DIVISOR CLASSES AND THE VIRTUAL CANONICAL BUNDLE FOR GENUS 0 MAPS

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Abstract. We prove divisor class relations for families of genus 0 curves and used them to compute the divisor class of the "virtual" canonical bundle of the Kontsevich space of genus 0 maps to a smooth target. This agrees with the canonical bundle in good cases. This work generalizes Pandharipande's results in the special case that the target is projective space, [7], [8]. Our method is completely different from Pandharipande's.

1. Statement of results

Much geometry of a higher-dimensional complex variety X is captured by the rational curves in X. For uniruled and rationally connected varieties the parameter spaces for rational curves in X are also interesting. These parameter spaces are rarely compact, but there are natural compacitifications: the Chow variety, the Hilbert scheme and the Kontsevich moduli space. Of these, the most manageable is the Kontsevich space. Briefly $\overline{\mathcal{M}}_{0,r}(X,\beta)$ parametrizes data $(C, p_1, \dots, p_r, f)$ of a proper, connected, at-worst-nodal, arithmetic genus 0 curve C, a collection $p_1, \dots, p_r$ of distinct, smooth points of C, and a morphism $f : C \to X$ with $f_*[C] = \beta$ and satisfying a natural stability condition.

When $\overline{\mathcal{M}}_{0,r}(X,\beta)$ is irreducible and reduced and when the dimension equals the expected, or *virtual*, dimension, one can ask more refined questions about the geometry of $\overline{\mathcal{M}}_{0,r}(X,\beta)$. For instance, what is its Kodaira dimension? The first step in answering this and other questions is understanding the canonical bundle.

In this article we give a formula for the *virtual canonical bundle* of $\overline{\mathcal{M}}_{0,r}(X,\beta)$. The virtual canonical bundle is a naturally defined line bundle which equals the actual canonical bundle if $\overline{\mathcal{M}}_{0,r}(X,\beta)$ is irreducible and reduced and its dimension equals the virtual dimension.

Theorem 1.1. *Assume that $e := \langle C_1(T_X), \beta\rangle \neq 0$. For $\overline{\mathcal{M}}_{0,0}(X,\beta)$, the virtual canonical bundle equals*

$$\frac{1}{2e}[2e\pi_* f^* C_2(T_X) - (e+1)\pi_* f^* C_1(T_X)^2 +$$
$$\sum_{\{\beta',\beta''\},\beta'+\beta''=\beta} (\langle f^*C_1(T_X), \beta'\rangle\langle f^*C_1(T_X),\beta''\rangle - 4e)\Delta_{\beta',\beta''}].$$

For $\overline{\mathcal{M}}_{0,1}(X,\beta)$, the virtual canonical bundle equals

$$\frac{1}{2e}[2e\pi_* f^* C_2(T_X) - (e+1)\pi_* f^* C_1(T_X)^2 +$$

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$$\sum_{\{\beta',\beta''\},\beta'+\beta''=\beta} (\langle f^*C_1(T_X),\beta'\rangle\langle f^*C_1(T_X),\beta''\rangle - 4e)\Delta_{\beta',\beta''}] + \psi_1.$$

Finally, for $r \geq 2$, the virtual canonical bundle of $\overline{\mathcal{M}}_{0,r}(X,\beta)$ equals

$$\frac{1}{2e}[2e\pi_*f^*C_2(T_X) - (e+1)\pi_*f^*C_1(T_X)^2 +$$

$$\sum_{\{\beta',\beta''\},\beta'+\beta''=\beta} (\langle f^*C_1(T_X),\beta'\rangle\langle f^*C_1(T_X),\beta''\rangle - 4e)\Delta_{\beta',\beta''}] +$$

$$\frac{1}{r-1}\sum_{(A,B),1\in A} \#B(r-\#B)\Delta_{(A,B)}.$$

In order to prove these formulas, we need to prove some divisor class relations for families of genus 0 curves. These relations are of some independent interest.

Proposition 1.2. *Let $\pi : \mathcal{C} \to M$ be a proper, flat family of connected, at-worst-nodal, arithmetic genus 0 curves over a quasi-projective variety M or over a Deligne-Mumford stack M with quasi-projective coarse moduli space. Let D be a $\mathbb{Q}$-Cartier divisor class on $\mathcal{C}$.*

(i) *There is an equality of $\mathbb{Q}$-divisor classes on M*

$$\pi_*(D\cdot D) + \langle D,\beta\rangle\pi_*(D\cdot C_1(\omega_\pi)) = \sum_{\{\beta',\beta''\},\beta'+\beta''=\beta} \langle D,\beta'\rangle\langle D,\beta''\rangle\Delta_{\beta',\beta''}.$$

(ii) *Also, there is an equality of $\mathbb{Q}$-divisor classes on $\mathcal{C}$*

$$2\langle D,\beta\rangle D - \pi^*\pi_*(D\cdot D) + \langle D,\beta\rangle^2 C_1(\omega_\pi) = \sum_{(\beta',\beta'')} \langle D,\beta''\rangle^2\widetilde{\Delta}_{\beta',\beta''}.$$

The notation Δ indicates a boundary divisor. The precise meaning of each term is given in Section 2. Roughly, $\Delta_{(A,B)}$ is the divisor in M parametrizing reducible curves where the marked points indexed by A lie in one component, and the marked points indexed by B lie in the other component. The divisor $\Delta_{\beta',\beta''}$ in M parametrizes reducible curves with one component of class β' and one component of class β''. And the divisor $\widetilde{\Delta}_{\beta',\beta''}$ in $\mathcal{C}$ is the closed subset of $\pi^{-1}(\Delta_{\beta',\beta''})$ which is the union of all components of fibers with class β'. The bundle ω_π on $\mathcal{C}$ is the relative dualizing sheaf of π. Finally the bundle ψ_i on M is the pullback of ω_π by the "i^{th} marked point" section.

The *virtual canonical bundle* is the determinant of the Behrend-Fantechi perfect obstruction theory for $\overline{\mathcal{M}}_{0,r}(X,\beta)$, cf. [2]. This is a complex $E^\bullet$ of $\mathcal{O}$-modules on $\overline{\mathcal{M}}_{0,r}(X,\beta)$ which is perfect of amplitude $[-1,0]$, together with a map to the cotangent complex

$$\phi : E^\bullet \to L^\bullet_{\overline{\mathcal{M}}_{0,r}(X,\beta)}$$

such that $h^0(\phi)$ is an isomorphism and $h^{-1}(\phi)$ is surjective. The determinant of a perfect complex is an invertible sheaf that, roughly, is the alternating tensor product of the determinants of the terms of the complex, cf. [5].

