FANO VARIETIES AND LINEAR SECTIONS OF HYPERSURFACES

JASON MICHAEL STARR

Abstract. When n satisfies an inequality which is almost best possible, we prove that the k-plane sections of every smooth, degree d, complex hypersurface in $\mathbb{P}^n$ dominate the moduli space of degree d hypersurfaces in $\mathbb{P}^k$. As a corollary we prove that, for n sufficiently large, every smooth, degree d hypersurface in $\mathbb{P}^n$ satisfies a version of "rational simple connectedness".

1. Statement of results

In their article [2], Harris, Mazur and Pandharipande prove that for fixed integers d and k, there exists an integer $n_0 = n_0(d,k)$ such that for every $n \geq n_0$, every smooth degree d hypersurface X in $\mathbb{P}^n_{\mathbb{C}}$ has a number of good properties:

(i) The hypersurface is unirational.
(ii) The Fano variety of k-planes in X has the expected dimension.
(iii) The k-plane sections of the hypersurface dominate the moduli space of degree d hypersurfaces in $\mathbb{P}^k$.

It is this last property which we consider. To be precise, the statement is that the following rational transformation

$$\Phi : \mathbb{G}(k,n) \dashrightarrow \mathbb{P}^{N_d}//\mathbf{PGL}_{k+1}$$

is dominant. Here $\mathbb{G}(k,n)$ is the Grassmannian parametrizing linear $\mathbb{P}^k$s in $\mathbb{P}^n$, $\mathbb{P}^{N_d}$ is the parameter space for degree d hypersurface in $\mathbb{P}^k$, $\mathbb{P}^{N_d}//\mathbf{PGL}_{k+1}$ is the moduli space of semistable degree k hypersurface in $\mathbb{P}^k$, and Φ is the rational transformation sending a k-plane Λ to the moduli point of the hypersurface $\Lambda \cap X \subset \Lambda$ (assuming $\Lambda \cap X$ is a semistable degree k hypersurface in $\mathbb{P}^k$).

The bound $n_0(d,k)$ is very large, roughly a d-fold iterated exponential. Our result is the following.

Theorem 1.1. *Let X be a smooth degree d hypersurface in $\mathbb{P}^n$. The map Φ is dominant if*

$$n \geq \binom{d+k-1}{k} + k - 1.$$

Question 1.2. For fixed d and k, what is the smallest integer $n_0 = n_0(d,k)$ such that for every $n \geq n_0$ and every smooth, degree d hypersurface in $\mathbb{P}^n$, the associated rational transformation Φ is dominant?

Theorem 1.1 is equivalent to the inequality

$$n_0(d,k) \leq \binom{d+k-1}{k} + k - 1.$$

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If Φ is dominant, then the dimension of the domain is at least the dimension of the target, i.e.,

$$(k+1)(n-k) = \dim \mathbb{G}(k,n) \geq \dim(\mathbb{P}^{N_d}//\mathbf{PGL}_{k+1}) = \binom{d+k}{k} - (k+1)^2.$$

This is equivalent to the condition

$$n_0(d,k) \geq \frac{1}{k+1}\binom{d+k}{k} - 1.$$

As far as we know, this is the correct bound. The bound from Theorem 1.1 differs from this optimal bound by roughly a factor of k.

The main step in the proof is a result of some independent interest.

Proposition 1.3. *Let X be a smooth degree d hypersurface in $\mathbb{P}^n$. Let $F_k(X)$ be the Fano variety of k-planes in X. There exists an irreducible component I of $F_k(X)$ of the expected dimension if*

$$n \geq \binom{d+k-1}{k} + k.$$

Moreover, if

$$n = \binom{d+k-1}{k} + k - 1$$

then there is a nonempty open subset $U_{k-1} \subset F_{k-1}(X)$ such that for every $[\Lambda_{k-1}] \in U_{k-1}$, there exists no k-plane in X containing Λ_{k-1}.

