrm{ev}_Y^{-1}(p)$ is a connected, locally complete intersection. Then by [Har77, Corollary III.7.9, Corollary III.11.5], $\mathrm{ev}_X^{-1}(p)$ is also connected.

To see that $\mathrm{ev}_X^{-1}(p)$ is an intersection of big, base-point-free divisors in $\mathrm{ev}_Y^{-1}(p)$, fix a hyperplane $H \subset \mathbb{P}^N$ containing $\{p_1, \ldots, p_m\}$. Associated to every $\mathcal{O}(1)$-degree m curve C containing $\{p_1, \ldots, p_m\}$, the span of C is an m-plane containing $\mathrm{span}(p_1, \ldots, p_m)$. In other words, it is a point in $\mathbb{P}^{N-m} \cong \mathbb{P}^N/\mathrm{span}(p_1, \ldots, p_m)$. The hyperplane H corresponds to a hyperplane in $\mathbb{P}^{N-m}$. Therefore the set of curves C contained in H forms a divisor in $\mathrm{ev}_Y - 1(p)$. Varying H, it is clear that this divisor class is big and base-point-free. □

cor-mpcdom4

Corollary 5.10. *Assume that* $c_1(T_X) = \langle c_1(T_X), \alpha\rangle c_1(\mathcal{O}(1))$ *for an* $\mathcal{O}(1)$*-degree* 1 *curve class* α*, i.e.,* $c_1(T_X)$ *is a multiple of* $c_1(\mathcal{O}(1))$*, resp. assume* $Pic(X)$ *is generated by* $c_1(\mathcal{O}(1))$*. Assume that* $(X, \mathcal{O}(1))$ *is neither a linear space nor a quadric hypersurface. Let* m *be an integer* $m \geq 2$*. Assume that*

$$dim(X) \geq m(dim(X)-\langle c_1(T_X), \alpha\rangle-1)+3, \ \textit{resp.} \ \geq m(dim(X)-\langle c_1(T_X), \alpha\rangle-1)+4.$$

Assume there exists a smooth, projective variety Y *such that* X *is a codimension* c *linear section of* Y*. If*

$$c \geq m(dim(X) - \langle c_1(T_X), \alpha\rangle - 1) - dim(X) + m - 2,$$

respectively

$$c \geq 2(m-1)(dim(X) - \langle c_1(T_X), \alpha\rangle - 1) - dim(X) + 2m - 6$$

and if

$$CH^p(Y) = \mathbb{Z}\{c_1(\mathcal{O}(1))^p\}, \ \textit{for } 0 \leq p \leq dim(X) + 1 - \langle c_1(T_X), \alpha\rangle,$$

respectively

$$\textit{for } 0 \leq p \leq m(dim(X) + 1 - \langle c_1(T_X), \alpha\rangle),$$

then

$$ev_X : \overline{\mathcal{M}}_{0,m}(X, m\alpha) \to X^m$$

is dominant, resp. dominant with connected fibers. By Barth's theorems, [Bar70], *this last condition holds if*

$$c \geq 2(dim(X) - \langle c_1(T_X), \alpha\rangle - 1) - dim(X) + (h^0(X, \mathcal{O}(1)) - dim(X) - 2) + 5,$$

respectively

$$c \geq 2m(dim(X) - \langle c_1(T_X), \alpha\rangle - 1) - dim(X) + (h^0(X, \mathcal{O}(1)) - dim(X) - 2) + 4m + 1.$$

Proof. This follows from Lemmas 5.8 and 5.9. □

The main application is to complete intersections.

Corollary 5.11. *Let* X *be a smooth complete intersection in* $\mathbb{P}^n$ *of multidegree* $\underline{d} = (d_1, \dots, d_c)$*. For simplicity, assume all* $d_i \geq 2$ *and assume* $\underline{d} \neq (2)$*. Let* m *be an integer* $1 \leq m \leq n$*. The fiber of* cor-mpcdom5

$$ev : \overline{\mathcal{M}}_{0,m}(X, m) \to X^m$$

over a general point in X^m *is nonempty, resp. nonempty and irreducible, if*

$$dim(X) \geq m(dim(X)-\langle c_1(T_X), \alpha\rangle-1)+3, \ \textit{resp.} \ \geq m(dim(X)-\langle c_1(T_X), \alpha\rangle-1)+4,$$

in other words, if

$$n \geq \sum_{i=1}^{c}(md_i - m + 1) - m + 3, \ \textit{resp.} \ \geq \sum_{i=1}^{c}(md_i - m + 1) - m + 4.$$

Proof. This follows from Corollary 5.10 since complete intersections are arbitrarily extendable by Lemma 2.5. □

6. Minimal pointed curves. Canonical bundle and uniruledness

sec-mpc2

The rational surfaces we will need in Section 7 arise from minimal free curves in the space of minimal pointed curves. We study these curves using the formula for the canonical bundle from [dJS05] together with the method of Kollár-Miyaoka-Mori, cf. [MM86], [Kol96, §V.1]. As in Sections 4 and 5, the existence of the curves follows from hypotheses on $\dim(X)$, $\langle c_1(T_X), \alpha\rangle$, and also the second graded piece of the Chern character $\langle \mathrm{ch}_2(T_X), \Pi\rangle$. The hypotheses are stronger than the weak hypotheses suggested by a parameter count. However, under when X is a linear section of sufficiently high codimension in a smooth, projective variety Y, the weaker hypotheses suffice (and, in fact, are also necessary conditions).

Because of the criterion in Lemma 5.5, we shall focus on the case of m-minimal classes of $\mathcal{O}(1)$-degree m. Moreover the varieties we will focus on have another feature formalized in the following definition/hypothesis.

defn-mcpU

Definition 6.1. Let m and N be integers satisfying $1 \leq m \leq N$. For a linearly general m-tuple $p = (p_1, \dots, p_m)$ in $(\mathbb{P}^N)^m$, the *linearly nondegenerate locus* U_p is the maximal open substack of the corresponding fiber of

$$\mathrm{ev} : \overline{\mathcal{M}}_{0,m}(\mathbb{P}^N, m) \to (\mathbb{P}^N)^m$$

parametrizing stable maps for which no irreducible component is mapped into the linear subspace $\mathrm{span}(p_1, \dots, p_m)$. By Bézout's theorem, this is equivalent to the condition that no non-contracted component is mapped into the linear subspace $\mathrm{span}(p_1, \dots, p_m)$.

lem-mpcU

Lemma 6.2. *Let $(X, \mathcal{O}(1))$ be a smooth, projective variety. Let α be an $\mathcal{O}(1)$-degree 1 curve class. Let m be a positive integer $\leq h^0(X, \mathcal{O}(1))$. If $m \geq 2$, assume $(X, \mathcal{O}(1))$ is not a linear variety. If $m \geq 3$, assume $(X, \mathcal{O}(1)$ is not a quadric hypersurface. Assume $c_1(T_X) = \langle c_1(T_X), \alpha\rangle c_1(\mathcal{O}(1))$, i.e., $c_1(T_X)$ is a multiple of $c_1(\mathcal{O}(1))$. Assume $m\alpha$ is an m-dominating class. Assume either*

(i)
$$dim(X) > \langle c_1(T_X), \alpha\rangle + 1,$$

(ii)
$$dim(X) = \langle c_1(T_X), \alpha\rangle + 1,$$

and
$$h^0(X, \mathcal{O}(1)) \geq dim(X) + m - 2,$$

(iii) *$(X, \mathcal{O}(1))$ is a quadric hypersurface and $m \leq 2$, or*

(iv) *$(X, \mathcal{O}(1))$ is linear and $m = 1$.*

Then the fiber of

$$ev : \overline{\mathcal{M}}_{0,m}(X, m\alpha) \to X^m$$

over a general point p of X^m is contained in the nondegenerate locus U_p.

Proof. Embed X in $\mathbb{P}^N$ by the complete linear system $|\mathcal{O}(1)|$, where $N = h^0(X, \mathcal{O}(1)) - 1$. First consider the case when

$$h^0(X, \mathcal{O}(1)) \geq \dim(X) + m.$$

The codimension of X is $c = h^0(X, \mathcal{O}(1)) - \dim(X) - 1 \geq m-1$. Also the degree of X is $\geq \mathrm{codim}(X) + 1 \geq m$. Therefore m general points $p_1, \dots, p_m$ of X are contained in a codimension c linear section of X, which is necessarily zero dimensional. For every curve C in X containing $p_1, \ldots, p_m$, $\text{span}(C)$ contains $\text{span}(p_1, \ldots, p_m)$, which is contained in the zero dimensional linear section of X. Since $\dim(C)$ is positive, $\text{span}(C)$ must properly contain $\text{span}(p_1, \ldots, p_m)$. In particular, if C is a degree m rational curve, C is linearly nondegenerate.

Next assume that

$$\dim(X) + m > h^0(X, \mathcal{O}(1)).$$

The claim is that

$$h^0(X, \mathcal{O}(1)) \geq \dim(X) - m(\dim(X) - \langle c_1(T_X), \alpha\rangle - 1) + m - 2.$$

Indeed, $h^0(X, \mathcal{O}(1)) \geq \dim(X) + 1$ always. So in case (i),

$$h^0(X, \mathcal{O}(1)) \geq \dim(X)+1 \geq \dim(X)-2 \geq \dim(X)-2-m(\dim(X)-\langle c_1(T_X), \alpha\rangle-2).$$

In case (ii), by hypothesis,

$$h^0(X, \mathcal{O}(1)) \geq \dim(X) + m - 2 = \dim(X) - m(\dim(X) - \langle c_1(T_X), \alpha\rangle - 1) + m - 2.$$

In case (iii), since $m \leq 2$,then

$$h^0(X, \mathcal{O}(1)) = \dim(X)+2 \geq \dim(X)-m(-1)+m-2 = \dim(X)-m(\dim(X)-\langle c_1(T_X), \alpha\rangle-1)+m-2.$$

And case (iv) is clear.

Let Z be the intersection of X with a general $\mathbb{P}^{m-1}$ in $\mathbb{P}^N$. Then

$$\dim(Z) = \dim(X) + m - h^0(X, \mathcal{O}(1)),$$

and

$$\langle c_1(T_Z), \alpha\rangle = \langle c_1(T_X), \alpha\rangle + m - h^0(X, \mathcal{O}(1)).$$

Thus,

$$\dim(Z) - \langle c_1(T_Z), \alpha\rangle - 1 = \dim(X) - \langle c_1(T_X), \alpha - 1.$$

Plugging this in,

$$\dim(Z)-m(\dim(Z)-\langle c_1(T_Z), \alpha\rangle-1)-3 = \dim(X)+m-h^0(X, \mathcal{O}(1))-m(\dim(X)-\langle c_1(T_X), \alpha\rangle-1)-3.$$

Because

$$h^0(X, \mathcal{O}(1)) \geq \dim(X) - m(\dim(X) - \langle c_1(T_X), \alpha\rangle - 1) - 3,$$

it follows that

$$\dim(Z) - m(\dim(Z) - \langle c_1(T_Z), \alpha\rangle - 1) - 3$$

is negative. By Lemma 5.1, there is no $\mathcal{O}(1)$-degree m curve class on Z that is m-dominating. Let $p_1, \ldots, p_m$ be m general points of Z. These span a $\mathbb{P}^{m-1}$, i.e., $\text{span}(p_1, \ldots, p_m) = \text{span}(Z)$. Let C be a degree m curve in X containing $p_1, \ldots, p_m$. Then $\text{span}(C)$ contains $\text{span}(p_1, \ldots, p_m) = \text{span}(Z)$. Because there is no $\mathcal{O}(1)$-degree m curve class on Z that is m-dominating, the curve C cannot be contained in Z, i.e., it cannot be contained in $\text{span}(Z)$. Therefore $\text{span}(C)$ properly contains $\text{span}(Z) \cong \mathbb{P}^{m-1}$, i.e., C is a linearly nondegenerate curve. $\square$

Because of Lemma 6.2, we introduce the following hypothesis.

hyp-mpcU

Hypothesis 6.3. Let $(X, \mathcal{O}(1), m)$ be a smooth, projective variety together with a positive integer m. Let α be an $\mathcal{O}(1)$-degree 1 curve class. Assume $m \leq h^0(X, \mathcal{O}(1))$. Assume $c_1(T_X) = \langle c_1(T_X), \alpha \rangle c_1(\mathcal{O}(1))$, i.e., $c_1(T_X)$ is a multiple of $c_1(\mathcal{O}(1))$. If $m \geq 2$, assume $(X, \mathcal{O}(1))$ is not a linear variety. If $m \geq 3$, assume $(X, \mathcal{O}(1))$ is not a quadric hypersurface. If $\dim(X) = \langle c_1(T_X), \alpha \rangle + 1$, assume $h^0(X, \mathcal{O}(1)) \geq \dim(X) + m - 2$. Assume $m\alpha$ is an m-dominating class. By Lemma 6.2, these assumptions imply the fiber M_p of

$$\mathrm{ev} : \overline{\mathcal{M}}_{0,m}(X, m\alpha) \to X^m \subset (\mathbb{P}^N)^m$$

over a general point p of X^m is contained in the linearly nondegenerate locus U_p.

The varieties we consider satisfy Hypothesis 6.3. The hypothesis implies consequences for the divisor theory of M_p. As usual, denote by

$$(\pi : \mathcal{C} \to U, (s_i : U \to \mathcal{C} | i = 1, \ldots, m), g : \mathcal{C} \to \mathbb{P}^N)$$

the universal family of stable maps over U.

lem-mpc4

Lemma 6.4. *The stack U_p is automorphism-free, i.e., U_p is a quasi-projective scheme. Moreover, there exists a big, base-point-free Cartier divisor class λ on U such that*

$$\pi_* g^* c_1(\mathcal{O}_{\mathbb{P}^N}(1))^2 = m\lambda,$$
$$\pi_*(s_i(U) \cdot s_i(U)) = -\lambda, \quad i = 1, \ldots, m.$$

If $m > 1$, also

$$\frac{1}{m-1} \sum_{\{A,B\}, A \cup B = \{1,\ldots,m\}} \#A \cdot \#B \Delta_{\{A,B\}} = m\lambda.$$

Proof. Denote by E the locally free sheaf

$$E := \pi_* g^* \mathcal{O}_{\mathbb{P}^N}(1).$$

Denote by F the vector space

$$F := \oplus_{i=1}^m \mathcal{O}_{\mathbb{P}^N}(1) \otimes \kappa(p_i)$$

Because $g \circ s_i$ is a constant morphism with image p_i, there is an evaluation morphism

$$E \to F \otimes \mathcal{O}_U.$$

This is surjective, because it factors the surjective evaluation

$$H^0(\mathbb{P}^N, \mathcal{O}(1)) \otimes \mathcal{O}_U \to E \to F \otimes \mathcal{O}_U.$$

Denote the kernel by

$$\mathcal{L} := \mathrm{Ker}(E \to F \otimes \mathcal{O}_U).$$

This is an invertible sheaf on U. Denote its divisor class by

$$\lambda := c_1(\mathcal{L}).$$

It is clear that λ is base-point-free. The complete linear system defines a morphism

$$\phi_\lambda : U_p \to \mathbb{P}^{N-m} = \mathbb{P}^N / \mathrm{span}(p_1, \ldots, p_m)$$

that associates to a stable map $(C, (q_1, \ldots, q_m), f)$ the point

$$\mathrm{span}(f(C)) \in \mathbb{P}^N / \mathrm{span}(p_1, \ldots, p_m).$$

It is clear that the only locus contracted by this morphism is the locus of curves with a contracted component (not mapping into $\mathrm{span}(p_1, \ldots, p_m)$!) containing at least 4 double points. Therefore ϕ_λ is big.

By adjunction there is a natural homomorphism

$$\pi^* E \to g^* \mathcal{O}_{\mathbb{P}^N}(1).$$

Restricting to $\mathcal{L}$ gives a morphism of invertible sheaves

$$\pi^* \mathcal{L} \to g^* \mathcal{O}_{\mathbb{P}^N}(1),$$

or equivalently, a global section of the twist

$$\sigma : \mathcal{O}_{\mathcal{C}} \to \pi^* \mathcal{L}^\vee \otimes g^* \mathcal{O}_{\mathbb{P}^N}(1).$$

The zero locus of this section is precisely

$$\text{Zero locus}(\sigma) = g^{-1}\mathrm{span}(p_1, \ldots, p_m).$$

Of course there is an inclusion

$$\cup_{i=1}^m s_i(U) \subset g^{-1}\mathrm{span}(p_1, \ldots, p_m).$$

Now a curve of degree m spans at most a $\mathbb{P}^m$. Thus $\mathrm{span}(p_1, \ldots, p_m)$ is a hyperplane in the span of the curve. By Bézout's theorem, if the intersection has degree $\geq m+1$, then some irreducible component of the curve is contained in $\mathrm{span}(p_1, \ldots, p_m)$. By hypothesis, every curve parametrized by U has no irreducible component contained in $\mathrm{span}(p_1, \ldots, p_m)$. Thus

$$\text{Zero locus}(\sigma) = \cup_{i=1}^m s_i(U)$$

as closed subschemes of $\mathcal{C}$. This implies an isomorphism of invertible sheaves

$$\mathcal{O}_{\mathcal{C}}(\sum_{i=1}^m s_i(U)) \cong \pi^* \mathcal{L}^\vee g^* \mathcal{O}(1).$$

Applying s_i^* to each side of this isomorphism gives

$$s_i^* \mathcal{O}_{\mathcal{C}}(s_i(U)) \cong \mathcal{L}^\vee,$$

or equivalently,

$$-\pi_*[s_i(U) \cdot s_i(U)] = \lambda$$

for every $i = 1, \ldots, m$. Combined with [dJS05, Lemma 5.8], this implies the divisor class relations

$$\frac{1}{m-1} \sum_{(A,B), 1 \in A, A \cup B = \{1,\ldots,m\}} \#A \cdot \#B \Delta_{(A,B)} = m\lambda$$

and

$$\frac{1}{m-1} \sum_{\{A,B\}} \#A \cdot \#B \Delta_{\{A,B\}} = m\lambda.$$

Also,

$$\pi_* g^* c_1(\mathcal{O}_{\mathbb{P}^N}(1))^2 = \pi_*[\sum_{i=1}^m s_i(U) + \pi^*\lambda]^2 = \sum_{i=1}^m \pi_*[s_i(U) \cdot s_i(U)] + 2\sum_{i=1}^m s_i^* \pi^* \lambda$$

which reduces to

$$\pi_* g^* c_1(\mathcal{O}_{\mathbb{P}^N}(1))^2 = m\lambda.$$

□

When m is small, this allows us to give a useful formula for the first Chern class of a general fiber of the evaluation morphism ev.

lem-mpc5

Lemma 6.5. *If $(X, \mathcal{O}(1), m)$ satisfies Hypothesis 6.3, then the fiber M_p of*

$$ev: \overline{\mathcal{M}}_{0,m}(X, m\alpha) \to X^m$$

over a general point p of X^m is a smooth, projective variety of the expected dimension

$$dim(X) - m(dim(X) - \langle c_1(T_X), \alpha\rangle - 1) - 3$$

whose first Chern class equals

$$c_1(T_M) = \pi_* g^*[ch_2(T_X) - \frac{(m-2)\langle c_1(T_X), \alpha\rangle + 2m}{2m} c_1(\mathcal{O}(1))^2] + 2\Delta.$$

In particular, for $m = 1$ this is the formula from Lemma 4.1

$$c_1(T_M) = \pi_* g^*[ch_2(T_X) + \frac{\langle c_1(T_X), \alpha\rangle - 2}{2} c_1(\mathcal{O}(1))^2],$$

for $m = 2$ this is

$$c_1(T_M) = \pi_* g^*[ch_2(T_X) + c_1(\mathcal{O}(1))^2],$$

and for $m = 3$ this is

$$c_1(T_M) = \pi_* g^*[ch_2(T_X) - \frac{\langle c_1(T_X), \alpha\rangle - 6}{6} c_1(\mathcal{O}(1))^2].$$

Proof. For $m = 1$, this follows from Lemma 4.1. Thus assume $m > 1$.