If the dimension of $\overline{\mathcal{M}}_{0,r}(X,\beta)$ equals the virtual dimension

$$\langle c_1(T_X),\beta\rangle + \dim(X) + r - 3$$

and if $\overline{\mathcal{M}}_{0,r}(X,\beta)$ is reduced at all codimension 1 points, then the virtual canonical bundle equals the canonical bundle. Even when they differ, the virtual canonical bundle can be quite useful. For instance, there is a lower bound for the dimension of the space of maps from a curve C into a smooth projective variety Y mapping an effective divisor $B \subset C$ to specified points of Y

$$\dim_{[f]}\mathrm{Hom}(C,Y;f|_B) \geq \langle -K_Y, f_*[C]\rangle + \dim(Y)(1-p_a(C)-\deg(B))$$

cf. [6, Theorem II.1.2, Theorem II.1.7]. An analogous results holds for $Y = \overline{\mathcal{M}}_{0,r}(X,\beta)$ when K_Y is replaced by the virtual canonical bundle and $\dim(Y)$ is replaced by the virtual dimension, cf. [4].

Proposition 1.3. [4, Lemma 2.2] *Let C be a projective Cohen-Macaulay curve, let $B \subset C$ be a divisor along which C is smooth, and let $f : C \to \overline{\mathcal{M}}_{0,r}(X,\beta)$ be a 1-morphism. Assume that every generic point of C parametrizes a smooth, free curve in X. Then for $Y = \overline{\mathcal{M}}_{0,r}(X,\beta)$*

$$dim_{[f]}Hom(C,Y;f|_B) \geq \langle -K_Y^{virt}, f_*[C]\rangle + dim^{virt}(Y)(1-p_a(C)-deg(B)).$$

1.1. **Outline of the article.** There is a universal family of stable maps over $\overline{\mathcal{M}}_{0,r}(X,\beta)$

$$(\pi : \mathcal{C} \to \overline{\mathcal{M}}_{0,r}(X,\beta), \sigma_1 : \overline{\mathcal{M}}_{0,r}(X,\beta) \to \mathcal{C}, \dots, \sigma_r : \overline{\mathcal{M}}_{0,r}(X,\beta), f : \mathcal{C} \to X).$$

The Behrend-Fantechi obstruction theory is defined in terms of total derived pushforwards under π of the relative cotangent sheaf of π and the pullback under f of the cotangent bundle of X. Thus the Grothendieck-Riemann-Roch theorem gives a formula for the virtual canonical bundle. Unfortunately it is not a very useful formula. For instance, using this formula it is difficult to determine whether the virtual canonical bundle is NEF, ample, etc. But combined with Proposition 1.2, Grothendieck-Riemann-Roch gives the formula from Theorem 1.1. The main work in this article is proving Proposition 1.2.

The proof reduces to local computations for the universal family over the Artin stack of all prestable curves of genus 0, cf. Section 4. Because of this, most results are stated for Artin stacks. This leads to one *ad hoc* consruction: since there is as yet no theory of cycle class groups for Artin stacks admitting Chern classes for all perfect complexes of bounded amplitude, a Riemann-Roch theorem for all perfect morphisms relatively representable by proper algebraic spaces, and arbitrary pullbacks for all cycles coming from Chern classes, a stand-in Q_π is used, cf. Section 3. (Also by avoiding Riemann-Roch, this allows some relations to be proved "integrally" rather than "modulo torsion").

In the special case $X = \mathbb{P}^n_k$, Pandharipande proved both Theorem 1.1 and Proposition 1.2 in [8] and [7]. Pandharipande's work was certainly our inspiration. But our proofs are completely different, yield a more general virtual canonical bundle formula, and hold modulo torsion (and sometimes "integrally") rather than modulo numerical equivalence.

2. Notation for moduli spaces and boundary divisor classes

Denote by $\mathfrak{M}_{0,0}$ the category whose objects are proper, flat families $\pi : C \to M$ of connected, at-worst-nodal, arithmetic genus 0 curves, and whose morphisms are Cartesian diagrams of such families. This category is a smooth Artin stack over

Spec $\mathbb{Z}$ (with the flat topology), cf. [1]. The subcategory Δ parametrizing families with reducible fibers is a closed substack. With respect to the smooth topology, the pair $(\mathfrak{M}_{0,0}, \Delta)$ is locally isomorphic to an irreducible simple normal crossings divisor in a smooth variety. Denote by Δ' the singular locus in Δ.

Let U_1 be $\mathfrak{M}_{0,0} - \Delta'$ and let U_2 be $\mathfrak{M}_{0,0} - \Delta$. There are smooth atlases for each of these stacks as follows. For U_2, the family $\pi : \mathbb{P}^1_{\mathbb{Z}} \to \text{Spec } \mathbb{Z}$ defines a 1-morphism $\zeta_2 : \text{Spec } \mathbb{Z} \to U_2$. This is a smooth surjective morphism. The 2-fibered product

$$\text{Spec } \mathbb{Z} \times_{\zeta_2, U_2, \zeta_2} \text{Spec } \mathbb{Z} = \text{Aut}(\mathbb{P}^1_{\mathbb{Z}})$$

is the group scheme $\mathbf{PGL}_2$. Thus U_2 is isomorphic to the quotient stack $[\text{Spec } \mathbb{Z}/\mathbf{PGL}_2]$.