Theorem 1.1 implies a result about rational curves on every smooth hypersurface of sufficiently small degree. The Kontsevich moduli space $\overline{\mathrm{M}}_{0,r}(X,e)$ parametrizes isomorphism classes of data $(C, q_1, \dots, q_r, f)$ of a proper, connected, at-worst-nodal, arithmetic genus 0 curve C, an ordered collection $q_1, \dots, q_r$ of distinct smooth points of C and a morphism $f : C \to X$ satisfying a stability condition. The space $\overline{\mathrm{M}}_{0,r}(X,e)$ is projective. There is an evaluation map

$$\mathrm{ev} : \overline{\mathrm{M}}_{0,r}(X,e) \to X^r$$

sending a datum $(C, q_1, \dots, q_r, f)$ to the ordered collection $(f(q_1), \dots, f(q_r))$.

Corollary 1.4. *Let X be a smooth degree d hypersurface in $\mathbb{P}^n$. If*

$$n \geq \binom{d^2+d-1}{d-1} + d^2 - 1$$

then for every integer $e \geq 2$ there exists a canonically defined irreducible component $\mathcal{M} \subset \overline{\mathrm{M}}_{0,2}(X,e)$ such that the evaluation morphism

$$ev : \mathcal{M} \to X \times X$$

is dominant with rationally connected generic fiber, i.e., X satisfies a version of rational simple connectedness. Moreover X has a very twisting family of pointed lines, cf. [4, Def. 3.7].

This is proved in [4] assuming n satisfies a much weaker hypothesis

$$n \geq d^2$$

but only for *general* hypersurfaces, not for *every* smooth hypersurface. The goal here is to find a stronger hypothesis on n that guarantees the theorem for every smooth hypersurface.

2. Flag Fano varieties

Naturally enough, the proof of Proposition 1.3 uses an induction on k. To set up the induction it is useful to consider not just k-planes in X, but flags of linear spaces

$$\mathbb{P}^0 \subset \mathbb{P}^1 \subset \mathbb{P}^2 \subset \cdots \subset \mathbb{P}^k \subset X.$$

The variety parametrizing such flags is the *flag Fano variety* of X. Also, although we are ultimately interested only in the case of a hypersurface in projective space, for the induction it is useful to allow a more general projective subvariety.

Let S be a scheme such that $H^0(S, \mathcal{O}_S)$ contains $\mathbb{Q}$. Let E be a locally free $\mathcal{O}_S$-module of rank $n+1$, and let $X \subset \mathbb{P}E$ be a closed subscheme such that the projection $\pi : X \to S$ is smooth and surjective of constant relative dimension $\dim(X/S)$. In other words, X is a family of smooth, $\dim(X/S)$-dimensional subvarieties of $\mathbb{P}^n$ parametrized by S.

Let $0 \leq k \leq n$ be an integer. Denote by $\mathrm{Fl}_k(E)$ the partial flag manifold representing the functor on S-schemes

$$T \mapsto \{(E_1 \subset E_2 \subset \cdots \subset E_{k+1} \subset E_T) | E_i \text{ locally free of rank } i, i = 1, \ldots, k+1\}.$$

For every $0 \leq j \leq k \leq n$, denote by $\rho^j_k : \mathrm{Fl}_k(E) \to \mathrm{Fl}_j(E)$ the obvious projection. The *flag Fano variety* is the locally closed subscheme $\mathrm{Fl}_k(X) \subset \mathrm{Fl}_k(E)$ parametrizing flags such that $\mathbb{P}(E_{k+1})$ is contained in X. In particular, $\mathrm{Fl}_0(X) = X$. Denote by $\rho^j_k : \mathrm{Fl}_k(X) \to \mathrm{Fl}_j(X)$ the restriction of ρ^j_k.

2.1. **Smoothness.** There are two elementary observations about the schemes $\mathrm{Fl}_k(X)$.