First of all, by Lemma 5.5, $m\alpha$ is m-minimal. Thus, by Lemma 5.3, M_p is smooth and automorphism-free. By [dJS05, Theorem 1.1], the first Chern class of M_p equals

$$c_1(T_M) = \pi_* g^*[\mathrm{ch}_2(T_X) + \frac{1}{2\langle c_1(T_X), m\alpha\rangle} c_1(T_X)^2] + 2\Delta$$

$$-\frac{1}{r-1} \sum_{\{A,B\}} \#A \cdot \#B \Delta_{\{A,B\}} - \frac{1}{2\langle c_1(T_X), m\alpha\rangle} \sum_{\{\beta', \beta''\}, \beta' + \beta'' = m\alpha} \langle c_1(T_X), \beta'\rangle \langle c_1(T_X), \beta''\rangle \Delta_{\{\beta', \beta''\}}.$$

Using the fact that every curve class is a multiple of α, this reduces to

$$\pi_* g^*[\mathrm{ch}_2(T_X) + \frac{1}{2\langle c_1(T_X), m\alpha\rangle} c_1(T_X)^2] + 2\Delta$$

$$-\frac{1}{r-1} \sum_{\{A,B\}} \#A \cdot \#B \Delta_{\{A,B\}} - \frac{\langle c_1(T_X), \alpha\rangle}{2m} \sum_{\{m', m''\} m' + m'' = m} m' m'' \Delta_{\{m'\alpha, m''\alpha\}}.$$

Next, because $m\beta$ is m-minimal,

$$\sum_{\{A,B\}, \#A = m', \#B = m''} \Delta_{\{A,B\}} = \Delta_{\{m'\alpha, m''\alpha\}}.$$

So the formula reduces further to

$$\pi_* g^*[\mathrm{ch}_2(T_X) + \frac{1}{2\langle c_1(T_X), m\alpha\rangle} c_1(T_X)^2] + 2\Delta$$

$$-\frac{1}{m-1} \sum_{m' + m'' = m} m' m'' \Delta_{\{m'\alpha, m''\alpha\}} - \frac{\langle c_1(T_X), \alpha\rangle}{2m} \sum_{m' + m'' = m} m' m'' \Delta_{\{m'\alpha, m''\alpha\}}.$$

Finally, by Lemma 6.4, there is a divisor class relation

$$\frac{1}{m-1}\sum_{m'+m''=m} m'm''\Delta_{\{m'\alpha,m''\alpha\}} = m\lambda = \pi_* g^* c_1(\mathcal{O}(1))^2.$$

Substituting this in and simplifying gives the formula.

The formulas in the cases $m = 2$ and 3 follow from the formula above together with the relations from Lemma 6.4,

$$\Delta_{1,1} = \frac{1\cdot 1}{1}\Delta_{1,1} = 2\lambda = \pi_* g^* c_1(\mathcal{O}(1))^2,$$
$$\Delta_{2,1} = \frac{2\cdot 1}{2}\Delta_{2,1} = 3\lambda = \pi_* g^* c_1(\mathcal{O}(1))^2.$$

When $m > 3$, the boundary Δ is not of a single type, i.e., $\Delta \neq \Delta_{i,j}$. Thus there is not a similar formula for $m > 3$. □

Because of the formula in the last lemma, $\mathrm{ch}_2(T_X)$ plays a significant role in the geometry of a general fiber of ev. It is useful to make the following hypothesis, which holds for all complete intersections.

hyp-mpcU2

Hypothesis 6.6. Assume $(X, \mathcal{O}(1), m)$ satisfies Hypothesis 6.3. Further assume that

$$\mathrm{ch}_2(T_X) = \frac{\langle 2\mathrm{ch}_2(T_X), \Pi\rangle}{2} c_1(\mathcal{O}(1))^2$$

for some integer $\langle 2\mathrm{ch}_2(T_X), \Pi\rangle$. If X contains the class of a linear 2-plane Π, then the integer is the intersection number indicated.

Kollár, Miyaoka and Mori proved uniruledness of varieties with positive first Chern class. In our setting, this gives the following.

cor-mpc6

Corollary 6.7. *Assume that $(X, \mathcal{O}(1), m)$ satisfies Hypothesis 6.6. If*

$$m\langle 2ch_2(T_X), \Pi\rangle - (m-2)\langle c_1(T_X), \alpha\rangle - 2m > 0$$

then the fiber M_p of

$$ev : \overline{\mathcal{M}}_{0,m}(X, m) \to X^m$$

over a general point p of X^m is uniruled. If

$$(m-1)dim(X) + m\langle 2ch_2(T_X), \Pi\rangle - 2(m-1)\langle c_1(T_X), \alpha\rangle - 3m + 2 > 0$$

then M_p is uniruled by rational curves of λ-degree 1.

Proof. When the first inequality holds, by Lemma 6.5, $c_1(T_X)$ is the sum of a nef, big divisor and an effective divisor. Therefore, by [MM86], M is uniruled.

Moreover, applying bend-and-break, the minimal class γ of a free rational curve in X satisfies the inequality from [Kol96, Theorem V.1.6.1],

$$\langle c_1(T_M), \gamma\rangle \leq \dim(M) + 1.$$

Every free rational curve class in X satisfies

$$\langle \Delta, \gamma\rangle \geq 0$$

since such curves deform out of Δ. If $\langle \lambda, \gamma\rangle \geq 2$ and if the second inequality holds, then

$$\langle c_1(T_M), \gamma\rangle > \dim(M) + 1.$$

Therefore, if the second inequality holds, M is uniruled by rational curves of λ-degree 1. $\square$

lem-mpc7

Lemma 6.8. *Assume $(X, \mathcal{O}(1), m)$ satisfies Hypothesis 6.3. Assume that the fiber M_p of*

$$ev : \overline{\mathcal{M}}_{0,m}(X, m\alpha) \to X^m$$

over a general point p of X^m is uniruled by rational curves of λ-degree 1. A general point of M_p parametrizes a curve which is either a hyperplane section of a degree m rational normal surface scroll $\Sigma \subset X$ or a hyperplane section of a Veronese surface $\Sigma \subset X$ (which can only occur if $m = 4$).

Proof. First of all, by the proof of Lemma 5.3, a general fiber M is smooth and automorphism-free, a general point of M parametrizes a smooth curve, and the boundary $\Delta \cap M$ is a simple normal crossings divisor.

Let D be a free rational curve in M of λ-degree 1. Denote by $\pi : \mathcal{C}_D \to D$ the restriction of the universal curve over D. Denote by $(\Sigma, \mathcal{O}(1))$ the linearly normal surface obtained by contracting all curves in $\mathcal{C}_D$ of $g^*\mathcal{O}(1)$-degree 0. In other words, if $\mathcal{C}_D \to \mathbb{P}^N$ is the morphism induced by the complete linear system of $g^*\mathcal{O}(1)$, then Σ is the linearly normal image of $\mathcal{C}_D$ in $\mathbb{P}^N$.

By Lemma 6.4, the surface Σ has $\mathcal{O}(1)$-degree m since D has λ-degree 1. Because the invertible sheaf $\mathcal{L}|_D$ has degree 1,

$$h^0(\Sigma, \mathcal{O}(1)) = h^0(D, \pi_* g^*\mathcal{O}(1)|_D) = m + 2.$$

Therefore Σ spans a linear $\mathbb{P}^{m+1}$. A surface of degree m spanning $\mathbb{P}^{m+1}$ is a surface of *minimal degree*. These were classified by Del Pezzo in 1885, [DP85]. For a modern account and generalization, see [EH87]. In particular, there are three possibilities for Σ: Σ is a cone over a rational normal curve, Σ is the Veronese surface, or Σ is a (smooth) rational normal surface scroll. In each of these cases, D is a pencil of hyperplane sections of Σ.

If Σ is a cone over a rational normal curve, then the hyperplane containing the vertex of the cone gives a point of D that intersects the boundary Δ with multiplicity > 1. Assuming D is general in its deformation class, D intersects any specified divisor transversally, cf. the proof of [Kol96, Proposition II.3.7]. Therefore Σ is not a cone over a rational normal surface scroll.

The final claim is that the morphism $\Sigma \to X$ is an embedding. If not, the image spans a $\mathbb{P}^m$. But then $\mathrm{span}(p_1, \dots, p_m) \cap \Sigma$ is a hyperplane section of Σ, thus a curve in Σ. In particular, it is strictly larger than $\{p_1, \dots, p_m\}$. Therefore a curve parametrized by D intersects $\mathrm{span}(p_1, \dots, p_m)$ in a subscheme of degree $\geq m + 1$. By Bézout's theorem, the curve has an irreducible component contained in $\mathrm{span}(p_1, \dots, p_m)$. This contradicts the hypothesis that D is contained in the open substack U of Lemma 6.4. Therefore $\Sigma \to X$ is an embedding. $\square$

Using the previous result, we can compute the dimension of the space of free λ-degree 1 curves in M_p. In particular, since the dimension must be positive if such curves exist, the dimension inequality gives a necessary condition for the existence of free λ-degree 1 curves in M_p.

lem-mpc8

Lemma 6.9. *Assume $(X, \mathcal{O}(1), m)$ satisfies Hypothesis 6.6. Assume that the fiber M_p of*

$$ev : \overline{\mathcal{M}}_{0,m}(X, m\alpha) \to X^m$$

over a general point p of X^m is uniruled by rational curves of λ-degree 1. With one exception, the space of λ-degree 1 curves in M_p containing a general point of M_p has dimension

$$\frac{m}{2}\langle 2ch_2(T_X), \Pi\rangle - \frac{m-2}{2}\langle c_1(T_X), \alpha\rangle + m - 2.$$

The one exception occurs when $m = 4$ and the λ-degree 1 curves sweep out Veronese surfaces. In this case the dimension equals

$$2\langle 2ch_2(T_X), \Pi\rangle - \langle c_1(T_X), \alpha\rangle.$$

Proof. For a free curve D in a smooth projective variety M, the dimension of the space of deformations of D containing a general point m equals

$$\langle c_1(T_M), D\rangle - 2,$$

cf. [Kol96, Theorem II.1.7, Corollary II.3.5.3]. The formula for $c_1(T_M)$ is given in Lemma 6.5. By hypothesis, the λ-degree is 1. Thus the only missing data to compute $\langle c_1(T_M), D\rangle$ is the intersection number with Δ. The two possibilities for D are given in Lemma 6.8: either D is a pencil of hyperplane sections of a rational normal surface scroll, or D is a pencil of hyperplane sections of a Veronese surface. In the first case, the intersection number with Δ equals m: the hyperplane becomes reducible precisely when it contains a line of ruling through one of the m points $p_1, \dots, p_m$. In the second case, the intersection number with Δ is 3 (even though $m = 4$): a pencil of conics in $\mathbb{P}^2$ has 3 reducible members. Substituting this in gives the two formulas above. $\square$

Assuming X is a linear section of sufficiently high codimension in a smooth projective variety Y, Lemma 6.9 implies the necessary condition above for the existence of a λ-degree 1 free curve is also a sufficient condition.

lem-mpc9

Lemma 6.10. *Assume $(X, \mathcal{O}(1), m)$ satisfies Hypothesis 6.6. If $m \neq 4$, assume that*

$$\frac{m}{2}\langle 2ch_2(T_X), \Pi\rangle - \frac{m-2}{2}\langle c_1(T_X), \alpha\rangle + m - 2 \geq 0.$$

If $m = 4$, assume that

$$2\langle 2ch_2(T_X), \Pi\rangle - \langle c_1(T_X), \alpha\rangle \geq 0.$$

Assume there exists a smooth projective variety Y such that X is a codimension c linear section of Y. If

$$c > 2(m-1)\langle c_1(T_X), \alpha\rangle - m\langle 2ch_2(T_X), \Pi\rangle - (m-1)dim(X) + 3m - 2$$

then a general fiber M of

$$ev : \overline{\mathcal{M}}_{0,m}(X, m\alpha) \to X^m$$

is uniruled by rational curves of λ-degree 1.

Proof. First of all, the hypotheses of Corollary 6.7 for X imply the hypotheses for Y. There is a formula

$$(m-1)\dim(Y) + m\langle 2\mathrm{ch}_2(T_Y), \Pi\rangle - 2(m-1)\langle c_1(T_Y), \alpha\rangle - 3m + 2 =$$
$$c + (m-1)\dim(X) + m\langle 2\mathrm{ch}_2(T_X), \Pi\rangle - 2(m-1)\langle c_1(T_X), \alpha\rangle - 3m + 2.$$

Therefore the inequality on c implies that

$$(m-1)\dim(Y) + m\langle 2\mathrm{ch}_2(T_Y), \Pi\rangle - 2(m-1)\langle c_1(T_Y), \alpha\rangle - 3m + 2 > 0.$$

By Corollary 6.7, a general fiber of

$$\mathrm{ev} : \overline{\mathcal{M}}_{0,m}(Y, m\alpha) \to Y^m$$

is uniruled by λ-degree 1 rational curves. By Lemma 6.9, the dimension of the space of such rational curves is at least

$$a_Y = \frac{m}{2}\langle 2\mathrm{ch}_2(T_Y), \Pi\rangle - \frac{m-2}{2}\langle c_1(T_Y), \alpha\rangle + m - 2$$

if $m \neq 4$, and is at least

$$a_Y = 2\langle 2\mathrm{ch}_2(T_Y), \Pi\rangle - \langle c_1(T_Y), \alpha\rangle$$

if $m = 4$. Of course there are formulas

$$\frac{m}{2}\langle 2\mathrm{ch}_2(T_Y), \Pi\rangle - \frac{m-2}{2}\langle c_1(T_Y), \alpha\rangle + m - 2 =$$
$$c + \frac{m}{2}\langle 2\mathrm{ch}_2(T_X), \Pi\rangle - \frac{m-2}{2}\langle c_1(T_X), \alpha\rangle + m - 2,$$

respectively

$$2\langle 2\mathrm{ch}_2(T_Y), \Pi\rangle - \langle c_1(T_Y), \alpha\rangle =$$
$$c + 2\langle 2\mathrm{ch}_2(T_X), \Pi\rangle - \langle c_1(T_X), \alpha\rangle.$$

In other words,

$$a_Y = c + a_X.$$

By upper semicontinuity, for every point of $\overline{\mathcal{M}}_{0,m}(Y, m\alpha)$ contained in an irreducible component dominating Y^m, the dimension of the space of λ-degree 1 curves is at least a_Y. Let $[C] \in \overline{\mathcal{M}}_{0,m}(X, m\alpha)$ be a general point of a general fiber of

$$\mathrm{ev} : \overline{\mathcal{M}}_{0,m}(X, m\alpha) \to X^m.$$

By Lemma 5.3, C is a smooth curve and

$$h^1(C, T_X|_C(-m)) = 0.$$

There is a short exact sequence

$$0 \longrightarrow T_X \longrightarrow T_Y|_X \longrightarrow \mathcal{O}(1)^{\oplus c} \longrightarrow 0$$

giving rise to a short exact sequence

$$0 \longrightarrow T_X|_C \longrightarrow T_Y|_C \longrightarrow \mathcal{O}(1)|_C^{\oplus c} \longrightarrow 0.$$

Since C has $\mathcal{O}(1)$-degree m, $\mathcal{O}(1)|_C \cong \mathcal{O}_C(m)$. Therefore twisting this sequence by $\mathcal{O}_C(-m)$ gives

$$0 \longrightarrow T_X|_C(-m) \longrightarrow T_Y|_C(-m) \longrightarrow \mathcal{O}_C^{\oplus c} \longrightarrow 0.$$

Since

$$h^1(C, \mathcal{O}_C) = 0$$

by the long exact sequence of cohomology, also

$$h^1(C, T_Y|_C(-m)) = 0$$

Thus C deforms to a curve in Y passing through m general points. Therefore the dimension of the space of λ-degree 1 rational curves containing C, for Y, is at least a_Y.

Denote by M_X the fiber of

$$\text{ev} : \overline{\mathcal{M}}_{0,m}(X, m\alpha) \to X^m$$

over a general point $(p_1, \ldots, p_m)$ and denote by M_Y the fiber of

$$\text{ev} : \overline{\mathcal{M}}_{0,m}(Y, m\alpha) \to Y^m$$

over the same point $(p_1, \ldots, p_m)$. Recall there is a morphism $M_Y \to \mathbb{P}^{N-m}$ defined as follows. Associated to a curve C in $\mathbb{P}^N$ containing $p_1, \ldots, p_m$, there is the point $v = \text{span}(C)$ in the projective space $\mathbb{P}^{N-m} = \mathbb{P}^N/\text{span}(p_1, \ldots, p_m)$. Each λ-degree 1 curve in M_Y maps to a line in $\mathbb{P}^{N-m}$. Because X is a codimension c linear section of Y, there is a codimension c linear subspace Λ of $\mathbb{P}^{N-m}$ such that M_X equals $M_Y \times_{\mathbb{P}^{N-m}} \Lambda$. Let $[C]$ be a general point of M_X mapping to a point v of $\mathbb{P}^{N-m}$. By the previous paragraph, the space of λ-degree 1 curves in M_Y containing $[C]$ is at least a_Y. These map to lines in $\mathbb{P}^{N-m}$ containing v, i.e., these correspond to points in the quotient projective space $\mathbb{P}^{N-m-1} = \mathbb{P}^{N-m}/v$. Of course $v \in \Lambda$ and Λ/v is a codimension c linear subspace of $\mathbb{P}^{N-m-1}$. Therefore, as long as

$$a_Y \geq c$$

there exists a λ-degree 1 curve in M_Y containing $[C]$ whose image is containing in Λ, i.e., there exists a λ-degree 1 curve in M_X. Because $a_Y = c + a_X$, the inequality above is precisely,

$$a_X \geq 0$$

which is the inequality in the statement of the lemma. □

The main application is to complete intersections.

cor-mpc10

Corollary 6.11. *Let X be a smooth complete intersection in $\mathbb{P}^n$ of type $\underline{d} = (d_1, \ldots, d_c)$. Without loss of generality, assume all $d_i > 1$. Also assume $\underline{d} \neq (2)$. If $\underline{d} = (3)$, assume $m \leq 4$, and if $\underline{d} = (2,2)$, assume $m \leq 5$.*

If $m \neq 4$, assume

$$n \geq \sum_{i=1}^{c} \left(\frac{m}{2} d_i^2 - \frac{m-2}{2} d_i\right) + 1 - m.$$

If $m = 4$, assume

$$n \geq \sum_{i=1}^{c} (2d_i^2 - d_i) - 1.$$

Then $(X, \mathcal{O}(1), m)$ satisfies Hypothesis 6.6 and a general fiber of

$$ev : \overline{\mathcal{M}}_{0,m}(X, m[\mathit{line}]) \to X^m$$

is uniruled by λ-degree 1 rational curves.

Proof. First of all, the inequalities above implies the inequality in Corollary 5.11. Thus, Corollary 5.11 together with Lemma 5.5 implies $m[\text{line}]$ is an m-minimal class. If $\dim(X) = \langle c_1(T_X), \alpha\rangle + 1$, then $\underline{d} = (3)$ or $\underline{d} = (2,2)$. In the first case, $h^0(X, \mathcal{O}(1)) \geq \dim(X) + 2 = \dim(X) + 4 - 2$. In the second case, $h^0(X, \mathcal{O}(1)) \geq \dim(X) + 3 = \dim(X) + 5 - 2$. Thus, in every case $(X, \mathcal{O}(1), m)$ satisfies Hypothesis 6.3. Moreover,

$$\mathrm{ch}_2(T_X) = \frac{\langle 2\mathrm{ch}_2(T_X), \Pi\rangle}{2} c_1(\mathcal{O}(1))^2,$$

for the integer

$$\langle 2\mathrm{ch}_2(T_X), \Pi\rangle = n + 1 - \sum_{i=1}^{c} d_i^2.$$

Therefore X satisfies Hypothesis 6.6.