There is a similar atlas for U_2. Let $V = \mathbb{Z}\{\mathbf{e}_0, \mathbf{e}_1\}$ be a free module of rank 2. Choose dual coordinates y_0, y_1 for $V^\vee$. Let $\mathbb{P}^1_{\mathbb{Z}} = \mathbb{P}(V)$ be the projective space with homogeneous coordinates y_0, y_1. Let $\mathbb{A}^1_{\mathbb{Z}}$ be the affine space with coordinate x. Denote by $Z \subset \mathbb{A}^1_{\mathbb{Z}} \times \mathbb{P}^1_{\mathbb{Z}}$ the closed subscheme $\mathbb{V}(x, y_1)$, i.e., the image of the section $(0, [1, 0])$. Let $\nu : C \to \mathbb{A}^1_{\mathbb{Z}} \times \mathbb{P}^1_{\mathbb{Z}}$ be the blowing-up along Z. Denote by $E \subset C$ the exceptional divisor. Define $\pi : C \to \mathbb{A}^1_{\mathbb{Z}}$ to be $\text{pr}_{\mathring{A}^1} \circ \nu$. This family defines a 1-morphism $\zeta_1 : \mathbb{A}^1_{\mathbb{Z}} \to U_1$. This is a smooth surjective morphism. The 2-fibered product

$$\mathbb{A}^1_{\mathbb{Z}} \times_{\zeta_1, U_1, \zeta_1} \mathbb{A}^1_{\mathbb{Z}} = \text{Isom}_{\mathbb{A}^2_{\mathbb{Z}}}(\text{pr}_1^* C, \text{pr}_2^* C)$$

is equivalent to a group scheme G over $\mathbb{A}^1_{\mathbb{Z}}$ whose restriction to $U = \mathbb{A}^1_{\mathbb{Z}} - \{0\}$ is $U \times \mathbf{G}_m \times \mathbf{PGL}_2$ and whose restriction to $\{0\} \cong \text{Spec } \mathbb{Z}$ is the wreath product $G_0 := (B \times B) \rtimes \mathfrak{S}_2$. Here $B \subset \mathbf{PGL}_2$ is a Borel subgroup, i.e., the stabilizer of a point in $\mathbb{P}^1$, and $\mathfrak{S}_2$ acts by interchanging the two components of C_0. Altogether U_1 is isomorphic to the quotient stack $[\mathbb{A}^1_{\mathbb{Z}}/G]$.

Denote by $\pi : \mathfrak{C} \to \mathfrak{M}_{0,0}$ the universal family. This is a proper, flat 1-morphism of Artin stacks, representable by algebraic spaces. Denote by Pic_π the stack parametrizing proper, flat families of connected, at-worst-nodal, arithmetic genus 0 curves together with a section of the relative Picard functor of the family. This is also an Artin stack, and the natural 1-morphism $\text{Pic}_\pi \to \mathfrak{M}_{0,0}$ is representable by (highly nonseparated) étale group schemes, cf. [9, Prop. 9.3.1].

In fact Pic_π has a covering by open substacks, each of which maps isomorphically to an open substack in $\mathfrak{M}_{0,0}$. The combinatorics of how these open pieces are glued to form Pic_π is straightforward. In particular, the inverse image in Pic_π of U_2 is canonically isomorphic to $U_2 \times \mathbb{Z}$.

The intersection $U_1 \cap \Delta$ parametrizes nodal, arithmetic genus 0 curves with two components, $C = C' \cup C''$. For every pair of integers e', e'' there is a unique invertible sheaf L on C whose restriction to C', resp. C'', has degree e', resp. e''. The rule $[C] \mapsto [(C, L)]$ defines a 1-morphism $U_1 \cap \Delta \to \text{Pic}_\pi$ representable by open immersions. Denote by $\Delta_{(e', e'')}$ the closure in Pic_π of the image of this 1-morphism. This is a Cartier divisor in Pic_π. The union over all (e', e'') of the Cartier divisors $\Delta_{(e', e'')}$ is precisely the inverse image in Pic_π of Δ.

More generally, for every integer $r \geq 0$, denote by $\Pi^r(\text{Pic}_\pi)$ the r-fold 2-fibered product of Pic_π with itself over $\mathfrak{M}_{0,0}$. Equivalently, $\Pi^r(\text{Pic}_\pi)$ is the stack of families of genus 0 curves together with r sections of the relative Picard functor of the family.

Let $(e_1', e_1'', \dots, e_r', e_r'')$ be an ordered $2r$-tuple of integers. Define $\Delta_{(e_1',e_1'',\dots,e_r',e_r'')}$ to be the iterated 2-fiber product

$$\Delta_{(e_1',e_1'',\dots,e_r',e_r'')} := \Delta_{(e_1',e_1'')} \times_\Delta \cdots \times_\Delta \Delta_{(e_r',e_r'')}.$$

This is a Cartier divisor in $\Pi^r(\mathrm{Pic}_\pi)$.

Let $\pi : C \to M$ be a flat 1-morphism representable by proper algebraic spaces whose geometric fibers are connected, at-worst-nodal curves of arithmetic genus 0. This defines a 1-morphism $\xi_0 : M \to \mathfrak{M}_{0,0}$. Let $D_1, \dots, D_r$ be Cartier divisor classes on C. This defines a 1-morphism $\xi : M \to \Pi^r(\mathrm{Pic}_\pi)$ lifting ξ_0. Let $f(e_1', e_1'', \dots, e_r', e_r'')$ be a function on $\mathbb{Z}^{2r}$ with values in $\mathbb{Z}$, resp. $\mathbb{Q}$, etc.

Notation 2.1. Denote by

$$\sum_{(\beta',\beta'')} f(\langle D_1, \beta'\rangle, \langle D_1, \beta''\rangle, \dots, \langle D_r, \beta'\rangle, \langle D_r, \beta''\rangle)\Delta_{\beta',\beta''}$$

the Cartier divisor class, resp. $\mathbb{Q}$-Cartier divisor class, etc., that is the pullback by ξ of the Cartier divisor class, etc.,

$$\sum_{(e_1',e_1'',\dots,e_r',e_r'')} f(e_1', e_1'', \dots, e_r', e_r'')\Delta_{(e_1',e_1'',\dots,e_r',e_r'')},$$

the summation over all sequences $(e_1', e_1'', \dots, e_r', e_r'')$. If $f(e_1', e_1'', \dots, e_r', e_r'') = f(e_1'', e_1', \dots, e_r'', e_r')$, denote by,

$$\sum_{(\beta',\beta'')}{}' f(\langle D_1, \beta'\rangle, \langle D_1, \beta''\rangle, \dots, \langle D_r, \beta'\rangle, \langle D_r, \beta''\rangle)\Delta_{\beta',\beta''}$$

the pullback by ξ of,

$$\sum_{(e_1',e_1'',\dots,e_r',e_r'')}{}' f(e_1', e_1'', \dots, e_r', e_r'')\Delta_{(e_1',e_1'',\dots,e_r',e_r'')},$$

where the summation is over equivalence classes of sequences $(e_1', e_1'', \dots, e_r', e_r'')$ such that $e_i' + e_i'' = e_i$ under the equivalence relation $(e_1', e_1'', \dots, e_r', e_r'') \sim (e_1'', e_1', \dots, e_r'', e_r')$.