Lemma 2.1. [3, 1.1] *There exists an open dense subset $U \subset X$ such that $U \times_X Fl_1(X)$ is smooth over U.*

Lemma 2.2. *Set $S^{new} = U$, the open subset from Lemma 2.1. Set E^{new} to be the universal rank n quotient bundle of $\pi^* E|_U$ so that $\mathbb{P}(E^{new}) = U \times_{\mathbb{P}(E)} Fl_1(E)$ and set $X^{new} = Fl_1(U)$. Then for every $0 \leq k \leq n-1$, $Fl_k(X^{new}) = U \times_X Fl_{k+1}(X)$.*

Proof. This is obvious. □

Proposition 2.3. *There exists a sequence of open subschemes $(U_k \subset Fl_k(X))_{0 \leq k \leq n}$ satisfying the following conditions.*

- (i) *The open subset U_0 is dense in $Fl_0(X)$, and for every $1 \leq k \leq n$, U_k is dense in $(\rho^{k-1}_k)^{-1}(U_{k-1})$.*
- (ii) *For every $1 \leq k \leq n$, $\rho^{k-1}_k : (\rho^{k-1}_k)^{-1}(U_{k-1}) \to U_{k-1}$ is smooth.*

Proof. Let U_0 be the open subscheme from Lemma 2.1. By way of induction, assume $k > 0$ and the open subscheme U_{k-1} has been constructed. As in Lemma 2.2, replace S by U_{k-1}, replace E by the universal quotient bundle, and replace X by $(\rho^{k-1}_k)^{-1}(U_{k-1})$. Now define $U_k \subset (\rho^{k-1}_k)^{-1}(U_{k-1})$ to be the open subscheme from Lemma 2.1. □

2.2. **Dimension.** Using the Grothendieck-Riemann-Roch formula, it is possible to express the Chern classes of $U \times_X F_1(X)$ in terms of the Chern classes of U. Iterating this leads, in particular, to a formula for the dimension of U_k. Denote by G_1, resp. G_2, the restriction to $\mathrm{Fl}_1(U)$ of E_1, resp. E_2. Denote by L the invertible sheaf

$$L := (G_2/G_1)^\vee.$$

Denote by

$$\pi : \mathbb{P}G_2 \to \mathrm{Fl}_1(U),$$

$$\sigma : \mathrm{Fl}_1(U) = \mathbb{P}G_1 \to \mathbb{P}G_2,$$

and

$$f : \mathbb{P}G_2 \to X$$

the obvious morphisms. In other words, $\mathbb{P}G_2$ is a family of $\mathbb{P}^1$s over $\mathrm{Fl}_1(U)$, σ is a marked point on each $\mathbb{P}^1$, and f is an embedding of each $\mathbb{P}^1$ as a line in X. The formula for the Chern character of the vertical tangent bundle of ρ_1^0 is,

$$\mathrm{ch}(T_{\mathrm{Fl}_1(U)/U}) = \pi_* f^*[(\mathrm{ch}(T_{X/S}) - \dim(X/S))\mathrm{Todd}(\mathcal{O}_{\mathbb{P}E}(1)|_X)] - \mathrm{ch}(L) - 1.$$

Given a flag $\mathbb{P} = (\mathbb{P}^1 \subset \mathbb{P}^2 \subset \cdots \subset \mathbb{P}^k \subset \mathbb{P}^n)$ in U_k, the formula for the fiber dimension of ρ_k^{k-1} at $\mathbb{P}$ is

$$\dim(U_k/U_{k-1}) = \sum_{m=1}^{k} b_{k,m} \langle \mathrm{ch}_m(T_{X/S}), \mathbb{P}^m \rangle - k - 1$$

where $\mathrm{ch}_m(E)$ is the m^{th} graded piece of the Chern character of E, and where the coefficients $b_{k,m}$ are the unique rational numbers such that

$$\binom{x+k-1}{k} = \sum_{m=1}^{k} \frac{b_{k,m}}{m!} x^m.$$

Now define the numbers $a_{k,m}$ to be

$$a_{k,m} = \sum_{l=m}^{k} b_{l,m},$$

in other words,

$$\sum_{m=1}^{k} \frac{a_{k,m}}{m!} x^m = \sum_{l=1}^{k} \binom{x+l-1}{l}.$$

Then it follows from the previous formula that the dimension of U_k at $\mathbb{P}$ equals

$$\dim(U_k) = \sum_{m=1}^{k} a_{k,m} \langle \mathrm{ch}_m(T_{X/S}), \mathbb{P}^m \rangle + \dim(X) - k^2.$$