Because a complete intersection is infinitely extendable, Lemma 2.5, the second hypothesis of Lemma 6.10 holds. Substituting in

$$\langle 2\mathrm{ch}_2(T_X), \Pi\rangle = n + 1 - \sum_{i=1}^{c} d_i^2$$

and

$$\langle c_1(T_X), [\text{line}]\rangle = n + 1 - \sum_{i=1}^{c} d_i,$$

the first hypothesis of Lemma 6.10 is precisely

$$n \geq \sum_{i=1}^{c} \left(\frac{m}{2} d_i^2 - \frac{m-2}{2} d_i\right) + 1 - m$$

if $m \neq 4$, respectively

$$n \geq \sum_{i=1}^{c} (2d_i^2 - d_i) - 1$$

if $m = 4$. Thus, by Lemma 6.10, a general fiber of

$$\mathrm{ev} : \overline{\mathcal{M}}_{0,m}(X, m[\text{line}]) \to X^m$$

is uniruled by λ-degree 1 rational curves. $\square$

sec-q

7. Twisting surfaces. Existence

Every rational ruled surface

$$\pi : \Sigma \to \mathbb{P}^1$$

is isomorphic to the projective bundle

$$\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-h)) \to \mathbb{P}^1$$

for some nonnegative integer h. We will call the integer h the *Hirzebruch type* of the ruled surface, or *H-type* for short. We are mostly interested in the case that $h = 0$, i.e., $\Sigma = \mathbb{P}^1 \times \mathbb{P}^1$. But for some arguments it is convenient to have a more general setup.

Denote by F the divisor class of a fiber of π in Σ. Let E be the divisor class of a section of π with minimal self-intersection, i.e., E is the directrix. If this section is unique, also denote by E the unique effective curve in this divisor class. The self-intersection of the section is

$$(E \cdot E)_\Sigma = -h.$$

Denote by F' the divisor class

$$F' := E - (E \cdot E)_\Sigma F = E + hF$$

i.e., F' is the unique divisor class of a section such that

$$(F' \cdot E)_\Sigma = 0.$$

Associated to every morphism from a rational ruled surface to X,

$$f : \Sigma \to X,$$

there is a morphism

$$(\pi, f) : \Sigma \to \mathbb{P}^1 \times X.$$

In other words, consider f as a family of morphisms from $\mathbb{P}^1$ – the fibers of π – to X parametrized by the base of π, which is also $\mathbb{P}^1$. If (π, f) is finite, the *vertical normal bundle* is defined to be

$$N_{(\pi,f)} := \mathrm{Coker}(d(\pi, f) : T_\Sigma \to (\pi, f)^* T_{\mathbb{P}^1 \times X}).$$

Although $N_{(\pi,f)}$ need not be locally free, it is π-flat.

Lemma 7.1. *The sheaf $N_{(\pi,f)}$ is π-flat.* lem-q0

Proof. By the local flatness criterion, [Mat89, Theorem 22.5], it suffices to prove that for every geometric point t of $\mathbb{P}^1$, the induced morphism

$$d(f_t) : \Sigma_t \to f_t^* T_X$$

is injective, where f_t is the restriction of f to $\Sigma_t = \pi^{-1}(t)$. By hypothesis, f_t is finite. Since the characteristic is 0, this implies f_t is generically unramified and thus $d(f_t)$ is injective. $\square$

There is a slightly technical lemma which is useful. Let

$$\pi : \Sigma \to \mathbb{P}^1$$

be a ruled surface. Let n be a positive integer. Let N be a coherent sheaf on Σ. There is a cup-product map

$$H^0(\Sigma, \mathcal{O}_\Sigma(F' + nF)) \times H^1(\Sigma, N(-F' - nF)) \to H^1(\Sigma, N),$$

or equivalently a map

$$c : H^0(\Sigma, \mathcal{O}(F' + nF)) \to \mathrm{Hom}(H^1(\Sigma, N(-F' - nF)), H^1(\Sigma, N)).$$

Lemma 7.2. *Let n be a positive integer. Let N be a π-flat coherent sheaf on Σ. Assume there exists an N-regular section α of $H^0(\Sigma, \mathcal{O}_\Sigma(F' + nF))$, for which $c(\alpha)$ is injective. Also assume there exists an N-regular divisor $D \in |\mathcal{O}_\Sigma(F' + nF)|$ such that $N|_D$ is generated by global sections. Then N is generated by global sections and $h^1(\Sigma, N(-F' - nF)) = 0$.* lem-qtech

Proof. Let U be the maximal open subscheme of Σ on which N is generated by global sections. The goal is to prove that U equals all of Σ.

If $c(\alpha)$ is injective for one N-regular α, then it is injective for a general choice of α. Similarly, if $N|_D$ is generated by global sections for one N-regular D, it is generated by global sections for a general choice of D. Thus there exists α and D as above so that D is the zero locus of α. Also, we may assume that D is irreducible (hence smooth).

Consider the short exact sequence

$$0 \longrightarrow N(-D) \longrightarrow N \longrightarrow N|_D \longrightarrow 0.$$

There is a long exact sequence of cohomology, some of whose terms are

$$H^0(\Sigma, N) \longrightarrow H^0(D, N|_D) \longrightarrow H^1(\Sigma, N(-n,-1)) \xrightarrow{c(\alpha)} H^1(\Sigma, N).$$

By the hypothesis that $c(\alpha)$ is injective, the first map is surjective. By the hypothesis that $N|_D$ is generated by global sections, D is contained in U. Since D is an ample divisor, the complement of U consists of finitely many closed points of Σ.

claim-qtecha

Claim 7.3. *Both*

$$R^1\pi_*N \text{ and } R^1\pi_*N(-D)$$

are zero.

Because N is π-flat, by cohomology and base change [Har77, Theorem III.12.11], it suffices to prove that for every fiber F of π,

$$h^1(F, N|_F) = h^1(F, N(-D)|_F) = 0.$$

Now $N|_F$ is a coherent sheaf on $F \cong \mathbb{P}^1$ whose global sections generate N on the dense open $F \cap U$. Every coherent sheaf on $\mathbb{P}^1$ that is generically generated is globally generated. And for a globally generated coherent sheaf $N|_F$ on $\mathbb{P}^1$,

$$h^1(F, N|_F) = h^1(F, N|_F(-1)) = 0.$$

This is easy to see using Grothendieck's lemma on splitting of vector bundles together with the cohomology of invertible sheaves on $\mathbb{P}^1$. Therefore

$$R^1\pi_*N = R^1\pi_*N(-D) = (0),$$

i.e., Claim 7.3 is valid.

Because of Claim 7.3, the Leray spectral sequence associated to π proves that $c(\alpha)$ is a morphism

$$H^1(\mathbb{P}^1, \pi_*N(-D)) \to H^1(\mathbb{P}^1, \pi_*N).$$

Because $c(\alpha)$ is injective

$$h^1(\mathbb{P}^1, \pi_*N(-D)) \leq h^1(\mathbb{P}^1, \pi_*N).$$

There is an isomorphism

$$\pi_*N(-D) \cong [\pi_*(N(-F'))] \otimes \mathcal{O}(-n),$$

where F' is a general member in its linear equivalence class. In particular, F' is contained in U, thus $N|_{F'}$ is generated by global sections. Since there is a short exact sequence

$$0 \longrightarrow N(-F') \longrightarrow N \longrightarrow N|_{F'} \longrightarrow 0$$

and since

$$R^1\pi_*N(-F') \cong (R^1\pi_*N(-D))(n)$$

is zero by Claim 7.3, there is a short exact sequence

$$0 \longrightarrow \pi_*N(-F') \longrightarrow \pi_*N \longrightarrow N|_{F'} \longrightarrow 0.$$

Since $N|_{F'}$ is generated by global sections,

$$h^1(\mathbb{P}^1, N|_{F'}) = 0.$$

Thus the long exact sequence of cohomology gives

$$h^1(\mathbb{P}^1, \pi_*N(-F')) \geq h^1(\mathbb{P}^1, \pi_*N).$$

Since $\pi_*N(-F')$ is coherent, by Grothendieck's lemma it has the form

$$\pi_*N(-F') \cong (\text{Torsion}) \oplus \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_r)$$

for some integers $a_1, \ldots, a_r$. If h^1 is positive, some of the a_i are negative. Then, of course, twisting by $\mathcal{O}(-n)$ makes those a_i more negative. Therefore, if

$$h^1(\mathbb{P}^1, \pi_*N(-F')) \neq 0,$$

then

$$h^1(\mathbb{P}^1, [\pi_*N(-F')] \otimes \mathcal{O}(-m)) > h^1(\mathbb{P}^1, \pi_*N(-F')),$$

which implies

$$h^1(\mathbb{P}^1, \pi_*N(-D)) > h^1(\mathbb{P}^1, \pi_*N).$$

This contradicts the previous paragraph. Therefore

$$h^1(\mathbb{P}^1, \pi_*N(-F')) = 0.$$

By the previous inequalities this implies

$$h^1(\mathbb{P}^1, \pi_*N) = 0,$$

which in turn implies

$$h^1(\mathbb{P}^1, \pi_*N(-D)) = 0.$$

By Claim 7.3 and the Leray spectral sequence for π,

$$h^1(\Sigma, N(-D)) = h^1(\mathbb{P}^1, \pi_*N(-D)) = 0.$$

Thus it only remains to prove N is globally generated.

Since $\pi_*N(-D)$ is a coherent sheaf on $\mathbb{P}^1$ with vanishing h^1, the twist $[\pi_*N(-D)] \otimes \mathcal{O}(n)$ is globally generated, i.e., $\pi_*N(-F')$ is globally generated. As the first and last terms in the short exact sequence

$$0 \longrightarrow \pi_*N(-F') \longrightarrow \pi_*N \longrightarrow N|_{F'} \longrightarrow 0$$

are globally generated, also π_*E is globally generated. As established above, E is π-relatively globally generated. Therefore E is globally generated. $\square$

The geometric question behind these definitions is this: Given a rational ruled surface in X, and given a deformation in X of a rational curve in the surface, is there a corresponding deformation of the surface containing the deformation of the curve? The following lemma gives one answer.

lem-qmove

Lemma 7.4. *Let $\pi : \Sigma \to \mathbb{P}^1$ be a ruled surface. Let $f : \Sigma \to X$ be a morphism such that (π, f) is finite. Let n be a positive integer. Assume that $N_{(pr_1,f)}$ is globally generated and $h^1(\Sigma, N_{(pr_1,f)}(-F' - nF)) = 0$. Every free curve in Σ maps to a free curve in X. Moreover, for every reduced curve D in $|\mathcal{O}_\Sigma(F' + nF)|$, for every infinitesimal deformation of D in X, there exists an infinitesimal deformation of Σ in X containing the deformation of D.*

Proof. For a free curve C in Σ, both $N_{C/\Sigma}$ and $N_{(\mathrm{pr}_1,f)}|_C$ are globally generated. Therefore $N_{C/X}$ is globally generated, i.e., C is a free curve in X.

In particular, since a reduced curve D in $|\mathcal{O}_\Sigma(F' + nF)|$ is free in Σ, it is free in X. Thus the deformation space of the morphism $(D, f|_D : D \to X)$ (allowing both D and f to vary) is smooth. Because

$$h^1(D, N_{(\mathrm{pr}_1,f)}|_D) = h^1(\Sigma, N_{(\mathrm{pr}_1,f)}(-D)) = 0,$$

and by the long exact sequence of cohomology associated to the short exact sequence

$$0 \longrightarrow N_{(\mathrm{pr}_1,f)}(-D) \longrightarrow N_{(\mathrm{pr}_1,f)} \longrightarrow N_{(\mathrm{pr}_1,f)}|_D \longrightarrow 0,$$

it follows that $h^1(\Sigma, N_{(\mathrm{pr}_1,f)}) = 0$. Thus the deformation space of $(\pi, f) : \Sigma \to \mathbb{P}^1 \times X$ (allowing both Σ and (π, f) to vary) is also smooth. It follows that the deformation space of the datum,

$$(\Sigma, D \subset \Sigma, (\pi, f) : \Sigma \to \mathbb{P}^1 \times X),$$

(allowing Σ and (π, f) to vary, and allowing D to vary as a divisor in Σ) is also smooth.

There is a morphism from the deformation space of the datum $(\Sigma, D, (\pi, f))$ to the deformation space of $(D, f|_D)$. Because both are smooth, the morphism is smooth if and only if the induced map of Zariski tangent spaces is surjective, i.e., if and only if

$$H^0(\Sigma, N_{(\mathrm{pr}_1,f)}) \to H^0(D, N_{(\mathrm{pr}_1,f)}|_D)$$

is surjective. By the long exact sequence above, the cokernel is contained in $H^1(\Sigma, N_{(\mathrm{pr}_1,f)}(-D))$. This is zero by hypothesis, therefore the map of deformation spaces is surjective. $\square$

The central notion of this section is as follows. It is closely related to the notion of *twisting family of pointed curves* from [HS05] and [Sta04].

defn-q1

Definition 7.5. For an integer n an *n-twisting surface* in X is a ruled surface

$$\pi : \Sigma \to \mathbb{P}^1$$

together with a morphism

$$f : \Sigma \to X$$

such that

(i) f^*T_X is generated by global sections,
(ii) (π_1, f) is finite and

$$h^1(\Sigma, N_{(\pi,f)}(-F' - nF)) = 0.$$

If the pushforward of the divisor class F equals β_1 and the pushforward of F' equals β_2, then f *has class* (β_1, β_2).

Let α and $M = M_{\alpha,1}$ be given as in Lemma 3.5. If the pushforward of every element in $|\mathcal{O}_\Sigma(F)|$ is in $M_{e_1\cdot\alpha,0}$ and the pushforward of every element in $|\mathcal{O}_\Sigma(F')|$ is in $M_{e_2\cdot\alpha,0}$, then f *has M-class* $(e_1\cdot\alpha, e_2\cdot\alpha)$.

There are some elementary results comprising the "sorites" of twisting surfaces.

lem-qa

Lemma 7.6. (i) *For $l \leq n$, every n-twisting surface is also l-twisting.*

(ii) *Let f be an n-twisting surface. For all integers $l_1 \geq max(-1, -n)$ and $l_2 \geq 0$,*

$$h^1(\Sigma, N_{(\pi,f)}(l_1F' + l_2F)) = 0.$$

(iii) *If f is a 1-twisting surface of H-type 0, then the composition of f with the permutation morphism*

$$(pr_2, pr_1) : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$$

is also 1-twisting of H-type 0.

(iv) *Let $h = 0$ or $h = 1$ and let $n = 1 - h$. For $i = 1, 2$, let*

$$f_i : \Sigma_i \to X$$

be an n-twisting surface in X of H-type h and with class (β_i, β), respectively with M-class $(e_i \cdot \alpha, e\cdot\alpha)$. Assume there exist irreducible divisors D_1 of class E in Σ_1 and D_2 of class F' in Σ_2 such that $f_1|_{D_1}$ equals $f_2|_{D_2}$. Then there exists a n-twisting surface in X of H-type h and with class $(\beta_1+\beta_2, \beta)$, resp. with M-class $((e_1 + e_2)\cdot\alpha, e\cdot\alpha)$.

(v) *Let n_1 and n_2 be positive integers. For $i = 1, 2$, let*

$$f_i : \Sigma_i \to X$$

be an n_i-twisting surface in X of class (β, β_i), respectively of M-class $(e\cdot\alpha, e_i\cdot\alpha)$. Assume the H-type of f_1 is 0 and the H-type of f_2 is $h = 0$ or $h = 1$. For $i = 1, 2$, assume there exists fibers F_i of $\pi_i : \Sigma_i \to \mathbb{P}^1$ and an isomorphism $F_1 \cong F_2$ such that

$$f_1|_{F_1} = f_2|_{F_2}.$$

Then there exists an $(n_1 + n_2 - 1)$-twisting surface f in X of H-type h and with class $(\beta, \beta_1 + \beta_2)$, resp. with M-class $(e\cdot\alpha, (e_1 + e_2)\cdot\alpha)$. Moreover, the restrictions of f_1, f_2 and f to F-curves give points of $\overline{\mathcal{M}}_{0,0}(X)$ that are all parametrized by the same irreducible component.

Proof. **(i)** If $l = n$, this is just the hypothesis that f is n-twisting. Thus assume $l < n$. Let C be a general member of the linear system $|\mathcal{O}_\Sigma((n-l)F)|$. Thus C is a disjoint union of $n - l$ fibers, $C = C_1 \cup \cdots \cup C_{n-l}$. There is a short exact sequence

$$0 \longrightarrow N_{(\pi,f)}(-F' - nF) \longrightarrow N_{(\pi_1,f)}(-F' - lF) \longrightarrow N_{(\pi_1,f)}(-F' - lF)|_C \longrightarrow 0.$$

Applying the long exact sequence in cohomology and the induction hypothesis,

$$h^1(\Sigma, N_{(\pi,f)}(-F'-lF)) \leq h^1(C, N_{(\pi,f)}(-F'-lF)|_C) = \sum_{i=1}^{n-l} h^1(C_i, N_{(\pi,f)}(-F'-lF)|_{C_i}).$$

There is a natural isomorphism

$$N_{(\pi,f)}(-F' - lF)|_{C_i} \cong N_{(\pi,f)}|_{C_i}(-1).$$

Because f^*T_X is globally generated, the same holds for $N_{(\pi,f)}|_{C_i}$. By Grothendieck's lemma,

$$N_{(\pi,f)}|_{C_i} \cong \text{Torsion} \oplus \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_r)$$

for integers $a_1, \ldots, a_r \geq 0$. Therefore

$$N_{(\pi,f)}|_{C_i}(-1) \cong \text{Torsion} \oplus \mathcal{O}(a_1 - 1) \oplus \cdots \oplus \mathcal{O}(a_r - 1)$$

for integers $a_i - 1 \geq -1$. Since $h^1(C_i, \mathcal{O}(b)) = 0$ for $b \geq -1$, it follows that

$$h^1(C_i, N_{(\pi,f)}(-F' - lF)|_{C_i}) = 0.$$

Therefore also

$$h^1(\Sigma, N_{(\pi,f)}(-F' - lF)) = 0.$$

(ii) Let C be a general member of the nonempty linear system $|\mathcal{O}((l_1+1)F' + (l_1 + n)F)|$. There is a short exact sequence

$$0 \longrightarrow N_{(\pi,f)}(-F' - nF) \longrightarrow N_{(\pi,f)}(l_2F' + l_1F) \longrightarrow N_{(\pi,f)}(l_2F' + l_1F)|_C \longrightarrow 0.$$

By the long exact sequence in cohomology and the hypothesis that f is n-twisting, it suffices to prove that

$$h^1(C, N_{(\pi,f)}(l_2F' + l_1F)|_C) = 0.$$

Because f^*T_X is globally generated, also $N_{(\pi,f)}$ is globally generated. Thus $N_{(\pi,f)}(l_2F' + l_1F)|_C$ is a quotient of $\mathcal{O}_\Sigma(l_2F' + l_1F)|_C^{\oplus N}$, for some integer N. Since $H^1(C, -)$ is right exact, it suffices to prove that

$$h^1(C, \mathcal{O}_\Sigma(l_2F' + l_1F)|_C) = 0.$$

Again using the long exact sequence of cohomology associated to the short exact sequence

$$0 \longrightarrow \mathcal{O}_\Sigma(-F' - nF) \longrightarrow \mathcal{O}_\Sigma(l_2F' + l_1F) \longrightarrow \mathcal{O}_\Sigma(l_2F' + l_1F)|_C \longrightarrow 0$$

and the fact that

$$h^1(\Sigma, \mathcal{O}(-F' - nF)) = h^2(\Sigma, \mathcal{O}(-F' - nF)) = 0$$

(using the Leray spectral sequence of π), it is equivalent to prove that

$$h^1(\Sigma, \mathcal{O}(l_2F' + l_1F)) = 0.$$

This holds since $l_1 \geq -1$ and $l_2 \geq 0$.