Of course these are all infinite sums, and so appear not to give well-defined Cartier divisors. However, every quasi-compact open subset of $\Pi^r(\mathrm{Pic}_\pi)$ intersects only finitely many of the divisors. On this subset, this is a finite sum and thus well-defined. Since $\Pi^r(\mathrm{Pic}_\pi)$ has a covering by quasi-compact open subsets, the Cartier divisor is defined by gluing.

Example 2.2. Let $n \geq 0$ be an integer and let (A, B) be a partition of $\{1, \dots, n\}$. For the universal family over $\mathfrak{M}_{0,n}$, denote by $s_1, \dots, s_n$ the universal sections. Then

$$\sum_{\beta',\beta''} \prod_{i\in A} \langle \mathrm{Image}(s_i), \beta'\rangle \cdot \prod_{j\in B} \langle \mathrm{Image}(s_j), \beta''\rangle \Delta_{\beta',\beta''}$$

is the Cartier divisor class of the boundary divisor $\Delta_{(A,B)}$.

3. The functor Q_π

Let M be an Artin stack, and let $\pi : C \to M$ be a flat 1-morphism, relatively representable by proper algebraic spaces whose geometric fibers are connected, at-worst-nodal curves of arithmetic genus 0. There exists an invertible dualizing sheaf ω_π, and the relative trace map, $\mathrm{Tr}_\pi : R\pi_*\omega_\pi[1] \to \mathcal{O}_M$, is a quasi-isomorphism. In particular, $\mathrm{Ext}^1_{\mathcal{O}_C}(\omega_\pi, \mathcal{O}_C)$ is canonically isomorphic to $H^0(M, \mathcal{O}_M)$. Therefore $1 \in H^0(M, \mathcal{O}_M)$ determines an extension class, i.e., a short exact sequence,

$$0 \longrightarrow \omega_\pi \longrightarrow E_\pi \longrightarrow \mathcal{O}_C \longrightarrow 0.$$

The morphism π is perfect, so for every complex $F^\bullet$ perfect of bounded amplitude on C, $R\pi_*F^\bullet$ is a perfect complex of bounded amplitude on M. By [5], the determinant of a perfect complex of bounded amplitude is defined.

Definition 3.1. For every complex $F^\bullet$ perfect of bounded amplitude on C, define $Q_\pi(F^\bullet) = \det(R\pi_*E_\pi \otimes F^\bullet)$.

There is another interpretation of $Q_\pi(F^\bullet)$.

Lemma 3.2. *For every complex $F^\bullet$ perfect of bounded amplitude on C,*

$$Q_\pi(F^\bullet) \cong det(R\pi_*(F^\bullet)) \otimes det(R\pi_*((F^\bullet)^\vee))^\vee.$$

Proof. By the short exact sequence for E_π, $Q_\pi(F^\bullet) \cong \det(R\pi_*(F^\bullet))\otimes\det(R\pi_*(\omega_\pi\otimes F^\bullet))$. The lemma follows by duality. □

It is straightforward to compute $F^\bullet$ whenever there exist cycle class groups for C and M such that Chern classes are defined for all perfect complexes of bounded amplitude and such that Grothendieck-Riemann-Roch holds for π.

Lemma 3.3. *If there exist cycle class groups for C and M such that Chern classes exist for all perfect complexes of bounded amplitude and such that Grothendieck-Riemann-Roch holds for π, then modulo 2-power torsion, the first Chern class of $Q_\pi(F^\bullet)$ is $\pi_*(C_1(F^\bullet)^2 - 2C_2(F^\bullet))$.*

Proof. Denote the Todd class of π by $\tau = 1+\tau_1+\tau_2+\dots$. Of course $\tau_1 = -C_1(\omega_\pi)$. By GRR, $\mathrm{ch}(R\pi_*\mathcal{O}_C) = \pi_*(\tau)$. The canonical map $\mathcal{O}_M \to R\pi_*\mathcal{O}_C$ is a quasi-isomorphism. Therefore $\pi_*(\tau_2) = 0$, modulo 2-power torsion. By additivity of the Chern character, $\mathrm{ch}(E_\pi) = 2 + C_1(\omega_\pi) + \frac{1}{2}C_1(\omega_\pi)^2 + \dots$. Therefore,

$$\mathrm{ch}(E_\pi)\cdot\tau = 2 + 2\tau_2 + \dots$$

So for any complex $F^\bullet$ perfect of bounded amplitude,

$$\mathrm{ch}(E_\pi \otimes F^\bullet)\cdot\tau = \mathrm{ch}(F^\bullet)\cdot\mathrm{ch}(E_\pi)\cdot\tau = \\ (\mathrm{rk}(F^\bullet) + C_1(F^\bullet) + \tfrac{1}{2}(C_1(F^\bullet)^2 - 2C_2(F^\bullet)) + \dots)(2 + 2\tau_2 + \dots).$$

Applying π_* gives,

$$2\pi_*(C_1(F^\bullet)) + \pi_*(C_1(F^\bullet)^2 - 2C_2(F^\bullet)) + \dots$$

Therefore the first Chern class of $\det(R\pi_*(E_\pi \otimes F^\bullet))$ is $\pi_*(C_1(F^\bullet)^2 - 2C_2(F^\bullet))$, modulo 2-power torsion. □

Remark 3.4. The point is this. In every reasonable case, Q_π is just $\pi_*(C_1^2 - 2C_2)$. Moreover Q_π is compatible with base-change by arbitrary 1-morphisms. This allows to reduce certain computations to the Artin stack of all genus 0 curves. As far as we are aware, no one has written a definition of cycle class groups for all locally finitely presented Artin stacks that has Chern classes for all perfect complexes of bounded amplitude, has pushforward maps and Grothendieck-Riemann-Roch for perfect 1-morphisms representable by proper algebraic spaces, and has pullback maps by arbitrary 1-morphisms for cycles coming from Chern classes. Doubtless such a theory exists; whatever it is, $Q_\pi = \pi_*(C_1^2 - 2C_2)$.

Let the following diagram be 2-Cartesian,

$$\begin{array}{ccc} C' & \xrightarrow{\zeta_C} & C \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ M' & \xrightarrow{\zeta_M} & M \end{array}$$

together with a 2-equivalence $\theta : \pi \circ \zeta_C \Rightarrow \zeta_M \circ \pi'$.