In a related direction, there is a class of complex projective varieties that is stable under the operation of replacing X by a general fiber of $\mathrm{Fl}_1(X) \to X$. Call a subvariety X of $\mathbb{P}^n$ a *quasi-complete-intersection* of type

$$\underline{d} = (d_1, \ldots, d_c)$$

if there is a sequence

$$X = X_c \subset X_{c-1} \subset \cdots \subset X_1 \subset X_0 = \mathbb{P}^n$$

such that each X_k is a Cartier divisor in X_{k-1} in the linear equivalence class of $\mathcal{O}_{\mathbb{P}^n}(d_k)|_{X_{k-1}}$. If X is a quasi-complete-intersection, then every fiber of $U \times_X \mathrm{Fl}_1(X) \to U$ is also a quasi-complete-intersection in $\mathbb{P}^{n-1}$ of type

$$(1, 2, \ldots, d_1, 1, 2, \ldots, d_2, \ldots, 1, 2, \ldots, d_c).$$

Iterating this, every (non-empty) fiber of $(\rho_k^{k-1})^{-1}(U_{k-1}) \to U_{k-1}$ is a quasi-complete-intersection in $\mathbb{P}^{n-k}$ of dimension

$$N_k(n,\underline{d}) = n - k - \sum_{i=1}^{c} \binom{d_i + k - 1}{k}.$$

Since the m^{th} graded piece of the Chern character of T_X equals

$$\text{ch}_m(T_X) = (n + 1 - \sum_{i=1}^{c} d_i^m) c_1(\mathcal{O}(1))^m / m!$$

this agrees with the previous formula for the fiber dimension.

Corollary 2.4. *Let X be a smooth quasi-complete-intersection of type $\underline{d}$. If the integer $N_k(n,\underline{d})$ is nonnegative, there exists an irreducible component I of $Fl_k(X)$ having the expected dimension*

$$dim(I) = \sum_{m=0}^{k} N_m(n,\underline{d}).$$

Proof. Of course we define I to be the closure of any connected component of U_k. The issue is whether or not U_k is empty. By construction U_k is not empty if for every $m = 1, \dots, k$ the morphism ρ_m^{m-1} is surjective. By the argument above every fiber of ρ_m^{m-1} is an iterated intersection in $\mathbb{P}^{n-m}$ of pseudo-divisors (in the sense of [1, Def. 2.2.1]) in the linear equivalence class of an ample divisor. Thus the fiber is nonempty if the number of pseudo-divisors is $\leq n - m$. This follows from the hypothesis that $N_k(n,\underline{d}) \geq 0$. □

3. Proofs

Proof of Proposition 1.3. The first part follows from Corollary 2.4. For the second part, observe that if $N_k(n,d) = -1$, then $N_{k-1}(n,d)$ is nonnegative. Therefore, by the first part, the open subset U_{k-1} from Proposition 2.3 is nonempty. Since $(\rho_k^{k-1})^{-1}(U_{k-1}) \to U_{k-1}$ is smooth of the expected dimension, and since the expected dimension is negative, $(\rho_k^{k-1})^{-1}(U_{k-1})$ is empty. In other words, for every $[\Lambda_{k-1}] \in U_{k-1}$, there exists no k-plane in X containing Λ_{k-1}. □

Proof of Theorem 1.1. Let $(H_{k,n}, e)$ be the universal pair of a scheme $H_{k,n}$ and a closed immersion of $H_{k,n}$-schemes

$$(\text{pr}_H, e) : H_{k,n} \times \mathbb{P}^k \to H_{k,n} \times \mathbb{P}^n$$

whose restriction to each fiber $\{h\} \times \mathbb{P}^k$ is a linear embedding. In other words, $H_{k,n}$ is the open subset of $\mathbb{P}\text{Hom}(\mathbb{C}^{k+1}, \mathbb{C}^{n+1})$ parametrizing injective matrices. Of course there is a natural action of $\mathbf{PGL}_{k+1}$ on $H_{k,n}$, and the quotient is the Grassmannian $\mathbb{G}(k,n)$. Denote by $\widetilde{F}_k(X)$ the inverse image of $F_k(X)$ in $H_{k,n}$, i.e., $\widetilde{F}_k(X)$ parametrizes linear embeddings of $\mathbb{P}^k$ into X.