(iii) As

$$h^1(\mathbb{P}^1 \times \mathbb{P}^1, T_{\mathbb{P}^1 \times \mathbb{P}^1}(-1, -1)) = h^2(\mathbb{P}^1 \times \mathbb{P}^1, T_{\mathbb{P}^1 \times \mathbb{P}^1}(-1, -1)) = 0,$$

from the long exact sequence of cohomology,

$$h^1(\mathbb{P}^1 \times \mathbb{P}^1, N_{(\mathrm{pr}_1, f)}(-1, -1)) = h^1(\mathbb{P}^1 \times \mathbb{P}^1, f^*T_X(-1, -1)).$$

Therefore f is 1-twisting if and only if f^*T_X is globally generated and

$$h^1(\mathbb{P}^1 \times \mathbb{P}^1, f^*T_X(-1, -1)) = 0.$$

This is clearly symmetric in the two factors.

(iv) Define Σ' to be the coproduct of Σ_1 and Σ_2 attached along $D_1 \cong D_2$, the isomorphism being the unique one compatible with π_1 and π_2. There is a unique morphism

$$\pi' : \Sigma' \to \mathbb{P}^1$$

such that $\pi'|_{\Sigma_i} = \pi_i$ for $i = 1, 2$. There is a unique morphism

$$f' : \Sigma' \to X$$

such that $f'|_{\Sigma_i} = f_i$ for $i = 1, 2$. By (i) and (ii),

$$h^1(\Sigma_i, N_{(\pi_i, f_i)} = h^1(\Sigma_i, N_{(\pi_i, f_i)}(-E)) = 0$$

for $i = 1, 2$. Since $h = 0$ or 1, also

$$h^1(\Sigma_i, T_{\Sigma_i}(-E)) = h^1(\Sigma_i, T_{\Sigma_i}(-E))$$

for $i = 1, 2$. Thus

$$h^1(\Sigma_i, f_i^* T_X) = h^1(\Sigma_i, f_i^* T_X(-E)) = 0.$$

It follows from the long exact sequence of cohomology that

$$h^1(\Sigma', (f')^* T_X) = 0.$$

In particular, there are no obstructions to deforming f'. Since Σ' deforms to a ruled surface

$$\pi : \Sigma \to \mathbb{P}^1$$

of H-type h, it follows that f' deforms to a morphism

$$f : \Sigma \to \mathbb{P}^1.$$

Clearly the class of f is $(\beta_1 + \beta_2, \beta)$, resp. with M-class $((e_1 + e_2) \cdot \alpha, e \cdot \alpha)$. It only remains to prove f is 1-twisting.

Let $\mathcal{L}$ be the invertible sheaf on $\mathbb{P}^1 \times C$ whose restriction to Σ_1 is $\mathcal{O}(-F' - nF)$ and whose restriction to Σ_2 is $\mathcal{O}(-nF)$. There is a short exact sequence

$$0 \longrightarrow f_2^* T_X(-F' - nF) \longrightarrow (f')^* T_X \otimes \mathcal{L} \longrightarrow f_2^* T_X(-F' - nF) \longrightarrow 0.$$

As above,

$$h^1(\Sigma_i, T_{\Sigma_i}(-F' - nF)) = 0;$$

for $h = 1$ this is precisely the statement above, for $h = 0$, this follows from the fact that

$$h^1(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(-1, 1)) = 0.$$

By the long exact sequence of cohomology,

$$h^1(\Sigma_i, f_i^* T_X(-F' - nF)) = 0.$$

Therefore

$$h^1(\Sigma', (f')^* T_X \otimes \mathcal{L}) = 0.$$

The invertible sheaf $\mathcal{L}$ deforms to $\mathcal{O}(-F' - nF)$ on Σ. Therefore, by upper semi-continuity,

$$h^1(\Sigma, f^* T_X(-F' - nF)) = 0,$$

i.e., f is n-twisting.

(v) Let Σ' be the coproduct of Σ_1 and Σ_2 via the isomorphism $F_1 \cong F_2$. Let C' be the coproduct of $\mathbb{P}^1$ and $\mathbb{P}^1$ via $\pi_1(F_1) \sim \pi_2(F_2)$. There is a unique morphism

$$\pi' : \Sigma' \to C'$$

such that $\pi'|_{\Sigma_i} = \pi_i$. And there is a unique morphism

$$f' : \Sigma' \to X$$

such that $f'|_{\Sigma_i} = f_i$. By (ii)

$$h^1(\Sigma_1, N_{(\pi_1,f_1)}(-F)) = 0.$$

Since also

$$h^1(\Sigma_1, T_{\Sigma_1}(-F)) = 0,$$

by the long exact sequence of cohomology,

$$h^1(\Sigma_1, f_1^* T_X(-F)) = 0.$$

By a similar argument,

$$h^1(\Sigma_2, f_2^* T_X) = 0.$$

There is a short exact sequence

$$0 \longrightarrow f_1^* T_X(-F) \longrightarrow (f')^* T_X \longrightarrow f_2^* T_X \longrightarrow 0.$$

Using the corresponding long exact sequence and the vanishing above,

$$h^1(\Sigma, (f')^* T_X) = 0.$$

In particular, there are no obstructions to deforming f'. Since Σ' deforms to a ruled surface

$$\pi : \Sigma \to \mathbb{P}^1$$

of H-type h, f' deforms to a morphism

$$f : \Sigma \to X.$$

Clearly f has class $(\beta, \beta_1 + \beta_2)$, resp. M-class $(e \cdot \alpha, (e_1 + e_2) \cdot \alpha)$. It remains to prove that f is $(n_1 + n_2 - 1)$-twisting.

Let $\mathcal{L}$ be the invertible sheaf on Σ whose restriction to Σ_1 is $\mathcal{O}(-F' - (n_1 - 1)F)$ and whose restriction to Σ_2 is $\mathcal{O}(-F' - n_2 F)$. Because

$$N_{(\pi',f')}|_{\Sigma_i} \cong N_{(\pi_i,f_i)}$$

for $i = 1, 2$, there is a short exact sequence

$$0 \longrightarrow N_{(\pi_1,f_1)}(-F' - n_1 F) \longrightarrow N_{(\pi',f')} \otimes \mathcal{L} \longrightarrow N_{(\pi_2,f_2)}(-F' - n_2 F) \longrightarrow 0.$$

By the long exact sequence of cohomology and the hypothesis that f_i is n_i-twisting for $i = 1, 2$,

$$h^1(\Sigma, N_{(\pi',f')} \otimes \mathcal{L}) = 0.$$

Since $N_{(\pi',f')}$ deforms to $N_{(\pi,f)}$ and since $\mathcal{L}$ deforms to $\mathcal{O}(-F' - (n_1 + n_2 - 1)F)$, by upper semicontinuity it follows that

$$h^1(\Sigma, N_{(\pi,f)}(-F' - (n_1 + n_2 - 1)F)) = 0.$$

Therefore f is $(n_1 + n_2 - 1)$-twisting.

It is clear that the F curves in Σ are deformations of the F curves in Σ_1 and deformations of the F-curves in Σ_2. □

The next result is the basic "bootstrapping" result producing 1-twisting surfaces of larger class from 1-twisting surface of smaller class, and producing m-twisting surfaces for arbitrary m from a 1-twisting surface and a 2-twisting surface.

cor-qa

Corollary 7.7. (i) *Let n be a positive integer. Let $f : \Sigma \to X$ be an n-twisting surface of H-type 0. Let D be a reduced divisor in $|\mathcal{O}_\Sigma(F' + nF)|$. There exists an open subset of $\overline{\mathcal{M}}_{0,0}(X)$ containing $(D, f|_D)$ parametrizing maps (C, g_C) for which there exists an n-twisting surface $g : \Theta \to X$ of of H-type 0 and an embedding of C as a divisor in $|\mathcal{O}_\Theta(F'+nF)|$ such that $g_C = g|_C$. In other words, every small deformation of D is contained in a deformation of Σ that is n-twisting.*

(ii) *If there exists a 1-twisting surface $f_1 : \Sigma_1 \to X$ of H-type 0 and class (β_1, β_2), resp. M-class $(e_1 \cdot \alpha, e_2 \cdot \alpha)$, then for every positive integer d there exists a 1-twisting surface $f_d : \Sigma_d \to X$ of H-type 0 and class $(\beta_1, d\beta_2)$, resp. M-class $(e_1 \cdot \alpha, de_2 \cdot \alpha)$. Moreover, for curves C in $|\mathcal{O}_\Sigma(F)|$ and curves C_d in $|\mathcal{O}_{\Sigma_d}(F)|$, the stable maps $(C, f|_C)$ and $(C_d, f_d|_{C_d})$ are parametrized by the same irreducible component of $\overline{\mathcal{M}}_{0,0}(X)$.*

(iii) *If there exists a 1-twisting surface of H-type 0 and class (β_1, β_2), resp. M-class $(e_1 \cdot \alpha, e_2 \cdot \alpha)$, then for every pair of positive integers d_1, d_2, there exists a 1-twisting surface of H-type 0 and class $(d_1\beta_1, d_2\beta_2)$, resp. M-class $(d_1e_1 \cdot \alpha, d_2e_2 \cdot \alpha)$.*

(iv) *Assume there exists a 1-twisting $f_1 : \Sigma_1 \to X$ surface of H-type 0 and class (β, β_1), resp. M-class $(e \cdot \alpha, e_1 \cdot \alpha)$, and assume there exists a 2-twisting surface $f_2 : \Sigma_2 \to X$ of H-type 0 and class (β, β_2), resp. M-class $(e \cdot \alpha, e_2 \cdot \alpha)$, such that the restriction of f_1 to F-curves in Σ_1 and the restriction of f_2 to F-curves in Σ_2 give points of $\overline{\mathcal{M}}_{0,0}(X, \beta)$ parametrized by the same irreducible component (this is automatic if f_i has M-class $(e \cdot \alpha, e_i \cdot \alpha)$ for $i = 1, 2$). Then for every positive integer m and every nonnegative integer r, there exists an m-twisting surface $f_{m,r} : \Sigma_{m,r} \to X$ of H-type 0 and class $(\beta, r\beta_1 + (m-1)\beta_2)$, resp. M-class $(e \cdot \alpha, (re_1 + (m-1)e_2) \cdot \alpha)$. Moreover, the restriction of $f_{m,r}$ to the F-curves in $\Sigma_{m,r}$ give points of $\overline{\mathcal{M}}_{0,0}(X, \beta)$ parametrized by the same irreducible component as above.*

Proof. **(i)** This follows from Lemma 7.4.

(ii) This is proved by induction on d. The base case $d = 1$ is tautological. Therefore, by way of induction, assume $d > 1$ and the result is proved for $d - 1$. By (i), each of f_1 and f_{d-1} may be chosen so that its restriction to a general F-curve gives a point of $\overline{\mathcal{M}}_{0,0}(X, \beta)$ which is a general member of its irreducible component. By the induction hypothesis, the irreducible component for f_1 equals the irreducible component for f_{d-1}. Therefore, assume there exists an F-curve F_1 in Σ_1, an F-curve F_{d-1} in Σ_{d-1}, and an identification $F_1 \cong F_{d-1}$ such that $f_1|_{F_1} = f_{d-1}|_{F_{d-1}}$. Then, by Lemma 7.6(v), there exists a 1-twisting surface $f_d : \Sigma_d \to X$ of H-type 0 and class $(\beta_1, d\beta_2)$, resp. of M-class $(e_1 \cdot \alpha, de_2 \cdot \alpha)$. Moreover, the restriction of f_d to a general F-curve gives a point of $\overline{\mathcal{M}}_{0,0}(X, \beta)$ in the same irreducible component as for f_1 and f_{d-1}. Thus the result is proved by induction on d.

(iii) Let $f : \mathbb{P}^1 \times \mathbb{P}^1 \to X$ be a 1-twisting surface of class (β_1, β_2), resp. of M-class $(e_1 \cdot \alpha, e_2 \cdot \alpha)$. By Lemma 7.6(iii), the morphism $f \circ (\mathrm{pr}_2, \mathrm{pr}_1)$ is a 1-twisting surface of class (β_2, β_1), resp. of M-class $(e_2 \cdot \alpha, e_1 \cdot \alpha)$. Applying (ii) to this morphism with $d = d_1$, there exists a 1-twisting morphism f_{d_1} of H-type 0 and class $(\beta_2, d_1\beta_1)$, resp. of M-class $(e_2 \cdot \alpha, d_1e_1 \cdot \alpha)$. Applying Lemma 7.6(iii) again, $f_{d_1} \circ (\mathrm{pr}_2, \mathrm{pr}_1)$ is a 1-twisting morphism of H-type 0 and class $(d_1\beta_1, \beta_2)$, resp. M-class $(d_1e_1 \cdot \alpha, e_2 \cdot \alpha)$.

Applying (ii) to this morphism with $d = d_2$, there exists a 1-twisting morphism f_{d_1,d_2} of H-type 0 and class $(d_1\beta_1, d_2\beta_2)$, resp. M-class $(d_1e_1 \cdot \alpha, d_2e_2 \cdot \alpha)$.

(iv) This is proved by induction on m and r. The argument is very similar to the argument above. By Lemma 7.6(v), there exists an m-twisting surface $f_{m,0} : \Sigma_{m,0} \to X$ of H-type 0 and class $(\beta, (m-1)\beta_2)$, resp. M-class $(e\cdot\alpha, (m-1)e_2\cdot\alpha)$. By (ii), there exists a 1-twisting surface $f_r : \Sigma_r \to X$ of H-type 0 and class $(\beta, r\beta_1)$, resp. M-class $(e \cdot \alpha, re_1 \cdot \alpha)$. Finally, the restrictions of $f_{m,0}$ and f_r to F-curves give points in $\overline{\mathcal{M}}_{0,0}(X)$ in the same irreducible component. Thus, by (i), $f_{m,0}$ and f_r may be deformed so that there exist F-curves $F_{m,0}$ in $\Sigma_{m,0}$, F_r in Σ_r and an identification $F_{m,0} \cong F_r$ so that $f_{m,0}|_{F_{m,0}} = f_r|_{F_r}$. Now the result follows from Lemma 7.6(v). $\square$

By Corollary 7.7, it is clear that existence of a single 1-twisting surface $f_1 : \mathbb{P}^1\times\mathbb{P}^1 \to X$ and a single 2-twisting surface $f_2 : \mathbb{P}^1 \times \mathbb{P}^1 \to X$ can produce a plethora of n-twisting surfaces for every integer n. To produce a 1-twisting surface and a 2-twisting surface, we exploit a connection between n-twisting surfaces and λ-degree 1 free curves in a general fiber of

$$\mathrm{ev} : \overline{\mathcal{M}}_{0,m}(X, m\alpha) \to X^m,$$

for $m = 2n$.

lem-qmcp

Lemma 7.8. *Let n be a positive integer. Assume $(X, \mathcal{O}(1), m = 2n)$ satisfies Hypothesis 6.3. Assume that the fiber of*

$$ev : \overline{\mathcal{M}}_{0,m}(X, m\alpha) \to X^m$$

is uniruled by λ-degree 1 rational curves. If a general surface Σ from Lemma 6.8 is abstractly isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ then every isomorphism

$$f : \mathbb{P}^1 \times \mathbb{P}^1 \to \Sigma \subset X$$

satisfying

$$f^{-1}\mathcal{O}_\Sigma(C) \cong \mathcal{O}(n, 1)$$

is n-twisting of type $(\alpha, n\alpha)$ (and there exists such an isomorphism f). Here C is the image in Σ of a general fiber of the morphism

$$\mathcal{C}_D \to D$$

as in the proof of Lemma 6.8.

Proof. First of all, a degree $2n$ rational surface scroll that is abstractly isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ has hyperplane class $\mathcal{O}(n, 1)$ up to permutation of the factors. So an isomorphism f as above exists.

Because Σ is embedded in X, the normal bundle $N_{\Sigma/X}$ is a locally free sheaf on Σ. The space of first order deformations of $\Sigma \subset X$, i.e., the Zariski tangent space of the Hilbert scheme of X at $[\Sigma]$, is canonically isomorphic to

$$H^0(\Sigma, N_{\Sigma/X}).$$

The space of first order deformations of $C \subset X$ is

$$H^0(C, N_{C/X}).$$

There is a short exact sequence

$$0 \longrightarrow N_{C/\Sigma} \longrightarrow N_{C/X} \longrightarrow N_{\Sigma/X}|_C \longrightarrow 0.$$

This gives rise to a diagram

$$\begin{array}{ccc} & & H^0(C, N_{C/X}) \\ & & \downarrow u \\ H^0(\Sigma, N_{\Sigma/X}) & \xrightarrow{v} & H^0(C, N_{\Sigma/X}|_C) \end{array}$$

Given a first order deformation $\theta_C \in H^0(C, N_{C/X})$ of C and a first order deforamtion $\theta_\Sigma \in H^0(\Sigma, N_{\Sigma/X})$, the deformation of C is contained in the deformation of Σ if and only if

$$u(\theta_C) = v(\theta_\Sigma).$$

Because the λ-degree 1 curve is free, for every first order deformation of C, there exists a first order deformation of Σ containing the first order deformation of C. In other words,

$$\mathrm{Image}(u) \subset \mathrm{Image}(v).$$

On the other hand, because C is a hyperplane section of Σ, the normal sheaf $N_{C/\Sigma}$ is globally generated and thus

$$h^1(C, N_{C/\Sigma}) = 0.$$

Therefore, by the long exact sequence of cohomology, u is surjective. Thus, by the last paragraph, also v is surjective. Also

$$h^1(\Sigma, \mathrm{pr}_2^* T_{\mathbb{P}^1}(-n, -1)) = h^2(\Sigma, \mathrm{pr}_2^* T_{\mathbb{P}^1}(-n, -1)) = 0$$

and

$$h^1(C, \mathrm{pr}_2^* T_{\mathbb{P}^1}(-n, -1)|_C) = h^1(C, \mathcal{O}_C) = 0.$$

Therefore, from the long exact sequence of cohomology associated to the short exact sequence

$$0 \longrightarrow \mathrm{pr}_2^* T_{\mathbb{P}^1} \longrightarrow N_{(\mathrm{pr}_1, f)} \longrightarrow N_{\Sigma/X} \longrightarrow 0$$

it follows that

$$H^0(\Sigma, N_{(\mathrm{pr}_1, f)}) \to H^0(C, N_{(\mathrm{pr}_1, f)}|_C)$$

is surjective. Since $N_{\Sigma/X}|_C$ and $\mathrm{pr}_2^* T_{\mathbb{P}^1}|_C$ are globally generated, also $N_{(\mathrm{pr}_1, f)}|_C$ is globally generated. Therefore, by Lemma 7.2, $N_{(\mathrm{pr}_1, f)}$ is globally generated and

$$h^1(\Sigma, N_{(\mathrm{pr}_1, f)}(-n, -1)) = 0.$$

In other words, f is n-twisting. □

This method of producing n-twisting surfaces is compatible with the canonical irreducible components from Section 3.

lem-qM

Lemma 7.9. *For $e = 1$ and n, assume that $M_{e\alpha,1}$ is the unique irreducible component of $\overline{\mathcal{M}}_{0,1}(X, e)$ dominating X. Then the n-twisting surface from Lemma 7.8 has M-class $(\alpha, n \cdot \alpha)$.*

Proof. Because $N_{\Sigma/X}$ is globally generated, every rational curve in Σ is a free rational curve in X. In particular the fibers of the projections are free curves of $\mathcal{O}(1)$-degree e for $e = 1, n$. Assuming that $M_{e\alpha}$ is the unique irreducible component parametrizing free curves of $\mathcal{O}(1)$-degree e, it follows that the fibers have M-class $e \cdot \alpha$. □

This raises the question, when is the surface Σ abstractly isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$?

lem-qP1 **Lemma 7.10.** *Let n equal 1 or 2, i.e., m equals 2 or 4. Assume $(X, \mathcal{O}(1), m = 2n)$ satisfies Hypothesis 6.3. Assume that the fiber of*

$$ev : \overline{\mathcal{M}}_{0,m}(X, m\alpha) \to X^m$$

over a general point is uniruled by λ-degree 1 rational curves. Let Σ be a general surface as in Lemma 6.8. For $n = 1$, Σ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. For $n = 2$, Σ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ or a Veronese surface.