Lemma 3.5. *For every complex $F^\bullet$ perfect of bounded amplitude on C, $\zeta_M^* Q_\pi(F^\bullet)$ is isomorphic to $Q_{\pi'}(\zeta_C^* F^\bullet)$.*

Proof. Of course $\zeta_C^* E_\pi = E_{\pi'}$. And $\zeta_M^* R\pi_*$ is canonically equivalent to $R(\pi')_* \zeta_C^*$ for perfect complexes of bounded amplitude. Therefore $\zeta_M^* Q_\pi(F^\bullet)$ equals $\det(\zeta_M^* R\pi_*(E_\pi \otimes F^\bullet))$ equals $\det(R(\pi')_* \zeta_C^*(E_\pi \otimes F^\bullet))$ equals $\det(R(\pi')_* E_{\pi'} \otimes \zeta_C^* F^\bullet)$ equals $Q_{\pi'}(\zeta_C^* F^\bullet)$. □

Lemma 3.6. *Let L be an invertible sheaf on C of relative degree e over M. For every invertible sheaf L' on M, $Q_\pi(L \otimes \pi^* L') \cong Q_\pi(L) \otimes (L')^{2e}$. In particular, if $e = 0$, $Q_\pi(L \otimes \pi^* L') \cong Q_\pi(L)$.*

Proof. To compute the rank of $R\pi_*(E_\pi \otimes F^\bullet)$ over any connected component of M, it suffices to base-change to the spectrum of a field mapping to that component. Then, by Grothendieck-Riemann-Roch, the rank is $2\deg(C_1(F^\bullet))$. In particular, $R\pi_*(E_\pi \otimes L)$ has rank $2e$.

By the projection formula, $R\pi_*(E_\pi \otimes L \otimes \pi^* L') \cong R\pi_*(E_\pi \otimes L) \otimes L'$. Of course $\det(R\pi_*(E_\pi \otimes L) \otimes L') = Q_\pi(L) \otimes (L')^{\mathrm{rank}}$. This follows from the uniqueness of det: for any invertible sheaf L' the association $F^\bullet \mapsto \det(F^\bullet \otimes L') \otimes (L')^{-\mathrm{rank}(F^\bullet)}$ also satisfies the axioms for a determinant function and is hence canonically isomorphic to $\det(F^\bullet)$. Therefore $Q_\pi(L \otimes \pi^* L') = Q_\pi(L) \otimes (L')^{2e}$. □

4. Local computations

This section contains 2 computations: $Q_\pi(\omega_\pi)$ and $Q_\pi(L)$ for every invertible sheaf on C of relative degree 0. Because of Lemma 3.5 the first computation reduces to the universal case over $\mathfrak{M}_{0,0}$. Because of Lemma 3.5 and Lemma 3.6, the second compuation reduces to the universal case over Pic^0_π.

4.1. **Computation of $Q_\pi(\omega_\pi)$.** Associated to $\pi_C : C \to M$, there is a 1-morphism $\zeta_M : M \to \mathfrak{M}_{0,0}$, a 1-morphism $\zeta_C : C \to \mathcal{C}$, and a 2-equivalence $\theta : \pi_{\mathcal{C}} \circ \zeta_C \Rightarrow \zeta_M \circ \pi_C$ such that the following diagram is 2-Cartesian,

$$\begin{array}{ccc} C & \xrightarrow{\zeta_C} & \mathcal{C} \\ {\scriptstyle \pi_C}\downarrow & & \downarrow{\scriptstyle \pi_{\mathcal{C}}} \\ M & \xrightarrow{\zeta_M} & \mathfrak{M}_{0,0} \end{array}$$

Of course ω_{π_C} is isomorphic to $\zeta_C^*\omega_{\pi_{\mathcal{C}}}$. By Lemma 3.5, $Q_{\pi_C}(\omega_{\pi_C}) \cong \zeta_M^* Q_{\pi_{\mathcal{C}}}(\omega_{\pi_{\mathcal{C}}})$. So the computation of $Q_{\pi_C}(\omega_{\pi_C})$ is reduced to the universal family $\pi : \mathfrak{C} \to \mathfrak{M}_{0,0}$.

Proposition 4.1. (i) *Over the open substack U_1, $\omega_\pi^\vee$ is π-relatively ample.*

(ii) *Over U_1, $R^1\pi_*\omega_\pi^\vee|_{U_1} = (0)$ and $\pi_*\omega_\pi^\vee|_{U_1}$ is locally free of rank 3.*

(iii) *Over U_2, there is a canonical isomorphism $i : det(\pi_*\omega_\pi^\vee|_{U_2}) \to \mathcal{O}_{U_2}$.*

(iv) *The image of $det(\pi_*\omega_\pi^\vee|_{U_1}) \to det(\pi_*\omega_\pi^\vee|_{U_2}) \xrightarrow{i} \mathcal{O}_{U_2}$ is $\mathcal{O}_{U_1}(-\Delta) \subset \mathcal{O}_{U_2}$.*

(v) *Over U_1, $Q_\pi(\omega_\pi)|_{U_1} \cong \mathcal{O}_{U_1}(-\Delta)$. Therefore on all of $\mathfrak{M}_{0,0}$, $Q_\pi(\omega_\pi) \cong \mathcal{O}_{\mathfrak{M}_{0,0}}(-\Delta)$.*

Proof. Recall the 1-morphism $\zeta_1 : \mathbb{A}^1_{\mathbb{Z}} \to U_1$ from Section 2. Because ζ_1 is smooth and surjective, (i) and (ii) can be checked after base-change by ζ_1. Also (iv) will reduce to a computation over $\mathbb{A}^1_{\mathbb{Z}}$ after base-change by ζ_1.