Let F be a defining equation for the hypersurface X. Then e^*F is a global section of $e^*\mathcal{O}_{\mathbb{P}^n}(d)$. By definition, this is canonically isomorphic to $\text{pr}_{\mathbb{P}^k}^*\mathcal{O}_{\mathbb{P}^k}(d)$. Therefore e^*F determines a regular morphism

$$\widetilde{\Phi} : H_{k,n} \to H^0(\mathbb{P}^k, \mathcal{O}_{\mathbb{P}^k}(d)).$$

Denote by V the open subset of $H_{k,n}$ of points whose fiber dimension equals

$$\dim H_{k,n} - \dim H^0(\mathbb{P}^k, \mathcal{O}_{\mathbb{P}^k}(d)).$$

The rational transformation Φ is dominant if and only if $\widetilde{\Phi}$ is dominant. And the morphism $\widetilde{\Phi}$ is dominant if and only if V is nonempty.

The scheme $\widetilde{F}_k(X)$ is the fiber $\widetilde{\Phi}^{-1}(0)$. If

$$n \geq \binom{d+k-1}{k} + k$$

then Proposition 1.3 implies there exists an irreducible component I of $F_k(X)$ of the expected dimension. Thus the inverse image $\widetilde{I}$ in $H_{k,n}$ is an irreducible component of $\widetilde{F}_k(X)$ of the expected dimension, or what is equivalent, the expected codimension. But the expected codimension is precisely

$$h^0(\mathbb{P}^k, \mathcal{O}_{\mathbb{P}^k}(d)) = \binom{d+k}{k}.$$

Thus, the generic point of $\widetilde{I}$ is contained in V, i.e., V is not empty.

This only leaves the case when

$$n = \binom{d+k-1}{k} + k - 1.$$

The argument is very similar. Let y be a linear coordinate on $\mathbb{P}^k$, and let $\widetilde{G}_k(X)$ be the closed subscheme of $H_{k,d}$ where e^*F is a multiple of y^d. In other words, $\widetilde{G}_k(X)$ parametrizes linear embeddings of $\mathbb{P}^k$ into $\mathbb{P}^n$ whose intersection with X contains $d\mathbb{V}(y)$. There is a projection morphism $\widetilde{G}_k(X) \to F_{k-1}(X)$ associating to the linear embedding the $(k-1)$-plane

$$\Lambda_{k-1} = \text{Image}(\mathbb{V}(y)).$$

Denote by $G_k(X)$ the image of $\widetilde{G}_k(X)$ under the obvious morphism

$$\widetilde{G}_k(X) \to F_{k-1}(\mathbb{P}^n) \times F_k(\mathbb{P}^n).$$

Recall that for a quasi-complete-intersection X, the fiber of $F_1(X) \to X$ is an interated intersection of ample pseudo-divisors in projective space. By a very similar argument, every fiber of $G_k(X) \to F_{k-1}(X)$ is an iterated intersection of ample pseudo-divisors in the projective space $\mathbb{P}^n/\Lambda_{k-1} \cong \mathbb{P}^{n-k}$. Moreover, the fiber of $\text{Fl}_k(X) \to \text{Fl}_{k-1}(X)$ (for any extension of Λ_{k-1} to a flag in $\text{Fl}_{k-1}(X)$) is an ample pseudo-divisor in $G_k(X)$. By the second part of Proposition 1.3, there exists a nonempty open subset $U_{k-1} \subset \Lambda_{k-1}$ such that for every $\Lambda_{k-1} \in U_{k-1}$ this ample pseudo-divisor is empty. Therefore the fiber in $G_k(X)$ is finite or empty. But the equation

$$n = \binom{d+k-1}{k} + k - 1$$

implies the expected dimension of the fiber is 0. Since an intersection of ample pseudo-divisors is nonempty if the expected dimension is nonnegative, the fiber of $G_k(X) \to F_{k-1}(X)$ is not empty and has the expected dimension 0. Since U_{k-1} has the expected dimension, the open set $U_{k-1} \times_{F_{k-1}(X)} \widetilde{G}_k(X)$ is nonempty and has the expected dimension. Thus it has the expected codimension. Therefore a generic point of this nonempty open set is in V, i.e., V is not empty. $\square$