Proof. Every smooth quadric surface is abstractly isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. So for $n = 1$, Σ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

The case $n = 2$ is more difficult. From Lemma 6.8 it is possible that the surface Σ is abstractly isomorphic to the Hirzebruch surface

$$\Sigma \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)).$$

We need to prove that in this case, a general deformation of Σ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

The Hirzebruch surface contains a unique rational curve E with self-intersection -2. Denote by F the divisor class of the lines of ruling of Σ. The linear system embedding Σ as a quartic scroll is $\mathcal{O}_\Sigma(E + 3F)$. Because a general deformation of a curve $D \in |\mathcal{O}_\Sigma(E + 3F)|$ is contained in a general deformation of Σ, the normal bundle $N_{\Sigma/X}$ satisfies the hypotheses of Lemma 7.2 with $F' = E + 2F$ and $n = 1$. In particular, $N_{\Sigma/X}$ is globally generated and $h^1(\Sigma, N_{\Sigma/X}(-E - 3F)) = 0$. By Lemma 7.4, every free curve in Σ maps to a free curve in X and every deformation in X of a reduced member $D \in |\mathcal{O}_\Sigma(E + 3F)|$ is contained in a deformation of Σ.

Denote by α' the curve class of E in X. The goal, of course, is to argue that Σ deforms to a quartic scroll that is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. To prove this, we first show that if Σ is general then the curve E is free.

claim-qP1a **Claim 7.11.** *For a general surface Σ as above, E is a free curve.*

Let $C = C_1 \cup C_2$ be a general reducible curve on Σ with

$$C_1 \in |\mathcal{O}_\Sigma(F)|, \quad C_2 \in |\mathcal{O}_\Sigma(E + 2F)|.$$

By the argument above, C_1 and C_2 are free curves in X. Moreover, because Σ contains four general points of X and C_1 and C_2 can be chosen to contain one and three of these points respectively, C_2 is 3-dominating.

By Lemma 7.4, a general deformation of $C_1 \cup C_2$ is contained in a deformation of Σ. After replacing $C_1 \cup C_2$ by a general reducible deformation, let us assume C_1 and C_2 are general in their deformation classes. The curve E intersects both C_1 and C_2, and the intersection points are distinct. Thus these intersection points specify E uniquely in $X \subset \mathbb{P}^N$ since E maps to a line in $\mathbb{P}^N$.

Let Y denote the union in X of all non-free curves of class α. To prove that E is a free curve, it suffices to prove that E is not contained in Y. In other words, for every irreducible component Y_i, it suffices to prove that E is not contained in Y_i.

(i) codim$_X(Y_i) \geq 2$. Because the two irreducible components of C are general free curves, they do not intersect any component of Y having codimension ≥ 2 in X, cf. [Kol96, Proposition II.3.7]. Since E intersects C_1 and C_2, E is contained in no irreducible component Y_i of Y of having codimension ≥ 2 in X.

(ii) $\mathbf{codim}_X(Y_i) = 1$, Y_i not linear. Let Y_i be an irreducible component of Y that is a codimension 1 component of X. Assume that Y_i is not a linear space. Since C_1 is a free curve, C_1 intersects Y_i in general points. Because C_2 is 3-dominating, even after specifying that C_2 intersect C_1, C_2 intersects Y_i in points that are general with respect to $C_1 \cap Y$. Thus every pair of a point of $C_1 \cap Y$ and a point of $C_2 \cap Y$ is a general point of $Y \times Y$. Because Y is not a linear space, there is no line in Y_i that contains a general pair of points in Y. Thus there is no line in Y_i that intersects both C_1 and C_2. Since E is a line that intersects both C_1 and C_2, E is not contained in Y_i.

(iii) $\mathbf{codim}_X(Y_i) = 1$, Y is linear. Finally, let Y_i be an irreducible component of Y that is a codimension 1 component of X and is a linear space, i.e.,

$$(Y_i, \mathcal{O}(1)|_{Y_i}) \cong (\mathbb{P}^{d-1}, \mathcal{O}_{\mathbb{P}^{d-1}}(1))$$

where d is the dimension of X. By way of contradiction, assume E lies in Y_i. The intersection of Y_i with Σ contains E. Moreover, because the lines of ruling of Σ are free curves, they intersect Y_i transversally. Therefore the multiplicity of E in $Y_i \cap \Sigma$ is 1, i.e.,

$$Y_i \cap \Sigma = E + D'$$

where E is not an irreducible component of D'. Therefore,

$$(E \cdot (Y_i \cap \Sigma))_\Sigma = (E \cdot E)_\Sigma + (E \cdot D')_\Sigma \geq -2.$$

In other words,

$$\langle c_1(N_{Y_i/X}), [E] \rangle \geq -2.$$

Because $c_1(T_X) = \langle c_1(T_X), \alpha \rangle c_1(\mathcal{O}(1))$ and because E has $\mathcal{O}(1)$-degree 1,

$$\langle c_1(T_X), \alpha \rangle = \langle c_1(T_X), \alpha \rangle \langle c_1(\mathcal{O}(1)), [E] \rangle = \langle c_1(T_X), [E] \rangle.$$

On the one hand, restricting the short exact sequence

$$0 \longrightarrow T_{Y_i} \longrightarrow T_X|_{Y_i} \longrightarrow N_{Y_i/X} \longrightarrow 0$$

to E gives

$$\langle c_1(T_X), [E] \rangle = \langle c_1(T_{Y_i}), [E] \rangle + \langle c_1(N_{Y_i/X}), [E] \rangle \geq d + (-2) = d - 2.$$

On the other hand, because of the hypothesis that X is neither a linear hypersurface nor a quadric hypersurface,

$$\langle c_1(T_X), \alpha \rangle \leq d - 1.$$

Together this implies that

$$\langle c_1(T_X), \alpha \rangle = d - 1 \text{ or } d - 2.$$

We will show that each of these leads to a contradiction.

claim-qP1b

Claim 7.12.

$$\langle c_1(T_X), \alpha \rangle \neq d - 1.$$

By way of contradiction, assume that $\langle c_1(T_X), \alpha \rangle$ does equal $d-1$. Then the curve D' is nonempty:

$$(E \cdot D')_\Sigma = -2 - (E \cdot E)_\Sigma = 1.$$

Thus D' must be contained in Y_i. Since E is a line in the projective space Y_i, every curve in Y_i is rationally equivalent to a multiple of E, i.e.,

$$D' \overset{Y_i}{\sim} aE$$

for some positive integer a. Therefore

$$(D' \cdot Y_i)_X = \langle c_1(N_{Y_i/X}), D' \rangle = a\langle c_1(N_{Y_i/X}), E \rangle = a(E \cdot (E + D'))_\Sigma = -a.$$

On the other hand,

$$(D' \cdot Y_i)_X = (D' \cdot (E + D'))_\Sigma = (D' \cdot D')_\Sigma + 1 > 0.$$

This contradicts that a is positive. Therefore Claim 7.12 is valid.

claim-qP1c

Claim 7.13.

$$\langle c_1(T_X), \alpha \rangle \neq d - 2.$$

By way of contradiction, assume that $\langle c_1(T_X), \alpha \rangle$ does equal $d - 2$. Because 4α is 4-dominating, the dimension of a general fiber of ev is nonnegative, i.e.,

$$4\langle c_1(T_X), \alpha \rangle + 1 \geq 3\dim(X) = 3d.$$

Substituting in $\langle c_1(T_X), \alpha \rangle = d - 2$, this gives

$$d \geq 7.$$

But for $d \geq 7$,

$$\langle c_1(T_X), \alpha \rangle = d - 2 > \frac{1}{2}d + 1 = \frac{1}{2}\dim(X) + 1.$$

By a theorem of Wiśniewski, [Wiś90], this implies $\mathrm{Pic}(X) \cong \mathbb{Z}$. Therefore Y_i is either ample, trivial, or antiample. Because Y_i is effective, it is neither trivial nor antiample. But since

$$\langle [Y_i], [E] \rangle = \langle c_1(N_{Y_i/X}), [E] \rangle = -2,$$

Y_i is not ample. This contradiction proves Claim 7.13. This proves that E is not contained in Y_i. Since E is contained in no irreducible component Y_i of Y, E is a free curve, i.e., Claim 7.11 is valid.

Let D be a union of E and three general lines of ruling F_1, F_2, F_3. Because E and F are free lines in X, $E \cup F_1$ deforms to a conic in X. Therefore D deforms to a union of a conic and two lines. By Lemma 7.4, there is a deformation of Σ containing this deformation of D. A quartic surface scroll contains a conic if and only if it is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Therefore a general deformation of Σ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. $\square$

As proved, the constructions above are compatible with the canonical irreducible components of Section 3. In fact, existence of free λ-degree 1 rational curves often implies the basic hypothesis of Section 3.

lem-qM0.5

Lemma 7.14. *Let $n = 1$, i.e., $m = 2$. Assume $(X, \mathcal{O}(1), 2)$ satisfies Hypothesis 6.6. Assume that the fiber of*

$$ev : \overline{\mathcal{M}}_{0,2}(X, 2\alpha) \to X^2$$

over a general point is uniruled by λ-degree 1 rational curves.

(i) *If a general fiber has dimension ≥ 2, then there is a unique component $M_{\alpha,1}$ of $\overline{\mathcal{M}}_{0,1}(X, \alpha)$ dominating X and a general fiber of $M_{\alpha,1}$ over X is geometrically connected. In particular this holds if*

$$\langle 2ch_2(T_X), \Pi \rangle > 0.$$

(ii) *Assume there exists a unique component $M_{\alpha,1}$ of $\overline{\mathcal{M}}_{0,1}(X,\alpha)$ dominating X. With respect to $M_{\alpha,1}$, the surface Σ from Lemma 7.10 has M-class (α,α).*

Proof. **(i)** By Lemmas 5.1, 5.3, 5.5 and Hypothesis 6.3, the dimension of a general fiber of

$$\mathrm{ev} : \overline{\mathcal{M}}_{0,2}(X,2\alpha) \to X^2$$

equals

$$\dim(X) - 2(\dim(X) - \langle c_1(T_X),\alpha\rangle - 1) - 3.$$

If this dimension is ≥ 2, then

$$2\langle c_1(T_X),\alpha\rangle \geq \dim(X) + 3.$$

Then, by Lemma 4.2, a general fiber of

$$\mathrm{ev} : \overline{\mathcal{M}}_{0,1}(X,\alpha) \to X$$

is irreducible, provided it is nonempty. By hypothesis, a general pair of points of X is contained in a surface Σ. By Lemma 7.10, Σ is a smooth quadric surface, which is ruled by lines. Thus every point is contained in a line, i.e., a general fiber of ev is nonempty, thus irreducible.

Notice by Lemma 6.9, the dimension of the space of λ-degree 1 curves containing a general point of a general fiber of

$$\mathrm{ev} : \overline{\mathcal{M}}_{0,2}(X,2\alpha) \to X^2$$

equals

$$\langle 2\mathrm{ch}_2(T_X),\Pi\rangle.$$

If this is positive, then there is at least a 1-parameter family of λ-degree 1 curves in a general fiber. Thus the fiber must have dimension ≥ 2.

(ii) Because a general deformation of Σ contains a general point of X, the lines in Σ are in $M_{\alpha,0}$, i.e., Σ has M-class (α,α). □

To summarize, positivity hypotheses on the Chern character of X and conditions on the Chow ring of X imply the basic hypothesis of Section 3 as well as the existence of 1-twisting surfaces in X.

cor-qM0.5

Corollary 7.15. *Let $(X,\mathcal{O}(1))$ be a smooth, projective variety. Let α be an $\mathcal{O}(1)$-degree 1 curve class. Assume $c_1(T_X) = \langle c_1(T_X),\alpha\rangle c_1(\mathcal{O}(1))$, i.e., $c_1(T_X)$ is a multiple of $c_1(\mathcal{O}(1))$. Assume X is a Fano manifold, i.e., $\langle c_1(T_X),\alpha\rangle > 0$. Assume $(X,\mathcal{O}(1))$ is neither a linear variety nor a quadric hypersurface. Assume that*

$$ch_2(T_X) = \frac{\langle 2ch_2(T_X),\Pi\rangle}{2} c_1(\mathcal{O}(1))^2$$

for some integer $\langle 2ch_2(T_X),\Pi\rangle$. Assume that

$$\langle 2ch_2(T_X),\Pi\rangle \geq 0$$

and

$$dim(X) \geq 2(dim(X) - \langle c_1(T_X),\alpha\rangle - 1) + 4.$$

Assume there exists a smooth, projective variety Y such that X is a codimension c linear section of Y. Assume

$$c \geq 2(dim(X) - \langle c_1(T_X),\alpha\rangle - 1) - dim(X) + 5,$$

$$c > dim(X) - 2(dim(X) - \langle c_1(T_X), \alpha\rangle - 1) - 2\langle 2ch_2(T_X), \Pi\rangle + 2,$$

and

$$CH^p(Y) = \mathbb{Z}\{c_1(\mathcal{O}(1))^p\}, \text{ for } 0 \le p \le (dim(X) - \langle c_1(T_X), \alpha - 1) + 2.$$

By Barth's theorems, [Bar70]*, this last condition holds if*

$$c \ge 2(dim(X) - \langle c_1(T_X), \alpha\rangle - 1) - dim(X) + (h^0(X, \mathcal{O}(1)) - dim(X) - 2) + 5.$$

(i) *Then there exists a unique irreducible component $M_{\alpha,1}$ of $\overline{\mathcal{M}}_{0,1}(X,\alpha)$ dominating X and a general fiber of $\overline{\mathcal{M}}_{0,1}(X,\alpha)$ over X is geometrically connected.*

(ii) *And for every pair of positive integers d_1, d_2, there exists a 1-twisting surface of H-type 0 and M-class $(d_1 \cdot \alpha, d_2 \cdot \alpha)$ (with respect to $M_{\alpha,1}$).*

The same conclusion holds for every linear variety of dimension ≥ 2 and every smooth quadric variety of dimension ≥ 3.

Proof. **(i)** The proof uses Lemma 4.4. The hypotheses of that lemma are that

$$\langle c_1(T_X), \alpha\rangle \ge 3$$

and that X is a codimension c linear section of a smooth, projective variety with

$$c \ge 2(\dim(X) - \langle c_1(T_X), \alpha\rangle - 1) - \dim(X) + 5.$$

This second condition is one of the hypotheses above. For the first condition, observe that since $(X, \mathcal{O}(1))$ is neither linear nor quadric,

$$\dim(X) - \langle c_1(T_X), \alpha\rangle - 1 \ge 0.$$

Together with the inequality

$$\dim(X) \ge 2(\dim(X) - \langle c_1(T_X), \alpha\rangle - 1) + 4$$

this implies both that $\dim(X) \ge 4$ and $\langle c_1(T_X), \alpha\rangle \ge 3$.

(ii) The next claim is that 2α is 2-dominating. This is proved using Corollary 5.10. The first hypothesis of that lemma,

$$\dim(X) \ge 2(\dim(X) - \langle c_1(T_X), \alpha\rangle - 1) + 3,$$

is implied by one of the hypotheses above. The second hypothesis that

$$c \ge 2(\dim(X) - \langle c_1(T_X), \alpha\rangle - 1) - \dim(X)$$

follows from the first hypothesis: $c \ge -3$ holds for $c = 0$. The final condition,

$$\mathrm{CH}^p(Y) = \mathbb{Z}\{c_1(\mathcal{O}(1))^p\}, \text{ for } 0 \le p \le (\dim(X) - \langle c_1(T_X), \alpha\rangle - 1) + 2$$

is precisely one of the hypotheses above. Therefore 2α is 2-dominating by Corollary 5.10.

Because 2α is 2-dominating, $(X, \mathcal{O}(1), 2)$ satisfies Hypothesis 6.3. Together with the hypotheses above, this implies $(X, \mathcal{O}(1), 2)$ satisfies Hypothesis 6.6. The next claim is that a general fiber of

$$\mathrm{ev} : \overline{\mathcal{M}}_{0,2}(X, 2\alpha) \to X^2$$

is uniruled by rational curves of λ-degree 1. This is proved using Lemma 6.10. For $m = 2$, the first hypothesis of Lemma 6.10 is precisely that

$$\langle 2\mathrm{ch}_2(T_X), \Pi\rangle \ge 0,$$

which is one of the hypotheses above. The second hypothesis is that

$$c > \dim(X) - 2(\dim(X) - \langle c_1(T_X), \alpha\rangle - 1) - 2\langle 2\mathrm{ch}_2(T_X), \Pi\rangle + 2,$$

which is precisely one of the hypotheses above. Therefore a general fiber of ev is uniruled by rational curves of λ-degree 1 by Lemma 6.10.

By Lemma 7.10, the surface Σ swept out by a general λ-degree 1 curve in a fiber is a quadric surface. In particular, $\Sigma \cong \mathbb{P}^1 \times \mathbb{P}^1$. By Lemma 7.8, this surface Σ is a 1-twisting surface of H-type 0 and class (α, α). Moreover, by Lemma 7.9, the surface has M-class (α, α). Finally, by Corollary 7.7, for every pair of positive integers d_1, d_2, there exists a 1-twisting surface of H-type 0 and M-class $(d_1 \cdot \alpha, d_2 \cdot \alpha)$.

For a linear surface $(X, \mathcal{O}(1))$ of dimension ≥ 2, let $f : \mathbb{P}^1 \to \mathbb{P}^1 \to X$ be the composition of a degree 2 morphism $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2$ with a linear embedding $\mathbb{P}^2 \hookrightarrow X$. Then f is a 1-twisting surface of H-type 0 and M-class (α, α). And $\overline{\mathcal{M}}_{0,1}(X, \alpha)$ is irreducible. Similarly, if $(X, \mathcal{O}(1))$ is a quadric hypersurface of dimension ≥ 2, then a general 2-dimensional linear section of X is a quadric surface which is a 1-twisting surface of H-type 0 and M-class (α, α). And if the dimension ≥ 3, then $\overline{\mathcal{M}}_{0,1}(X, \alpha)$ is irreducible. Thus, by Corollary 7.7, for every pair of positive integers d_1, d_2, there exists a 1-twisting surface of H-type 0 and M-class $(d_1 \cdot \alpha, d_2 \cdot \alpha)$. □

Lemma 7.16. *Let $n = 2$, i.e., $m = 4$. Assume $(X, \mathcal{O}(1), 4)$ satisfies Hypothesis 6.6. Assume that the fiber of* lem-qM1

$$ev : \overline{\mathcal{M}}_{0,4}(X, 4\alpha) \to X^4$$

over a general point is uniruled by λ-degree 1 rational curves.

(i) *There is a unique component $M_{\alpha,1}$ of $\overline{\mathcal{M}}_{0,1}(X, \alpha)$ dominating X and a general fiber of $M_{\alpha,1}$ over X is geometrically connected.*
(ii) *Every $\mathcal{O}(1)$-degree 1 curve (assuming there are any) in a general surface Σ from Lemma 7.10 for $m = 4$ is parametrized by a point of $M_{\alpha,0}$, cf. Notation 3.7.*
(iii) *With respect to $M_{\alpha,1}$, every irreducible $\mathcal{O}(1)$-degree 2 curve in a general surface Σ from Lemma 7.10 is parametrized by a point of $M_{2\alpha,0}$, cf. Notation 3.7. In particular, if $\Sigma \cong \mathbb{P}^1 \times \mathbb{P}^1$, this surface has M-class $(\alpha, 2 \cdot \alpha)$.*

Proof. Because there exists λ-degree 1 curves in a general fiber of

$$\mathrm{ev} : \overline{\mathcal{M}}_{0,4}(X, 4\alpha) \to X^4,$$

the dimension of a general fiber is ≥ 1. By Lemmas 5.1, 5.3, 5.5 and Hypothesis 6.3, the dimension is

$$(\dim(X) - 4) - 4(\langle \dim(X) - 1 - \langle c_1(T_X), \alpha\rangle).$$

Also, by Lemma 6.9,

$$2\langle 2\mathrm{ch}_2(T_X), \Pi\rangle - \langle c_1(T_X), \alpha\rangle \geq 0.$$

Because X is neither linear nor a quadric hypersurface,

$$\dim(X) - 1 - \langle c_1(T_X), \alpha\rangle \geq 0.$$

Thus $\dim(X) \geq 4$. There is one extreme case that must be ruled out.