(i) and (ii): Denote by $\mathbb{P}^2_{\mathbb{Z}}$ the projective space with coordinates u_0, u_1, u_2. There is a rational transformation $f : \mathbb{A}^1_{\mathbb{Z}} \times \mathbb{P}^1_{\mathbb{Z}} \dashrightarrow \mathbb{A}^1_{\mathbb{Z}} \times \mathbb{P}^2_{\mathbb{Z}}$ by

$$\begin{array}{rcl} f^*x & = & x, \\ f^*u_0 & = & xy_0^2, \\ f^*u_1 & = & y_0y_1, \\ f^*u_2 & = & y_1^2 \end{array}$$

By local computation, this extends to a morphism $f : C \to \mathbb{A}^1_{\mathbb{Z}} \times \mathbb{P}^2_{\mathbb{Z}}$ that is a closed immersion and whose image is $\mathbb{V}(u_0u_2 - xu_1^2)$. By the adjunction formula, ω_π is the pullback of $\mathcal{O}_{\mathbb{P}^2}(-1)$. In particular, $\omega_\pi^\vee$ is very ample. Moreover, because $H^1(\mathbb{P}^2_{\mathbb{Z}}, \mathcal{O}_{\mathbb{P}^2}(-1)) = (0)$, also $H^1(C, \omega_\pi^\vee) = (0)$. By cohomology and base-change results, $R^1\pi_*(\omega_\pi^\vee) = (0)$ and $\pi_*(\omega_\pi^\vee)$ is locally free of rank 3.

(iii): The curve $\mathbb{P}^1_{\mathbb{Z}} = \mathbb{P}(V)$ determines a morphism $\eta : \text{Spec}\,(\mathbb{Z}) \to U_2$. This is smooth and surjective on geometric points. Moreover it gives a realization of U_2 as the classifying stack of the group scheme $\text{Aut}(\mathbb{P}(V)) = \mathbf{PGL}(V)$. Taking the exterior power of the Euler exact sequence, $\omega_{\mathbb{P}(V)/\mathbb{Z}} = \bigwedge^2(V^\vee) \otimes \mathcal{O}_{\mathbb{P}(V)}(-2)$. Therefore $H^0(\mathbb{P}(V), \omega^\vee_{\mathbb{P}(V)/\mathbb{Z}})$ equals $\bigwedge^2(V) \otimes \text{Sym}^2(V^\vee)$ as a representation of $\mathbf{GL}(V)$. The determinant of this representation is the trivial character of $\mathbf{GL}(V)$. Therefore it is the trivial character of $\mathbf{PGL}(V)$. This gives an isomorphism of $\det(\pi_*\omega_\pi|_{U_2})$ with $\mathcal{O}_{U_2}$.

(iv): This can be checked after pulling back by ζ_1. The pullback of U_2 is $\mathbb{G}_{m,\mathbb{Z}} \subset \mathbb{A}^1_{\mathbb{Z}}$. The pullback of i comes from the determinant of $H^0(\mathbb{G}_{m,\mathbb{Z}} \times \mathbb{P}^1_{\mathbb{Z}}, \omega_\pi^\vee) = \bigwedge^2(V) \otimes \text{Sym}^2(V^\vee) \otimes \mathcal{O}_{\mathbb{G}_m}$. By the adjunction formula, $\omega_{C/\mathbb{A}^1} = \nu^*\omega_{\mathbb{A}^1\times\mathbb{P}^1/\mathbb{A}^1}(E)$. Hence $\nu_*\omega^\vee_{C/\mathbb{A}^1} = I_Z\omega_{\mathbb{A}^1\times\mathbb{P}^1/\mathbb{A}^1}$. Therefore the canonical map,

$$H^0(C, \omega^\vee_{C/\mathbb{A}^1}) \to H^0(\mathbb{A}^1_{\mathbb{Z}} \times \mathbb{P}^1_{\mathbb{Z}}, \omega^\vee_{\mathbb{A}^1\times\mathbb{P}^1/\mathbb{A}^1}),$$

is given by,

$$\begin{aligned}
&\mathcal{O}_{\mathbb{A}^1}\{\mathbf{f}_0,\mathbf{f}_1,\mathbf{f}_2\} \to \bigwedge^2(V)\otimes \mathrm{Sym}^2(V^\vee)\otimes\mathcal{O}_{\mathbb{A}^1},\\
&\mathbf{f}_0 \mapsto x\cdot(\mathbf{e}_0\wedge\mathbf{e}_1)\otimes y_0^2,\\
&\mathbf{f}_1 \mapsto \quad\ (\mathbf{e}_0\wedge\mathbf{e}_1)\otimes y_0y_1,\\
&\mathbf{f}_2 \mapsto \quad\ (\mathbf{e}_0\wedge\mathbf{e}_1)\otimes y_1^2
\end{aligned}$$

It follows that $\det(\pi_*\omega_\pi^\vee)\to\mathcal{O}_{\mathbb{G}_m}$ has image $\langle x\rangle\mathcal{O}_{\mathbb{A}^1}$, i.e., $\zeta_1^*\mathcal{O}_{U_1}(-\Delta)$.

(v): By the short exact sequence for E_π, $Q_\pi(\omega_\pi)=\det(R\pi_*\omega_\pi)\otimes\deg(R\pi_*\omega_\pi^2)$. Because the trace map is a quasi-isomorphism, $\det(R\pi_*\omega_\pi)=\mathcal{O}_{U_1}$. By (ii) and duality,

$$\det(R\pi_*\omega_\pi^2)\cong\det(R^1\pi_*\omega_\pi^2)^vee\cong\det(\pi_*\omega_\pi^\vee).$$

By (iv), this is $\mathcal{O}_{U_1}(-\Delta)$. Therefore $Q_\pi(\omega_\pi)\cong\mathcal{O}_{U_1}(-\Delta)$ on U_1. Because $\mathfrak{M}_{0,0}$ is regular, and because the complement of U_1 has codimension 2, this isomorphism of invertible sheaves extends to all of $\mathfrak{M}_{0,0}$. $\square$

The sheaf of relative differentials Ω_π is a pure coherent sheaf on $\mathcal{C}$ of rank 1, flat over $\mathfrak{M}_{0,0}$ and is quasi-isomorphic to a perfect complex of amplitude $[-1,0]$.