Proof of Corollary 1.4. Let $\mathcal{M}_e$ be an irreducible component of $\overline{\mathrm{M}}_{0,0}(X,e)$ not entirely contained in the boundary Δ. Then for every integer $r \geq 0$ there exists a unique irreducible component $\mathcal{M}_{e,r}$ of $\overline{\mathrm{M}}_{0,r}(X,e)$ whose image in $\overline{\mathrm{M}}_{0,0}(X,e)$ equals $\mathcal{M}_e$. Before defining the irreducible component $\mathcal{M}$ of $\overline{\mathrm{M}}_{0,2}(X,e)$, we will first inductively define an irreducible component $\mathcal{M}_e$ of $\overline{\mathrm{M}}_{0,0}(X,e)$ which is not entirely contained in the boundary Δ and such that the evaluation morphism

$$\mathrm{ev} : \mathcal{M}_{e,1} \to X$$

is surjective. Then we define $\mathcal{M}$ to be $\mathcal{M}_{e,2}$.

Let U denote the open subset of $\overline{\mathrm{M}}_{0,1}(X,1)$ where the evaluation morphism

$$\mathrm{ev} : \overline{\mathrm{M}}_{0,1}(X,1) \to X$$

is smooth, i.e., U parametrizes *free* pointed lines. By [3, 1.1], U contains every general fiber of ev. By the argument in Subsection 2.2 (or any number of other references), a general fiber of ev is connected if $d \leq n-2$. Therefore $U \times_X U$ is irreducible. There is an obvious morphism $U \times_X U \to \overline{\mathrm{M}}_{0,0}(X,2)$. By elementary deformation theory, the morphism is unramified and $\overline{\mathrm{M}}_{0,0}(X,2)$ is smooth at every point of the image. Therefore there is a unique irreducible component $\mathcal{M}_2$ of $\overline{\mathrm{M}}_{0,0}(X,2)$ containing the image of $U \times_X U$. Because $U \to X$ is dominant, $\mathcal{M}_2 \to X$ is also dominant.

By way of induction assume $e \geq 3$ and $\mathcal{M}_{e-1}$ is given. Form the fiber product $\mathcal{M}_{e-1,1} \times_X U$. As above this is irreducible, and there is an unramified morphism

$$\mathcal{M}_{e-1,1} \times_X U \to \overline{\mathrm{M}}_{0,0}(X,e)$$

whose image is in the smooth locus. Therefore there exists a unique irreducible component $\mathcal{M}_e$ of $\overline{\mathrm{M}}_{0,0}(X,e)$ containing the image of $\mathcal{M}_{e-1,1} \times_X U$. Because $\mathcal{M}_{e-1,1} \to X$ is dominant, $\mathcal{M}_{e,1} \to X$ is also dominant. This finishes the inductive construction of the irreducible components $\mathcal{M}_e$, and thus also of $\mathcal{M}_{e,2}$.

It remains to prove that

$$\mathrm{ev} : \mathcal{M}_{e,2} \to X \times X$$

is dominant with rationally connected generic fiber. The article [4] gives an inductive argument for proving this. To carry out the induction, one needs two results: the base of the induction and an important component of the induction argument. Set k to be d^2. For a general degree d hypersurface Y in $\mathbb{P}^k$, [4, Prop. 4.6, Prop. 10.1] prove the two results for Y. By Theorem 1.1, since

$$n \geq \binom{d+k-1}{k} + k - 1,$$

for a general $\mathbb{P}^k \subset \mathbb{P}^n$ the intersection $Y = \mathbb{P}^k \cap X$ is a general degree d hypersurface in $\mathbb{P}^k$. Thus the two results hold for Y. As is clear from the proofs of [4, Prop. 4.6, Prop. 10.1], the results for Y imply the corresponding results for X. □

References

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Department of Mathematics, Massachusetts Institute of Technology, Cambridge MA 02139

E-mail address: jstarr@math.mit.edu