Claim 7.17. *Either $\dim(X) > 4$ or $\dim(X) > \langle c_1(T_X), \alpha\rangle + 1$.*

By way of contradiction, assume there exists X satisfying the hypotheses of the lemma with

$$\dim(X) = 4 \text{ and } \langle c_1(T_X), \alpha\rangle = 3.$$

Notice it also follows that

$$\langle 2\mathrm{ch}_2(T_X), \Pi\rangle \geq \frac{\langle c_1(T_X), \alpha\rangle}{2} = \frac{3}{2}.$$

Since $\langle 2\mathrm{ch}_2(T_X), \Pi\rangle$ is an integer, in fact this implies that

$$\langle 2\mathrm{ch}_2(T_X), \Pi\rangle \geq 2.$$

Now, by Lemma 4.3, the morphism

$$\mathrm{ev} : \overline{\mathcal{M}}_{0,1}(X, \alpha) \to X$$

is dominant. By Lemma 4.1(ii), a general fiber is 1-dimensional. By Lemma 4.3(iii), every connected component M of the fiber has

$$2c_1(T_M) = (\langle 2\mathrm{ch}_2(T_X), \Pi\rangle + 3)\pi_* f^*[c_1(\mathcal{O}(1))^2].$$

This is positive. Therefore $M \cong \mathbb{P}^1$ which implies $c_1(T_M) = 2$. Substituting in gives,

$$4 = 2c_1(T_M) = (\langle 2\mathrm{ch}_2(T_X), \Pi\rangle + 3)\pi_* f^*[c_1(\mathcal{O}(1))^2] \geq 5\pi_* f^*[c_1(\mathcal{O}(1))^2].$$

Since $\pi_* f^*[c_1(\mathcal{O}(1))^2]$ is a positive integer, this is impossible. This contradiction proves that X cannot have both

$$\dim(X) = 4 \text{ and } \langle c_1(T_X), \alpha\rangle = 3.$$

Therefore Claim 7.17 is valid.

(i) Claim 7.17 implies that

$$(\dim(X) - 5) - 2(\dim(X) - 1 - \langle c_1(T_X), \alpha\rangle) \geq 0.$$

Therefore, by Lemma 4.2, a general fiber of

$$\mathrm{ev} : \overline{\mathcal{M}}_{0,1}(X, \alpha) \to X$$

is irreducible. In particular, there exists a unique irreducible component $M_{\alpha,1}$ of $\overline{\mathcal{M}}_{0,1}(X, \alpha)$ dominating X.

(ii) By Lemma 7.10, either $\Sigma \cong \mathbb{P}^1 \times \mathbb{P}^1$ or Σ is a Veronese surface. A Veronese surface contains no lines, so the result is vacuous. If $\Sigma \cong \mathbb{P}^1 \times \mathbb{P}^1$, then the lines are the fibers of one of the two projections. In particular, a general point of Σ is contained in a line. Since a general point of Σ is a general point of X, every line in Σ containing a general point of Σ is a line in X containing a general point of X. Since $M_{\alpha,1}$ is the unique irreducible component dominating X, these lines are in $M_{\alpha,0}$. Since the space of lines in Σ is irreducible, every line in Σ is in $M_{\alpha,0}$.

(iii) Because

$$(\dim(X) - 5) - 2(\dim(X) - 1 - \langle c_1(T_X), \alpha\rangle) \geq 0,$$

by the same bend-and-break argument used to prove the second part of Lemma 5.6, every fiber of

$$\mathrm{ev} : \overline{\mathcal{M}}_{0,2}(X, 2\alpha) \to X \times X$$

intersects the boundary Δ. Thus, for every curve C in X with class 2α, and for every pair of points p_1, p_2 of C, there is a deformation of C containing p_1 and p_2 that specializes to a union of curves $C_1 \cup C_2$ of class α containing p_1 and p_2. Because

α is the minimal curve class, both C_1 and C_2 are smooth. By Lemma 3.5, to prove that $[C]$ is in $M_{2\alpha,0}$, it suffices to show that C_1 and C_2 are free curves.

Now, if Σ is the surface coming from a general λ-degree 1 curve, then a general point of Σ is a general point of X. Thus, for a general conic C in Σ and general points p_1, p_2 in C, every irreducible curve in X containing p_i is a free curve. Thus the curves C_1, C_2 are free curves, unless one of them contains both p_1 and p_2. Therefore, to finish the proof of the lemma, it suffices to prove that for a general Σ, a general conic C in Σ and a general pair p_1, p_2 of points in C, there is no line in X containing both p_1 and p_2.

By Lemma 7.10, either $\Sigma \cong \mathbb{P}^2$ or $\Sigma \cong \mathbb{P}^1 \times \mathbb{P}^1$. First consider the case that $\Sigma \cong \mathbb{P}^2$, i.e., Σ is a Veronese surface. Every pair of points of Σ is contained in a conic. Moreover, a general quadruple of points of X is contained in a surface Σ, by construction. Therefore, a general pair of points of X is a general pair of points on a general conic in a general Σ. Thus, if every such pair is contained in a line, then every general pair of points of X is contained in a line. This implies X is a linear variety, contrary to hypothesis. Therefore, for a general Σ, a general conic C in Σ and a general pair of points p_1, p_2 of C, there is no line in X containing p_1, p_2, and thus $[C]$ is in $M_{2\cdot\alpha,0}$.

Next consider the case $\Sigma \cong \mathbb{P}^1 \times \mathbb{P}^1$. The conics in Σ are the fibers of one of the projections. The union of all lines connecting a pair of points of one of these fibers is a linear projection of a cubic Segre variety $\mathbb{P}^1 \times \mathbb{P}^2$, where the quartic scroll $\mathbb{P}^1 \times \mathbb{P}^1$ is embedded in the cubic Segre variety by

$$\mathrm{Id}_{\mathbb{P}^1} \times (2-\text{uple Veronese}) : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^2.$$

In particular, any four general points in $\mathbb{P}^1 \times \mathbb{P}^1$ are contained in a twisted cubic in $\mathbb{P}^1 \times \mathbb{P}^2$. Since four general points of X are contained in a surface Σ, this implies that 3α is 4-dominating. By Lemma 5.5, this implies X is either a linear variety or a quadric hypersurface, contrary to hypothesis. Therefore, for a general Σ, a general conic C in Σ and a general pair of points p_1, p_2 of C, there is no line in X containing p_1, p_2, and thus $[C]$ is in $M_{2\cdot\alpha,0}$. □

lem-qV

Lemma 7.18. *Assume $(X, \mathcal{O}(1), m = 4)$ satisfies Hypothesis 6.3. Assume that the fiber of*

$$ev : \overline{\mathcal{M}}_{0,4}(X, 4\alpha) \to X^4$$

over a general point is uniruled by λ-degree 1 rational curves. Let Σ be a general surface as in Lemma 6.8. If Σ is a Veronese surface, then every morphism

$$f : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2 \cong \Sigma \subset X$$

with $f^*\mathcal{O}_{\mathbb{P}^2}(1) \cong \mathcal{O}(1,1)$ is a 2-twisting surface of type $(2\alpha, 2\alpha)$.

If X further satisfies Hypothesis 6.6, then there exists a unique irreducible component $M_{\alpha,1}$ of $\overline{\mathcal{M}}_{0,1}(X, \alpha)$ dominating X, and f has M-class $(2\cdot\alpha, 2\cdot\alpha)$.

Proof. Let C be a general conic in $\mathbb{P}^2 \cong \Sigma$. By the same argument as in the proof of Lemma 7.8, the map

$$H^0(\Sigma, N_{\Sigma/X}) \to H^0(C, N_{\Sigma/X}|_C)$$

is surjective. Denote by

$$\nu : \widetilde{\Sigma} \to \Sigma$$

the blowing up of Σ at a point p in C. The surface $\widetilde{\Sigma}$ is a rational ruled surface of Hirzebruch type 1. The sheaf

$$N := \nu^* N_{\Sigma/X}$$

satisfies the hypotheses of Lemma 7.2 for $n = 1$: for the divisor D use the strict transform of C. Thus by Lemma 7.2, N is globally generated on $\widetilde{\Sigma}$. Therefore $N_{\Sigma/X}$ is globally generated on Σ. Moreover

$$h^1(\Sigma, N_{\Sigma/X}(-C)) = h^1(\Sigma, \nu_*(N(-D))) = h^1(\widetilde{\Sigma}, N(-D)) = 0.$$

In other words,

$$h^1(\Sigma, N_{\Sigma/X}(-2)) = 0.$$

By an argument similar to the proof of Lemma 7.6(i), also

$$h^1(\Sigma, N_{\Sigma/X}(-1)) = h^1(\Sigma, N_{\Sigma/X}) = 0.$$

Moreover, because

$$h^1(\Sigma, T_\Sigma(-2)) = h^1(\Sigma, T_\Sigma(-1)) = h^1(\Sigma, T_\Sigma) = 0,$$

it follows from the long exact sequence of cohomology that $T_X|_\Sigma$ is globally generated and

$$h^1(\Sigma, T_X|_\Sigma(-2)) = h^1(\Sigma, T_X|_\Sigma(-1)) = h^1(\Sigma, T_X|_\Sigma) = 0.$$

For every morphism

$$f : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2$$

with $f^*\mathcal{O}(1) \cong \mathcal{O}(1,1)$,

$$f_*\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,0) \cong \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$$

and thus

$$f_*\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1})(-2,-1) \cong \mathcal{O}_{\mathbb{P}^2}(-2) \oplus \mathcal{O}_{\mathbb{P}^2}(-2).$$

Since $T_X|_\Sigma$ is globally generated, also f^*T_X is globally generated. And, by the computation above

$$h^1(\mathbb{P}^1\times\mathbb{P}^1, f^*T_X(-2,-1)) = h^1(\mathbb{P}^2, T_X \otimes f_*\mathcal{O}(-2,-1)) = h^1(\mathbb{P}^1, T_X(-2)) + h^1(\mathbb{P}^1, T_X(-2)) = 0+0 = 0.$$

Therefore f is 2-twisting.

By Lemma 7.16, if X satisfies Hypothesis 6.6, then there exists a unique irreducible component $M_{\alpha,1}$ of $\overline{\mathcal{M}}_{0,1}(X,\alpha)$ dominating X, and every conic in Σ (which is the image of a line in $\mathbb{P}^2$ under the Veronese morphism) is in $M_{2\cdot\alpha,0}$. Since the fibers of the two projections of $\mathbb{P}^1 \times \mathbb{P}^1$ map to lines in $\mathbb{P}^2$ under f, i.e., map to conics in Σ, f has M-class $(2\cdot\alpha, 2\cdot\alpha)$. $\square$

cor-qV

Corollary 7.19. *Let $(X, \mathcal{O}(1))$ be a smooth, projective variety. Let α be an $\mathcal{O}(1)$-degree 1 curve class. Assume $c_1(T_X) = \langle c_1(T_X), \alpha\rangle c_1(\mathcal{O}(1))$, i.e., $c_1(T_X)$ is a multiple of $c_1(\mathcal{O}(1))$. Assume X is a Fano manifold, i.e., $\langle c_1(T_X), \alpha\rangle > 0$. Assume $(X, \mathcal{O}(1))$ is neither a linear variety nor a quadric hypersurface. Assume that*

$$ch_2(T_X) = \frac{\langle 2ch_2(T_X), \Pi\rangle}{2} c_1(\mathcal{O}(1))^2$$

for some integer $\langle 2ch_2(T_X), \Pi\rangle$. Assume that

$$2\langle 2ch_2(T_X), \Pi\rangle - \langle c_1(T_X), \alpha\rangle \geq 0$$

and

$$dim(X) \geq 4(dim(X) - \langle c_1(T_X), \alpha\rangle - 1) + 4.$$

Assume there exists a smooth, projective variety Y such that X is a codimension c linear section of Y. Assume

$$c > dim(X) - 2(2\langle 2ch_2(T_X), \Pi\rangle - \langle c_1(T_X), \alpha\rangle) - 4(dim(X) - \langle c_1(T_X), \alpha\rangle - 1) + 6$$

and

$$CH^p(Y) = \mathbb{Z}\{c_1(\mathcal{O}(1))^p\}, \text{ for } 0 \leq p \leq (dim(X) - \langle c_1(T_X), \alpha\rangle - 1) + 2.$$

By Barth's theorems, [Bar70]*, this last condition holds if*

$$c \geq 2(dim(X) - \langle c_1(T_X), \alpha\rangle - 1) - dim(X) + (h^0(X, \mathcal{O}(1)) - dim(X) - 2) + 5.$$

(i) *Then there exists a unique irreducible component $M_{\alpha,1}$ of $\overline{\mathcal{M}}_{0,1}(X, \alpha)$ dominating X and a general fiber of $\overline{\mathcal{M}}_{0,1}(X, \alpha)$ over X is geometrically connected.*

(ii) *For every pair of positive integers d_1, d_2, there exists a 1-twisting surface of H-type 0 and M-class $(d_1 \cdot \alpha, d_2 \cdot \alpha)$ (with respect to $M_{\alpha,1}$).*

(iii) *And there exists a 2-twisting surface of H-type 0 and M-class either $(\alpha, 2\cdot\alpha)$ or $(2\cdot\alpha, 2\cdot\alpha)$. Thus for every positive integer m and nonnegative integer r, there exists an m-twisting surface of H-type 0 and M-class either $(\alpha, (2m+r-2)\cdot\alpha)$ or $(2\cdot\alpha, (2m+r-2)\cdot\alpha)$.*

The same conclusion holds for every linear variety of dimension ≥ 2 and every smooth quadric variety of dimension ≥ 3.

Proof. **(i) and (ii)** First of all, the hypotheses above imply the Hypotheses of Corollary 7.15. The one hypothesis of Corollary 7.15 that is a bit less obvious is,

$$\dim(X) \geq 2(\dim(X) - \langle c_1(T_X), \alpha\rangle - 1) + 5.$$

By Lemmas 5.1, 5.3, 5.5 and Hypothesis 6.3, this is equivalent to the hypothesis that

$$\text{ev} : \overline{\mathcal{M}}_{0,2}(X, 2\alpha) \to X^2$$

has fiber dimension ≥ 2. By the hypothesis above,

$$2\langle 2\text{ch}_2(T_X), \Pi\rangle \geq \langle c_1(T_X), \alpha\rangle > 0,$$

and thus $\langle 2\text{ch}_2(T_X), \Pi\rangle > 0$. By Lemma 7.14(i), this implies the fiber dimension of ev is ≥ 2. Therefore Corollary 7.15 implies (i) and (ii).

(iii) The next claim is that 4α is 4-dominating. This is proved using Corollary 5.10. The first hypothesis of that lemma,

$$\dim(X) \geq 4(\dim(X) - \langle c_1(T_X), \alpha\rangle - 1) + 3$$

is implied by one of the hypotheses above. The second hypothesis that

$$c \geq 4(\dim(X) - \langle c_1(T_X), \alpha\rangle - 1) - \dim(X) + 2$$

follows from the first hypothesis: $c \geq -1$ holds for $c = 0$. The final condition,

$$\text{CH}^p(Y) = \mathbb{Z}\{c_1(\mathcal{O}(1))^p\}, \text{ for } 0 \leq p \leq (\dim(X) - \langle c_1(T_X), \alpha\rangle - 1) + 2$$

is precisely one of the hypotheses above. Therefore 4α is 4-dominating by Corollary 5.10.

The claim is that $(X, \mathcal{O}(1), 4)$ satisfies Hypothesis 6.3. By the last paragraph, 4α is 4-dominating. Because $(X, \mathcal{O}(1))$ is neither linear nor quadric,

$$\dim(X) - \langle c_1(T_X) - 1 \geq 0.$$

Therefore the hypothesis that

$$\dim(X) \geq 4(\dim(X) - \langle c_1(T_X), \alpha\rangle - 1) + 4$$

implies that $\dim(X) \geq 4$. Since $(X, \mathcal{O}(1))$ is not linear,

$$h^0(X, \mathcal{O}(1)) \geq \dim(X) + 2.$$

Therefore $4 \leq h^0(X, \mathcal{O}(1))$ and

$$h^0(X, \mathcal{O}(1)) \geq \dim(X) + 4 - 2.$$

Therefore $(X, \mathcal{O}(1))$ satisfies Hypothesis 6.3. Together with the hypotheses above, this implies $(X, \mathcal{O}(1), 4)$ satisfies Hypothesis 6.6.

The next claim is that a general fiber of

$$\mathrm{ev} : \overline{\mathcal{M}}_{0,4}(X, 4\alpha) \to X^4$$

is uniruled by rational curves of λ-degree 1. This is proved using Lemma 6.10. For $m = 4$, the first hypothesis of Lemma 6.10 is precisely that

$$2\langle 2\mathrm{ch}_2(T_X), \Pi\rangle - \langle c_1(T_X), \alpha\rangle \geq 0,$$

which is one of the hypotheses above. The second hypothesis is that

$$c > \dim(X) - 2(2\langle 2\mathrm{ch}_2(T_X), \Pi\rangle - \langle c_1(T_X), \alpha\rangle) - 4(\dim(X) - \langle c_1(T_X), \alpha\rangle - 1) + 6$$

which is precisely one of the hypotheses above. Therefore a general fiber of ev is uniruled by rational curves of λ-degree 1 by Lemma 6.10.

By Lemma 7.10, the surface Σ swept out by a general λ-degree 1 curve in a fiber is either a quartic scroll abstractly isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ or else a Veronese surface. By Lemmas 7.8 and 7.16, in the first case Σ is a 2-twisting surface of H-type 0 and M-class $(\alpha, 2 \cdot \alpha)$. By Lemma 7.18, in the second case there is a morphism $\mathbb{P}^1 \times \mathbb{P}^1 \to \Sigma \subset X$ which is a 2-twisting surface of H-type 0 and M-class $(2 \cdot \alpha, 2 \cdot \alpha)$.

In the first case, since there exists a 1-twisting surface of H-type 0 and M-class (α, α) and a 2-twisting surface of H-type 0 and M-class $(\alpha, 2 \cdot \alpha)$, Corollary 7.7(iv) implies that for every positive integer m and nonnegative integer r, there exists an m-twisting surface of H-type 0 and M-class $(\alpha, (2m + r - 2) \cdot \alpha)$. In the second case, by (ii) there exists a 1-twisting surface of H-type 0 and M-class $(2 \cdot \alpha, \alpha)$ and a 2-twisting surface of H-type 0 and M-class $(2 \cdot \alpha, 2 \cdot \alpha)$. Thus Corollary 7.7(iv) implies that for every positive integer m and nonnegative integer r, there exists an m-twisting surface of H-type 0 and M-class $(2 \cdot \alpha, (2m + r - 2) \cdot \alpha)$.