Lemma 4.2. *The perfect complex $R\pi_*\Omega_\pi$ has rank -1 and determinant $\cong\mathcal{O}_{\mathfrak{M}_{0,0}}(-\Delta)$. The perfect complex $R\pi_*RHom_{\mathcal{O}_\mathcal{C}}(\Omega_\pi,\mathcal{O}_\mathcal{C})$ has rank 3 and determinant $\cong\mathcal{O}_{\mathfrak{M}_{0,0}}(-2\Delta)$.*

Proof. There is a canonical injective sheaf homomorphism $\Omega_\pi\to\omega_\pi$ and the support of the cokernel, $Z\subset\mathcal{C}$, is a closed substack that is smooth and such that $\pi:Z\to\mathfrak{M}_{0,0}$ is unramified and is the normalization of Δ. Over U_1, the lemma immediately follows from this and the arguments in the proof of Proposition 4.1. As in that case, it suffices to establish the lemma over U_1. $\square$

4.2. **Computation of $Q_\pi(L)$ for invertible sheaves of degree 0.** Let M be an Artin stack, let $\pi:C\to M$ be a flat 1-morphism, relatively representable by proper algebraic spaces whose geometric fibers are connected, at-worst-nodal curves of arithmetic genus 0. Let L be an invertible sheaf on C of relative degree 0 over M. This determines a 1-morphism to the relative Picard of the universal family over $\mathfrak{M}_{0,0}$,

$$\zeta_M : M\to \mathrm{Pic}^0_\pi.$$

The pullback of the universal family $\mathfrak{C}$ is equivalent to C and the pullback of the universal bundle $\mathcal{O}_\mathfrak{C}(\mathcal{D})$ differs from L by π^*L' for an invertible sheaf L' on M. By Lemma 3.5 and Lemma 3.6, $Q_\pi(L)\cong\zeta_M^*Q_\pi(\mathcal{O}_\mathfrak{C}(\mathcal{D}))$.

Proposition 4.3. *Over Pic^0_π, $\pi_*E_\pi(\mathcal{D})=(0)$ and $R^1\pi_*E_\pi(\mathcal{D})$ is a sheaf supported on the inverse image of Δ. The stalk of $R^1\pi_*E_\pi(\mathcal{D})$ at the generic point of $\Delta_{(a,-a)}$ is a torsion sheaf of length a^2. The filtration by order of vanishing at the generic point has associated graded pieces of length $2a-1,2a-3,\ldots,3,1$.*

Proof. Over the open complement of Δ, the divisor $\mathcal{D}$ is 0. So the first part of the proposition reduces to the statement that $R\pi_*E_\pi$ is quasi-isomorphic to 0. By definition of E_π, there is an exact triangle,

$$R\pi_*E_\pi \longrightarrow R\pi_*\mathcal{O}_\mathcal{C}\xrightarrow{\ \delta\ } R\pi_*\omega_\pi[1]\longrightarrow R\pi_*E_\pi[1].$$

Of course the bundle E_π and the canonical isomorphism $R\pi_*\mathcal{O}_\mathcal{C}\cong\mathcal{O}_\mathfrak{M}$ were defined so that the composition of δ with the trace map, which is a quasi-isomorphism in this case, would be the identity. Therefore δ is a quasi-isomorphism, so $R\pi_* E_\pi$ is quasi-isomorphic to 0.

The second part can be proved (and perhaps only makes sense) after smooth base-change to a scheme. Let $\mathbb{P}^1_s$ be a copy of $\mathbb{P}^1$ with homogeneous coordinates S_0, S_1. Let $\mathbb{P}^1_x$ be a copy of $\mathbb{P}^1$ with homogeneous coordinates X_0, X_1. Let $\mathbb{P}^1_y$ be a copy of $\mathbb{P}^1$ with homogeneous coordinates Y_0, Y_1. Denote by $C \subset \mathbb{P}^1_s \times \mathbb{P}^1_x \times \mathbb{P}^1_y$ the divisor with defining equation $F = S_0X_0Y_0 - S_1X_1Y_1$. The projection $\mathrm{pr}_s : C \to \mathbb{P}^1_s$ is a proper, flat morphism whose geometric fibers are connected, at-worst-nodal curves of arithmetic genus 0. Denote by L the invertible sheaf on C that is the restriction of $\mathrm{pr}_x^*\mathcal{O}_{\mathbb{P}^1_x}(a) \otimes \mathrm{pr}_y^*\mathcal{O}_{\mathbb{P}^1_y}(-a)$. This is an invertible sheaf of relative degree 0. Therefore there is an induced 1-morphism $\zeta : \mathbb{P}^1_s \to \mathfrak{M}_{\mathbb{Z}}(\tau_{0,0}(0))$.

It is straightforward that ζ is smooth, and the image intersects $\Delta_{b,-b}$ iff $b = a$. Moreover, $\zeta^*\Delta_{a,-a}$ is the reduced Cartier divisor $\mathbb{V}(S_0S_1) \subset \mathbb{P}^1_s$. There is an obvious involution $i : \mathbb{P}^1_s \to \mathbb{P}^1_s$ by $i(S_0, S_1) = (S_1, S_0)$, and $\zeta \circ i$ is 2-equivalent to ζ. Therefore the length of the $R^1\mathrm{pr}_{s,*}E_{\mathrm{pr}_s} \otimes L$ is 2 times the length of the stalk of $R^1\pi_*E_\pi(\mathcal{D})$ at the generic point of Δ_a; more precisely, the length of the stalk at each of $(1,0), (0,1) \in \mathbb{P}^1_s$ is the length of the stalk at Δ_a. Similarly for the lengths of the associated graded pieces of the filtration.

Because E_{pr_s} is the extension class of the trace mapping, $R^1\mathrm{pr}_{s,*}E_{\mathrm{pr}_s} \otimes L$ is the cokernel of the $\mathcal{O}_{\mathbb{P}^1_s}$-homomorphisms,

$$\gamma : \mathrm{pr}_{s,*}(L) \to \mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^1_s}}(\mathrm{pr}_{s,*}(L^\vee), \mathcal{O}_{\mathbb{P}^1_s}),$$

induced via adjointness from the multiplication map,

$$\mathrm{pr}_{s,*}(L) \otimes \mathrm{pr}_{s,*}(L^\vee) \to \mathrm{pr}_{s,*}(\mathcal{O}_C) = \mathcal{O}_{\mathbb{P}^1_s}.$$

On $\mathbb{P}^1_s \times \mathbb{P}^1_x \times \mathbb{P}^1_y$ there is a locally free resolution of the push-forward of L, resp. $L^\vee$,

$$0 \to \mathcal{O}_{\mathbb{P}^1_s}(-1) \boxtimes \mathcal{O}_{\mathbb{P}^1_x}(a-1) \boxtimes \mathcal{O}_{\mathbb{P}^1_y}(-a-1) \xrightarrow{F} \mathcal{O}_{\mathbb{P}^1_s}(0) \boxtimes \mathcal{O}_{\mathbb{P}^1_x}(a) \boxtimes \mathcal{O}_{\mathbb{P}^1_y}(-a) \to L \to 0,$$
$$0 \to \mathcal{O}_{\mathbb{P}^1_s}(-1) \boxtimes \mathcal{O}_{\mathbb{P}^1_x}(-a-1) \boxtimes \mathcal{O}_{\mathbb{P}^1_y}(a-1) \xrightarrow{F} \mathcal{O}_{\mathbb{P}^1_s}(0) \boxtimes \mathcal{O}_{\mathbb{P}^1_x}(-a) \boxtimes \mathcal{O}_{\mathbb{P}^1_y}(a) \to L^\vee \to 0$$