The conclusion for $\mathbb{P}^n$, $n \geq 2$ follows by considering any morphism $f : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^n$ such that $f^*\mathcal{O}(1) = \mathcal{O}(1, 2)$. The conclusion for a quadric hypersurface X of dimension ≥ 3 follows by considering any 2-dimensional linear section $\Sigma \cong \mathbb{P}^1 \times \mathbb{P}^1$ of X and then composing with a morphism

$$q \times \mathrm{Id}_{\mathbb{P}^1} : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1 \cong \Sigma \subset X$$

where q is a degree 2 morphism. $\square$

sec-qrc

8. Twisting surfaces. Rational simple connectedness

Let β_1 and β_2 be curve classes in X. Let m be a positive integer. Denote by $\tau_m(\beta_1;\beta_2)$ the genus 0, stable A-graph with one vertex v_0 of class β_2 and valence m, and m leaves v_i, $i=1,\dots,m$, of class β_1 connected to v_0. Denote by $\tau'_m(\beta_1;\beta_2)$ the A-graph obtained from $\tau_m(\beta_1;\beta_2)$ by attaching one tail to each of the leaf vertices only, v_i, $i=1,\dots,m$. And denote by $\tau''_m(\beta_1;\beta_2)$ the A-graph obtained from $\tau_m(\beta_1;\beta_2)$ by attaching a tail to *every* vertex v_i, $i=0,\dots,m$. In other words, $\tau_m(\beta_1;\beta_2)$ is the dual graph of a reducible, arithmetic genus 0 curve in X which is a comb with handle class β_2 and with m teeth of class β_1. And $\tau'_m(\beta_1;\beta_2)$ is the dual graph associated to the n-pointed curve obtained by putting one marked point on each tooth of the comb. And $\tau''_m(\beta_1;\beta_2)$ puts one marked point on each tooth and on the handle. [REFERENCES – DIAGRAMS]

For a Hirzebruch surface
$$\pi:\Sigma\to\mathbb{P}^1$$
of H-type h, an integer $m\geq h$ and a nonnegative integer n, denote by $U_{m,n}$ the open subset of $\overline{\mathcal{M}}_{0,n}(\Sigma,F'+mF)$ parametrizing smooth divisors
$$C\in|\mathcal{O}_\Sigma(F'+mF)|$$
together with n distinct points
$$p_1,\dots,p_n\in C.$$
If $n=m$, denote $U_{m,m}$ by U_m.

lem-t1

Lemma 8.1. *The variety $\overline{\mathcal{M}}(\Sigma,\tau'_m(F;F'))$ is smooth and irreducible. The image in $\overline{\mathcal{M}}_{0,m}(\Sigma,F'+mF)$ intersects the smooth locus of the morphism*
$$ev:\overline{\mathcal{M}}_{0,m}(\Sigma,F'+mF)\to\Sigma^m$$
and is contained in $\overline{U}_m$. A general fiber of the restriction $ev|_{\overline{U}_m}$ is rationally connected. A general point of $\overline{U}_m$ is rationally connected to the image of $\overline{\mathcal{M}}(\Sigma,\tau'_m(F;F'))$ by a rational curve in a fiber of ev. Moreover, the intersection points may be chosen general.

Proof. A point of $\overline{\mathcal{M}}(\Sigma,\tau'_m(F;F'))$ is equivalent to the data of a divisor D_0 in the linear system $|\mathcal{O}_\Sigma(F')|$, divisors D_i, $i=1,\dots,m$, in the linear system $|\mathcal{O}_\Sigma(F)|$, and a point p_i of D_i for each $i=1,\dots,m$. The parameter space for such data is clearly smooth and irreducible. Moreover, for a general such datum, the divisor $D=D_0+D_1+\cdots+D_m$ is a reduced member of the linear system $|\mathcal{O}_\Sigma(F'+mF)|$, and each point p_i is distinct from the intersection point q_i of D_i and D_0.

There is a short exact sequence
$$0\to\oplus_{i=1}^m N_{D/\Sigma}(-(p_1+\cdots+p_m))|_{D_i/\Sigma}(-q_i)\to N_{D/\Sigma}(-(p_1+\cdots+p_m))\to N_{D/\Sigma}(-(p_1+\cdots+p_m))|_{D_0}\to 0.$$
There are also isomorphisms
$$N_{D/\Sigma}(-(p_1+\cdots+p_m))|_{D_0}\cong N_{D/\Sigma}|_{D_0}\cong N_{D_0/\Sigma}(q_1+\cdots+q_m)=\mathcal{O}_{D_0}(m-h)$$
and
$$N_{D/\Sigma}(-(p_1+\cdots+p_m))|_{D_i}(-q_i)\cong N_{D_i/\Sigma}(-p_i)\cong\mathcal{O}_{D_i}(-1)$$
for $i=1,\dots,m$. Since
$$h^1(D_0,\mathcal{O}_{D_0}(m-h))=h^1(D_i,\mathcal{O}_{D_i}(-1))=0,$$

by the long exact sequence of cohomology

$$h^1(D, N_{D/\Sigma}(-(p_1 + \cdots + p_m))) = 0.$$

Thus both $\overline{\mathcal{M}}_{0,m}(\Sigma, F' + mF)$ and the evaluation morphism

$$\text{ev} : \overline{\mathcal{M}}_{0,m}(\Sigma, F' + mF) \to \Sigma^m$$

are smooth at the point $(D, p_1, \dots, p_m)$. Moreover, by a parameter count, the divisor D deforms to a smooth divisor C in $|\mathcal{O}_\Sigma(F' + mF)|$ containing $p_1, \dots, p_m$. Therefore the image of $\overline{\mathcal{M}}(X, \tau'_m(F; F'))$ is contained in $\overline{U}_m$.

The fiber of

$$\text{ev} : \overline{U}_m \to \Sigma^m$$

over a general point $(p_1, \dots, p_m)$ is birational to the linear system of curves in $|\mathcal{O}_\Sigma(F' + mF)|$ with base locus $p_1, \dots, p_m$. Therefore it is rationally connected.

Let C be a smooth member of the linear system of curves in $|\mathcal{O}_\Sigma(F'+mF)|$ with base locus $p_1, \dots, p_m$. The divisors D and C span a pencil Λ of divisors in $|\mathcal{O}_\Sigma(F'+mF)|$ containing $p_1, \dots, p_m$. Marking each divisor in the pencil by $p_1, \dots, p_n$, there is a 1-morphism

$$\zeta : \Lambda \to \overline{\mathcal{M}}_{0,m}(\Sigma, F' + mF)$$

whose image contains $(C, p_1, \dots, p_m)$ and $(D, p_1, \dots, p_m)$ and is contained in a fiber of ev. By construction, $(D, p_1, \dots, p_m)$ is general in $\overline{\mathcal{M}}(X, \tau'_m(F; F'))$ and $(C, p_1, \dots, p_m)$ is general in $\overline{\mathcal{M}}_{0,m}(X, F' + mF)$. □

lem-t1.5

Lemma 8.2. *Assume the H-type of Σ is $h = 0$, i.e., $\Sigma \cong \mathbb{P}^1 \times \mathbb{P}^1$. The variety $\overline{\mathcal{M}}(\Sigma, \tau''_m(F, F'))$ is smooth and irreducible. The image in $\overline{\mathcal{M}}_{0,m+1}(\Sigma, F' + mF)$ intersects the smooth locus of the morphism*

$$ev : \overline{\mathcal{M}}_{0,m+1}(\Sigma, F' + mF) \to \Sigma^{m+1}$$

and is contained in $\overline{U}_{m,m+1}$. A general fiber of the restriction $ev|_{\overline{U}_{m,m+1}}$ is rationally connected. A general point of $\overline{U}_{m,m+1}$ is rationally connected to the image of $\overline{\mathcal{M}}(\Sigma, \tau''_m(F; F'))$ by a rational curve in a fiber of ev. Moreover, the intersection points may be chosen general.

Proof. The proof is very similar to the proof of Lemma 8.1. The one difference is that, because $\Sigma \cong \mathbb{P}^1 \times \mathbb{P}^1$, the curve D_0 can be chosen to contain any point p_0 of Σ. □

lem-t2

Lemma 8.3. *Let h equal 0 or 1 and let m be a positive integer. Let*

$$\pi : \Sigma \to \mathbb{P}^1, \quad f : \Sigma \to X$$

be an m-twisting surface of H-type h and class (β_1, β_2). Denote $\beta_2 + m\beta_1$ by β. By functoriality of the Kontsevich space, for each nonnegative integer n there is a 1-morphism

$$\overline{\mathcal{M}}_{0,n}(f) : \overline{\mathcal{M}}_{0,n}(\Sigma, F' + mF) \to \overline{\mathcal{M}}_{0,n}(X, \beta).$$

Every point of $\overline{\mathcal{M}}_{0,n}(\Sigma, F' + nF)$ corresponding to a reduced divisor D in $|\mathcal{O}_\Sigma(F' + nF)|$ with n distinct, smooth marked points is mapped to a smooth point of $\overline{\mathcal{M}}_{0,n}(X, \beta)$.

Denote by M_m the unique irreducible component of $\overline{\mathcal{M}}_{0,m}(X, \beta)$ containing the image of $\overline{U}_m$. A general point of M_m is contained in the image of a 1-morphism

$$\zeta : \mathbb{P}^1 \to M_m$$

contained in a fiber of

$$ev : \overline{\mathcal{M}}_{0,m}(X,\beta) \to X^m$$

and intersecting $\overline{\mathcal{M}}(X, \tau'_m(\beta_1;\beta_2))$ *in a general point specializing to the image of* $\overline{\mathcal{M}}(\Sigma, \tau'_m(F;F'))$.

Finally, assume $h = 0$*, i.e.,* $\Sigma \cong \mathbb{P}^1 \times \mathbb{P}^1$*. Denote by* M_{m+1} *the unique irreducible component of* $\overline{\mathcal{M}}_{0,m+1}(X,\beta)$ *containing the image of* $\overline{U}_{m,m+1}$*. A general point of* M_{m+1} *is contained in the image of a* 1*-morphism*

$$\zeta : \mathbb{P}^1 \to M_{m+1}$$

contained in a fiber of

$$ev : \overline{\mathcal{M}}_{0,m+1}(X,\beta) \to X^{m+1}$$

and intersecting $\overline{\mathcal{M}}(X, \tau''_m(\beta_1;\beta_2))$ *in an general point specializing to the image of* $\overline{\mathcal{M}}(\Sigma, \tau''_m(F;F'))$.

Proof. For every reduced divisor C in $|\mathcal{O}_\Sigma(F' + mF)|$, the normal bundle $N_{C/\Sigma}$ is globally generated. Since also $N_{\Sigma/X}$ is globally generated, $N_{C/X}$ is globally generated. Therefore C is a smooth point of $\overline{\mathcal{M}}_{0,n}(X, \beta_2 + m\beta_1)$.

By Lemma 7.4, a general deformation in X of a reduced divisor C in $|\mathcal{O}_\Sigma(F'+mF)|$ is contained in a deformation of Σ. Because the H-type h equals 0 or 1, every small deformation of Σ is again a Hirzebruch surface of H-type h. By Lemma 8.1, a small deformation of C is contained in the image of a morphism ζ as above.

Because the deformation of C may be taken to be a general point of M_m, or a general deformation of a point of $\overline{\mathcal{M}}(\Sigma, \tau'_m(F;F'))$ to a point of $\overline{\mathcal{M}}(X, \tau'_m(\beta_1;\beta_2))$, the image of a general morphism ζ as above intersects both loci in general points.

The version with M_m replaced by M_{m+1} and $\overline{U}_n$ replaced by $\overline{U}_{m,m+1}$ follows in the same way from Lemma 8.2. $\square$

lem-tM

Lemma 8.4. *Let* $M_{\alpha,1}$ *be an irreducible component of* $\overline{\mathcal{M}}_{0,1}(X,\alpha)$ *dominating* X *and whose general point is a smooth rational curve. If the* m*-twisting surface in Lemma 8.3 has H-type* 0*, i.e.,* $\Sigma \cong \mathbb{P}^1 \times \mathbb{P}^1$*, and* M*-class* $(e_1 \cdot \alpha, e_2 \cdot \alpha)$*, then denoting* $e = e_2 + me_1$*,* M_m*, resp.* $M_{m,m+1}$ *equals* $M_{e\cdot\alpha,m}$*, resp.* $M_{e\cdot\alpha,m+1}$*. And the image of* $\overline{\mathcal{M}}(\Sigma; \tau'_m(F;F'))$*, resp.* $\overline{\mathcal{M}}(\Sigma; \tau''_m(F;F'))$*, is contained in the image of* $M_{\tau'_m(e_1\cdot\alpha;e_2\cdot\alpha)}$*, resp.* $M_{\tau''_m(e_1\cdot\alpha;e_2\cdot\alpha)}$*, cf. Notation 3.8.*

Proof. Because the H-type is 0, every curve in Σ is free. Because the M-class is $(e_1 \cdot \alpha, e_2 \cdot \alpha)$, there is a curve in M_m, resp. $M_{m,m+1}$, whose irreducible components are free, smooth curves parametrized by components $M_{e_i\cdot\alpha,0}$ for various positive integer e_i. Therefore the lemma follows from Lemma 3.6. $\square$

lem-t2.5

Lemma 8.5. *Let* Y *be a proper algebraic space over* k*, let* $\nu : \widetilde{Y} \to Y$ *be a strong resolution of singularities, and let*

$$\phi : \widetilde{Y} \dashrightarrow Q$$

be the MRC fibration.

Let Z *be a closed subset of* Y *such that* $Z \cap Y_{smooth}$ *is dense in* Z*. If every general point of* Y *is contained in a rational curve intersecting* $Z \cap Y_{smooth}$*, then the strict transform* $\widetilde{Z}$ *of* Z *intersects the domain of definition of* ϕ*, and* $\phi(\widetilde{Z})$ *is dense in* Q*.*

Proof. By definition of the MRC quotient, there exists an open subset U of $\widetilde{Y}$ such that the restriction of ϕ,

$$\phi_U : U \to Q,$$

is regular and proper and every curve in $\widetilde{Y}$ intersecting the geometric generic fiber of ϕ_U is contained in the geometric generic fiber of ϕ_U (in particular, it is contained in U).

Because ν is an isomorphism over Y_{smooth}, the strict transform of the rational curve in the statement intersects $\widetilde{Z}$. Therefore every general point of $\widetilde{Y}$ is contained in a rational curve intersecting $\widetilde{Z}$. By the previous paragraph, $\widetilde{Z}$ intersects the geometric generic fiber of ϕ_U. In other words, $\widetilde{Z}$ intersects U and $\phi_U(U \cap \widetilde{Z})$ is dense in Q. $\square$

lem-t3 **Lemma 8.6.** *Let $M_{\alpha,1}$ be an irreducible component of $\overline{\mathcal{M}}_{0,1}(X,\alpha)$ dominating X and whose generic fiber over X is geometrically irreducible. Let m be a positive integer. Assume there exists an m-twisting surface of H-type 0 and M-class $(e_1 \cdot \alpha, e_2 \cdot \alpha)$.*

Denoting $e = e_2 + me_1$, a general point of a general fiber of the restriction

$$ev|_M : M_{e\cdot\alpha,m+1} \subset \overline{\mathcal{M}}_{0,m+1}(X, e\alpha) \to X^{m+1}$$

is contained in a rational curve in the fiber intersecting the image of the boundary map

$$M_{e_1\cdot\alpha,2} \times_{pr_2\circ ev, X, pr_1\circ ev} M_{(e-e_1)\cdot\alpha,m+1} \to M_{e\cdot\alpha,m+1},$$

in a smooth point of the fiber. Thus the image $\Delta_{(e_1\cdot\alpha,\{0\}),((e-e_1)\cdot\alpha,\{1,\dots,m\})}$ of the boundary map intersects the domain of definition of the MRC fibration of a strong resolution of the fiber, and this intersection dominates the MRC quotient of a strong resolution of the fiber.

In particular, if a general fiber of $\Delta_{(e_1\cdot\alpha,\{0\}),((e-e_1)\cdot\alpha,\{1,\dots,m\})}$ over X^{m+1} is geometrically connected and geometrically rationally connected, then a general fiber of $ev|_M$ is geometrically connected and geometrically rationally connected.

Proof. By the last part of Lemma 8.3 and by Lemma 8.4, a general point of a general fiber of $\text{ev}|_M$ is contained in a rational curve intersecting $M_{\tau''_m(e_1\cdot\alpha;e_2\cdot\alpha)}$ in a smooth point. Recall that $M_{\tau'_m(e_1\cdot\alpha;e_2\cdot\alpha)}$ parametrizes reducible combs $C = C_0 \cup C_1 \cup \dots \cup C_m$ whose handle C_0 has M-class $e_2 \cdot \alpha$ and whose teeth C_i, $i = 1, \dots, m$, have M-class $e_1 \cdot \alpha$. For each $i = 1, \dots, m$, there is a point $q_{i,0}$ of C_0 that is attached to a point q_i of C_i. Also, there is a marked point p_0 of C_0 and for each $i = 1, \dots, m$, there is a marked point p_i of C_i. Altogether, $(C, p_0, \dots, p_m)$ is a marked curve parametrized by $M_{e\cdot\alpha,m+1}$.

Now let B_0 be one of the teeth C_i, $i = 1, \dots, m$ and mark it by $r_0 = p_i$ and $s_0 = q_i$ so that (B_0, r_0, s_0) is parametrized by $M_{e_1\cdot\alpha,2}$. Next let B_1 be the union of C_0 and all of the teeth C_j with $j \neq i$. Mark B_1 by $s_1 = q_{i,0}$ and

$$r_j = \begin{cases} p_j, & j \neq i \\ p_0, & j = i \end{cases}$$

so that $(B_1, s_1, p_1, \dots, p_m)$ is parametrized by $M_{(e-e_1)\cdot\alpha,m+1}$. Thus the pair $((B_0, r_0, s_0), (B_1, s_1, r_1, \dots, r_m))$ is parametrized by a point of

$$M_{e_1\cdot\alpha,2} \times_{\text{pr}_2\circ\text{ev},X,\text{pr}_1\circ\text{ev}} M_{(e-e_1)\cdot\alpha,m+1}.$$

Moreover, up to a reordering of the marked points, the image of this point under the boundary map

$$M_{e_1\cdot\alpha,2} \times_{\mathrm{pr}_2\circ\mathrm{ev},X,\mathrm{pr}_1\circ\mathrm{ev}} M_{(e-e_1)\cdot\alpha,m+1} \to M_{e\cdot\alpha,m+1}$$

is precisely $(C, p_0, \ldots, p_m)$. In other words, $M_{\tau''_m(e_1\cdot\alpha;e_2\cdot\alpha)}$ is contained in $\Delta_{(e_1\cdot\alpha,\{0\}),((e-e_1)\cdot\alpha,\{1,\ldots,m\})}$ up to reordering of the marked points. Therefore the rational curve from the previous paragraph intersects $\Delta_{(e_1\cdot\alpha,\{0\}),((e-e_1)\cdot\alpha,\{1,\ldots,m\})}$.

Also, since a general point of $M_{\tau''_m(e_1\cdot\alpha;e_2\cdot\alpha)}$ parametrizes a curve whose irreducible components are all free, a general point is a smooth point of the fiber. Thus, by Lemma 8.5, for a general geometric point (x_1, x_2) of $X \times X$, the boundary $\Delta \cap \mathrm{ev}|_M^{-1}(x_1, x_2)$ intersects the domain of definition of the MRC fibration of $\mathrm{ev}|_M^{-1}(x_1, x_2)$, and the intersection dominates the MRC quotient. If $\Delta \cap \mathrm{ev}|_M^{-1}(x_1, x_2)$ is connected and rationally connected, the same holds for the MRC quotient. Then, by [GHS03], $\mathrm{ev}|_M^{-1}(x_1, x_2)$ is connected and rationally connected. $\square$

lem-t4

Lemma 8.7. *Let $M_{\alpha,1}$ be an irreducible component of $\overline{\mathcal{M}}_{0,1}(X, \alpha)$ dominating X and whose generic fiber over X is geometrically irreducible. Let m, e'_1, e''_1 and e be nonnegative integers such that $e \geq e'_1 + e''_1$.*

Assume that for both $(c, c_1) = (e, e'_1)$ and $(c, c_1) = (e - e'_1, e'_2)$ a general point of a general fiber of the restriction

$$ev|_M : M_{c\cdot\alpha,m_1} \subset \overline{\mathcal{M}}_{0,m+1}(X, c\alpha) \to X^{m+1}$$

is contained in a rational curve in the fiber intersecting the image of

$$M_{c_1\cdot\alpha,2} \times_{pr_2,\circ ev,X,pr_1\circ ev} M_{(c-c_1)\cdot\alpha,m+1} \to M_{c\cdot\alpha,m+1}$$

in a smooth point. Then a general point of a general fiber of the restriction

$$ev|_M : M_{e\cdot\alpha,m+1} \subset \overline{\mathcal{M}}_{0,m+1}(X, e\alpha) \to X^{m+1}$$

is contained in a rational curve in the fiber intersecting the image of the boundary map

$$M_{(e'_1+e''_1)\cdot\alpha,2} \times_{pr_2\circ ev,X,pr_1\circ ev} M_{(e-e'_1-e'_2)\cdot\alpha,m+1} \to M_{e\cdot\alpha,m+1}$$

in a smooth point.

Proof. Consider a general fiber of the three restrictions of the evaluation morphism

$$\mathrm{ev}_I = \mathrm{ev}|_M : M_{e\cdot\alpha,m+1} \subset \overline{\mathcal{M}}_{0,m+1}(X, e\alpha) \to X^{m+1},$$

$$\mathrm{ev}_{II} = (\mathrm{pr}_1\circ\mathrm{ev}\circ\mathrm{pr}_1, \mathrm{pr}_{2,\ldots,m+1}\circ\mathrm{ev}\circ\mathrm{pr}_2) : M_{e'_1\cdot\alpha,2}\times_{\mathrm{pr}_2,\circ\mathrm{ev},X,\mathrm{pr}_1\circ\mathrm{ev}}M_{(e-e'_1)\cdot\alpha,m+1} \to X^{m+1},$$

and

$$\mathrm{ev}_{III} = (\mathrm{pr}_1\circ\mathrm{ev}\circ\mathrm{pr}_1, \mathrm{pr}_{2,\ldots,m+1}\circ\mathrm{ev}\circ\mathrm{pr}_3) : M_{e'_1\cdot\alpha,2}\times_{\mathrm{pr}_2\circ\mathrm{ev},X,\mathrm{pr}_1\circ\mathrm{ev}}M_{e''_1\cdot\alpha,2}\times_{\mathrm{pr}_2\circ\mathrm{ev},X,\mathrm{pr}_1\circ\mathrm{ev}}M_{(e-e'_1-e''_1)\cdot\alpha,m+1} \to X^{m}$$

Of course ev_{II} is the restriction of ev_I to the domain of ev_{II}, and likewise ev_{III} is the restriction of ev_{II} to the domain of ev_{III}. Thus the fibers of the three evaluation maps form a triple of nested subvarieties.

Using the hypothesis for $(c, c_1) = (e, e'_1)$, Lemma 8.5 implies that the MRC quotient of a strong desingularization of a general fiber of ev_I is dominated by the MRC quotient of a strong desingularization of the corresponding fiber of ev_{II}. Using the hypothesis for $(c, c_1) = (e - e'_1, e''_1)$, Lemma 8.5 implies that the MRC quotient of a strong desingularization of a general fiber of ev_{II} is dominated by the MRC quotient of a strong desingularization of the corresponding fiber of ev_{III}. Thus the MRC quotient of a strong desingularization of a general fiber of ev_I is dominated by the strict transform of the corresponding fiber of ev_{III}. Also every general point of the domain of ev_{III} is a smooth point of the image of

$$M_{(e_1'+e_1'')\cdot\alpha,2}\times_{\mathrm{pr}_2\circ\mathrm{ev},X,\mathrm{pr}_1\circ\mathrm{ev}} M_{(e-e_1'-e_2')\cdot\alpha,m+1}\to M_{e\cdot\alpha,m+1}.$$

Thus the MRC quotient of a general fiber of ev_I is dominated by the strict transform of the corresponding fiber of $\Delta_{((e_1'+e_1'')\cdot\alpha,\{0\}),((e-e_1'-e_1'')\cdot\alpha,\{1,\dots,m\})}$. □

sec-pf

9. Proofs of the main theorems

Proof of Theorem 1.7. First of all, the hypotheses of Corollary 7.15 are implied by the hypotheses of Theorem 1.7. Therefore, by Corollary 7.15, there is a unique irreducible component $M_{\alpha,1}$ of $\overline{\mathcal{M}}_{0,1}(X,\alpha)$ dominating X, and a general fiber of $M_{\alpha,1}$ over X is geometrically connected. Moreover, by Lemma 4.1(iii), a general fiber is Fano and thus rationally connected. Also by Corollary 7.15, for every positive integer r there exists a 1-twisting surface of H-type 0 and M-class $(\alpha, r\cdot\alpha)$.

The proof that the fiber of

$$\mathrm{ev}|_M : M_{e\cdot\alpha,2}\to X^2$$

over a general point of X^2 is geometrically rationally connected is by induction on $e\geq 2$. The base case is $e=2$. First of all, by Lemmas 5.1, 5.3 and 5.5, the fiber is smooth (but possibly empty). The claim is that the fiber is nonempty and geometrically connected. This will be proved by Corollary 5.10. The first hypothesis of Corollary 5.10 for $m=2$ is,

$$\dim(X)\geq 2(\dim(X)-\langle c_1(T_X),\alpha\rangle-1)+4.$$

This is implied by one of the hypotheses of Theorem 1.7. The second hypothesis is

$$c\geq 2(\dim(X)-\langle c_1(T_X),\alpha\rangle-1)-\dim(X)-2,$$

which is implied by the first hypothesis: $c\geq -2$ holds for $c=0$. The final hypothesis,

$$\mathrm{CH}^p(Y)=\mathbb{Z}\{c_1(\mathcal{O}(1))^p\},\ \text{for } 0\leq p\leq 2(\dim(X)-\langle c_1(T_X),\alpha\rangle-1)+4,$$

is one of the hypotheses of Theorem 1.7. Therefore, by Corollary 5.10, a general fiber of ev is nonempty and geometrically connected. So the fiber is nonempty, smooth and geometrically connected.

The next claim is that a general fiber of

$$\mathrm{ev}: M_{2\cdot\alpha,2}\to X^2$$

is a Fano manifold, and thus geometrically rationally connected. Since the fiber is nonempty, smooth and geometrically connected, it only remains to prove the first Chern class is positive. This follows from Lemma 6.5. The hypothesis of Lemma 6.5 is

$$\langle 2\mathrm{ch}_2(T_X),\Pi\rangle\geq -1.$$

Since one of the hypotheses of Theorem 1.7 is that

$$\langle 2\mathrm{ch}_2(T_X),\Pi\rangle\geq 0,$$

Lemma 6.5 implies that a general fiber of ev is Fano. Therefore by [KMM92] and [Cam92], a general fiber is geometrically rationally connected. This establishes the base case of the induction.

Now, by way of induction, assume $e > 2$ and assume the result is known for $e-1$. There exists a 1-twisting surface of H-type 0 and M-class $(\alpha, (e-1)\cdot\alpha)$. Thus, by Lemma 8.6, the fiber of

$$\mathrm{ev}|_M : M_{e\cdot\alpha,2} \to X^2$$

over a general point is geometrically connected and geometrically rationally connected if the fiber of

$$\mathrm{ev}_\Delta : M_{\alpha,2} \times_{\mathrm{pr}_2\circ\mathrm{ev},X,\mathrm{pr}_1\circ\mathrm{ev}} M_{(e-1)\cdot\alpha,2} \to X \times X$$

over a general point is geometrically connected and geometrically rationally connected.

Fix a general geometric point (x_1, x_2) of $X \times X$. Projection onto the first factor

$$\mathrm{pr}_1 : M_{\alpha,2} \times_{\mathrm{pr}_2\circ\mathrm{ev},X,\mathrm{pr}_1\circ\mathrm{ev}} M_{(e-1)\cdot\alpha,2} \to M_{\alpha,2}$$

restricts to give a projection of the fiber $\mathrm{ev}_\Delta^{-1}(x_1, x_2)$ onto the fiber of

$$\mathrm{pr}_1 \circ \mathrm{ev}|_M : M_{\alpha,2} \to X \times X \to X.$$

Of course this is the same as the composition of the forgetful morphism (forgetting the second marked point),

$$\Phi_1 : M_{\alpha,2} \to M_{\alpha,1}$$

with the evaluation morphism

$$\mathrm{ev}|_M : M_{\alpha,1} \to X.$$

By the first paragraph of the proof, the fiber of this last map over a general point x_1 is geometrically connected and geometrically rationally connected. Moreover, a general fiber of Φ_1 is a smooth rational curve. Therefore, by [GHS03] (or simpler arguments), the fiber F of

$$\mathrm{pr}_1 \circ \mathrm{ev}|_M : M_{\alpha,2} \to X \times X \to X$$

over a general point x_1 is rationally connected.

By [GHS03] again, to prove that $\mathrm{ev}_\Delta^{-1}(x_1, x_2)$ is connected and rationally connected, it suffices to prove that the fiber of

$$\mathrm{pr}_1|_{\mathrm{pr}_1^{-1}(F)} : \mathrm{ev}_\Delta^1(x_1, x_2) \to F$$

over a general point is geometrically connected and geometrically rationally connected. But the geometric generic fiber of $\mathrm{pr}_1|_{\mathrm{pr}_1^{-1}(F)}$ is precisely the same as the geometric generic fiber of

$$\mathrm{ev}|_M : M_{(e-1)\cdot\alpha,2} \to X^2.$$

By the induction hypothesis, this is connected and rationally connected. Therefore $\mathrm{ev}_\Delta^{-1}(x_1, x_2)$ is connected and rationally connected. By the arguments above, this implies the fiber of

$$\mathrm{ev}|_M : M_{e\cdot\alpha,2} \subset \overline{\mathcal{M}}_{0,2}(X, e\alpha) \to X^2$$

over a general point of X^2 is geometrically connected and geometrically rationally connected. Therefore the theorem is proved by induction on e. $\square$

Proof of Theorem 1.1. The case of a linear or quadric hypersurface follows immediately from Theorem 1.7. Thus assume the smooth complete intersection is neither linear nor quadric, i.e., all $d_i \geq 2$ and $\underline{d} \neq (2)$. To prove the theorem, it suffices to verify the hypotheses of Theorem 1.7. By Lemma 2.1,

$$\langle c_1(T_X), \alpha \rangle = n + 1 - \sum_{i=1}^{c} d_i,$$

and

$$\langle 2\mathrm{ch}_2(T_X), \Pi \rangle = n + 1 - \sum_{i=1}^{c} d_i^2.$$

Thus,

$$\dim(X) - \langle c_1(T_X), \alpha \rangle - 1 = \sum_{i=1}^{c} (d_i - 1) - 2.$$

So the first three inequalities in Theorem 1.7 are

$$n + 1 - \sum_{i=1}^{c} d_i > 0,$$

$$n + 1 - \sum_{i=1}^{c} d_i^2 \geq 0,$$

and

$$n \geq \sum_{i=1}^{c} (2d_i - 1).$$

Because $d_i > 1$ for all i, $\sum_i d_i < \sum_i d_i^2$. Also, the difference

$$[\sum_{i=1}^{c} d_i^2] - [\sum_{i=1}^{c} (2d_i - 1)] = \sum_{i=1}^{c} (d_i - 1)^2$$

is strictly positive. In particular, it is ≥ 1. Therefore the hypothesis

$$n + 1 \geq \sum_{i=1}^{c} d_i^2$$

in Theorem 1.1 implies all three of the hypotheses above.

The hypotheses on c all hold. In fact, for a smooth complete intersection X, for every sufficiently positive integer c, X is a codimension c linear section of a complete intersection Y by Lemma 2.5. $\square$

Proof of Theorem 1.8. First of all, the hypotheses of Theorem 1.8 imply the hypotheses of Theorem 1.7. The one hypothesis of Theorem 1.7 that is a bit less obvious is,

$$\dim(X) \geq 2(\dim(X) - \langle c_1(T_X), \alpha \rangle - 1) + 5.$$

This follows by the same argument as in the proof of Corollary 7.19. In particular, there exists a unique irreducible component $M_{\alpha,1}$ of $\overline{\mathcal{M}}_{0,1}(X, \alpha)$ dominating X, and a general fiber of $M_{\alpha,1}$ over X is geometrically connected and geometrically rationally connected.

The second part is proved by induction on m. The base case, $m = 2$, follows from Theorem 1.7. Thus, by way of induction, assume the result for m and consider the case $m+1$. The hypotheses of Theorem 1.8 imply the hypotheses of Corollary 7.19.

Thus, by Corollary 7.19, for every $e' \geq 2m-2$, there exists an m-twisting surface of H-type 0 and M-class either

(I). $(\alpha, e' \cdot \alpha)$ or

(II). $(2 \cdot \alpha, e' \cdot \alpha)$.

Then, by Lemma 8.6, one of the following hold

(I). For every $e' \geq 3m-2$, every general point of a general fiber of the restriction

$$\mathrm{ev}|_M : M_{e' \cdot \alpha, m+1} \subset \overline{\mathcal{M}}_{0,m+1}(X, e'\alpha) \to X^{m+1}$$

is contained in a rational curve in the fiber intersecting the boundary divisor $\Delta_{(\alpha,\{0\}),((e'-1)\cdot\alpha,\{1,\dots,m\})}$ in a smooth point.

(II). For every $e' \geq 4m-2$, every general point of a general fiber of the restriction

$$\mathrm{ev}|_M : M_{e' \cdot \alpha, m+1} \subset \overline{\mathcal{M}}_{0,m+1}(X, e'\alpha) \to X^{m+1}$$

is contained in a rational curve in the fiber intersecting the boundary divisor $\Delta_{(2\cdot\alpha,\{0\}),((e'-2)\cdot\alpha,\{1,\dots,m\})}$ in a smooth point.

Moreover, in Case (I), using the result for both e' and $e'-1$, Lemma 8.7 implies

(I). For every $e' \geq 3m-1$, every general point of a general fiber of the restriction

$$\mathrm{ev}|_M : M_{e' \cdot \alpha, m+1} \subset \overline{\mathcal{M}}_{0,m+1}(X, e'\alpha) \to X^{m+1}$$

is contained in a rational curve in the fiber intersecting the boundary divisor $\Delta_{(2\cdot\alpha,\{0\}),((e'-2)\cdot\alpha,\{1,\dots,m\})}$ in a smooth point.

Thus, both in Case (I) and Case (II), for every $e \geq 4(m+1)-6$, every general point of a general fiber of the restriction

$$\mathrm{ev}|_M : M_{e \cdot \alpha, m+1} \subset \overline{\mathcal{M}}_{0,m+1}(X, e\alpha) \to X^{m+1}$$

is contained in a rational curve in the fiber intersecting the boundary divisor $\Delta_{(2\cdot\alpha,\{0\}),((e-2)\cdot\alpha,\{1,\dots,m\})}$. Thus, a general fiber of $\mathrm{ev}|_M$ is nonempty, geometrically connected and geometrically rationally connected if and only if a general fiber of

$$\mathrm{ev}_\Delta = (\mathrm{pr}_1 \circ \mathrm{ev} \circ \mathrm{pr}_1, \mathrm{pr}_{2,\dots,m+1} \circ \mathrm{ev}, \mathrm{pr}_2) : M_{2\cdot\alpha,2} \times_{\mathrm{pr}_2 \circ \mathrm{ev}, X, \mathrm{pr}_1 \circ \mathrm{ev}} M_{(e-2)\cdot\alpha, m+1} \to X \times X^m$$

is nonempty, geometrically connected and geometrically rationally connected.

Fix a general (geometric) point $(p, (q_1, \dots, q_m))$ of $X \times X^m$. Because $e \geq 4(m+1)-6$, $e-2 \geq 4m-6$. Thus, by the induction hypothesis, the fiber of

$$\mathrm{ev}|_M : M_{(e-2)\cdot\alpha, m} \to X^m$$

over $(q_1, \dots, q_m)$ is nonempty, geometrically connected and geometrically rationally connected. Moreover, for the forgetful map

$$\Phi_1 : M_{(e-2)\cdot\alpha, m+1} \to M_{(e-2)\cdot\alpha, m}$$

forgetting the first marked point, the fiber of

$$\mathrm{ev}|_M \circ \Phi_1 : M_{(e-2)\cdot\alpha, m+1} \to M_{(e-2)\cdot\alpha, m} \to X^m$$

over $(q_1, \dots, q_m)$ is a flat, proper family of rationally connected curves over $\mathrm{ev}|_M^{-1}(q_1, \dots, q_m)$ whose general fiber is smooth. A conic bundle over a rationally connected variety is rationally connected, thus the fiber of $\mathrm{ev}|_M \circ \Phi_1$ over $(q_1, \dots, q_m)$ is rationally connected.

Projection π_2 onto the $M_{(e-2)\cdot\alpha,m+1}$ factor defines a projection from the fiber of

$$\mathrm{ev}_\Delta : M_{2\cdot\alpha,2} \times_{\mathrm{pr}_2\circ\mathrm{ev},X,\mathrm{pr}_1\circ\mathrm{ev}} M_{(e-2)\cdot\alpha,m+1} \to X \times X^m$$

over $(p,(q_1,\dots,q_m))$ to the fiber of

$$\mathrm{ev}|_M \circ \Phi_1 : M_{(e-2)\cdot\alpha,m+1} \to M_{(e-2)\cdot\alpha,m} \to X^m$$

over $(q_1,\dots,q_m)$. Since the second fiber is nonempty, geometrically connected and geometrically rationally connected, by [GHS03], to prove the first fiber is rationally connected, it suffices to prove the geometric generic fiber of π_2 is nonempty, geometrically connected and geometrically rationally connected.

Assuming that $(q_1,\dots,q_m)$ is general, the first marked point of a general curve parametrized by a point of the second fiber is a general point p' of X. Thus, for general p, the pair (p,p') is a general point of $X \times X$. Thus the geometric generic fiber of π_2 equals the geometric generic fiber of

$$\mathrm{ev}|_M : M_{2\cdot\alpha,2} \subset \overline{\mathcal{M}}_{0,2}(X,2) \to X^2.$$

By Theorem 1.7, this is nonempty, geometrically connected and geometrically rationally connected. Thus, by the argument above, the geometric generic fiber of

$$\mathrm{ev}|_M : M_{e\cdot\alpha,m+1} \subset \overline{\mathcal{M}}_{0,m+1}(X,e\alpha) \to X^{m+1}$$

is nonempty, connected and rationally connected. Therefore the theorem is proved by induction on m. $\square$

Proof of Theorem 1.2. The case of a linear or quadric hypersurface follows immediately from Theorem 1.8. Thus assume the smooth complete intersection is neither linear nor quadric, i.e., all $d_i \geq 2$ and $\underline{d} \neq (2)$. To prove the theorem, it suffices to verify the hypotheses of Theorem 1.8. By Lemma 2.1,

$$\langle c_1(T_X),\alpha\rangle = n+1-\sum_{i=1}^{c} d_i,$$

and

$$\langle 2\mathrm{ch}_2(T_X),\Pi\rangle = n+1-\sum_{i=1}^{c} d_i^2.$$

Thus,

$$\dim(X) - \langle c_1(T_X),\alpha\rangle - 1 = \sum_{i=1}^{c}(d_i-1) - 2.$$

So the first three inequalities in Theorem 1.8 are

$$n+1-\sum_{i=1}^{c} d_i > 0,$$

$$2(n+1-\sum_{i=1}^{c} d_i^2) \geq n+1-\sum_{i=1}^{c} d_i \ \Leftrightarrow n+1 \geq \sum_{i=1}^{c}(2d_i^2-d_i),$$

and

$$n+4 \geq \sum_{i=1}^{c}(4d_i-3).$$

Because $d_i > 1$ for all i, $\sum_i d_i < \sum_i d_i^2$, which implies $\sum_i d_i < \sum_i (2d_i^2 - d_i)$. Also, the difference

$$[\sum_{i=1}^{c}(2d_i^2 - d_i)] - [\sum_{i=1}^{c}(4d_i - 3)] = \sum_{i=1}^{c}(2d_i - 3)(d_i - 1)$$

is strictly positive. In particular, it is ≥ 1. Therefore the hypothesis

$$n + 1 \geq \sum_{i=1}^{c}(2d_i^2 - d_i)$$

in Theorem 1.1 implies all three of the hypotheses above.

The hypotheses on c all hold. In fact, for a smooth complete intersection X, for every sufficiently positive integer c, X is a codimension c linear section of a complete intersection Y by Lemma 2.5. □

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