Hence $R\mathrm{pr}_{s,*}L$ is the complex,

$$\mathcal{O}_{\mathbb{P}^1_s}(-1) \otimes_k H^0(\mathbb{P}^1_x, \mathcal{O}_{\mathbb{P}^1_x}(a-1)) \otimes_k H^1(\mathbb{P}^1_y, \mathcal{O}_{\mathbb{P}^1_y}(-a-1)) \xrightarrow{F} \mathcal{O}_{\mathbb{P}^1_s} \otimes_k H^0(\mathbb{P}^1_x, \mathcal{O}_{\mathbb{P}^1_x}(a)) \otimes_k H^1(\mathbb{P}^1_y, \mathcal{O}_{\mathbb{P}^1_y}(-a)).$$

A similar result holds for $R\mathrm{pr}_{s,*}L^\vee$. It is possible to write out this map explicitly in terms of bases for H^0 and H^1, but for the main statement just observe the complex has rank 1 and degree $-a^2$. A similar result holds for $R\mathrm{pr}_{s,*}L^\vee$. Therefore $R^1\pi_*E_\pi(L)$ is a torsion sheaf of length $2a^2$. Because it is equivariant for i, the localization at each of $(0,1)$ and $(1,0)$ has length a^2.

The lengths of the associated graded pieces of the filtration by order of vanishing at $\mathbb{V}(S_0S_1)$ can be computed from the complexes for $R\mathrm{pr}_{s,*}L$ and $R\mathrm{pr}_{s,*}L^\vee$. This is left to the reader. $\square$

Corollary 4.4. *In the universal case, $Q_\pi(\mathcal{D}) = -\sum_{a\geq 0} a^2\Delta_a$. Therefore in the general case of $\pi : C \to M$ and an invertible sheaf L of relative degree 0,*

$$Q_\pi(L) = {\sum_{\beta',\beta''}}' \langle C_1(L), \beta'\rangle\langle C_1(L), \beta''\rangle \Delta_{\beta',\beta''}.$$

5. Proof of Proposition 1.2

As usual, let M be an Artin stack and let $\pi : C \to M$ be a flat 1-morphism, relatively representable by proper algebraic spaces whose geometric fibers are connected, at-worst-nodal curves of genus 0.

Hypothesis 5.1. There are cycle class groups for C and M admitting Chern classes for locally free sheaves, and such that Grothendieck-Riemann-Roch holds for π. In particular, this holds if M is a Deligne-Mumford stack whose coarse moduli space is quasi-projective.

Proof of Proposition 1.2(i). Define $D' = 2D + \langle D, \beta\rangle C_1(\omega_\pi)$. This is a Cartier divisor class of relative degree 0. By Corollary 4.4,

$$Q_\pi(D') = {\sum_{\beta',\beta''}}'(\langle 2D, \beta'\rangle - \langle D, \beta\rangle)(\langle 2D, \beta''\rangle - \langle D, \beta\rangle)\Delta_{\beta',\beta''}.$$

By Lemma 3.3 this is,

$$4\pi_*(D \cdot D) + 4\langle D, \beta\rangle\pi_*(D \cdot C_1(\omega_\pi) + (\langle D, \beta\rangle)^2 Q_\pi(C_1(\omega_\pi)) = \\ {\sum_{\beta',\beta''}}'(4\langle D, \beta'\rangle\langle D, \beta''\rangle - (\langle D, \beta\rangle)^2)\Delta_{\beta',\beta''}.$$

By Proposition 4.1, $Q_\pi(\omega_\pi) = -{\sum_{\beta',\beta''}}'\Delta_{\beta',\beta''}$. Substituting this into the equation, simplifying, and dividing by 4 gives the relation. □

Lemma 5.2. *For every pair of Cartier divisor classes on C, D_1, D_2, of relative degrees $\langle D_1, \beta\rangle$, resp. $\langle D_2, \beta\rangle$, modulo 2-power torsion,*

$$2\pi_*(D_1 \cdot D_2) + \langle D_1, \beta\rangle\pi_*(D_2 \cdot C_1(\omega_\pi)) + \langle D_2, \beta\rangle\pi_*(D_1 \cdot C_1(\omega_\pi)) =$$

$${\sum_{\beta',\beta''}}'(\langle D_1, \beta'\rangle\langle D_2, \beta''\rangle + \langle D_2, \beta'\rangle\langle D_1, \beta''\rangle)\Delta_{\beta',\beta''}.$$

Proof. This follows from Proposition 1.2(i) and the polarization identity for quadratic forms. □

Lemma 5.3. *For every section of π, $s : M \to C$, whose image is contained in the smooth locus of π,*

$$s(M) \cdot s(M) + s(M) \cdot C_1(\omega_\pi).$$

Proof. This follows by adjunction since the relative dualizing sheaf of $s(M) \to M$ is trivial. □

Lemma 5.4. *For every section of π, $s : M \to C$, whose image is contained in the smooth locus of π and for every Cartier divisor class D on C of relative degree $\langle D, \beta\rangle$ over M, modulo 2-power torsion,*

$$2\langle D, \beta\rangle s^*D - \pi_*(D \cdot D) - \langle D, \beta\rangle^2\pi_*(s(M) \cdot s(M)) =$$

$${\sum_{\beta',\beta''}}'(\langle D, \beta'\rangle^2\langle s(M), \beta''\rangle + \langle D, \beta''\rangle^2\langle s(M), \beta'\rangle)\Delta_{\beta',\beta''}.$$

Proof. By Proposition 1.2(i),

$$2s^*D + \pi_*(D \cdot C_1(\omega_\pi)) + \langle D, \beta\rangle\pi_*(s(M) \cdot C_1(\omega_\pi)) =$$

$${\sum}'(\langle D, \beta'\rangle\langle s(M), \beta''\rangle + \langle D, \beta''\rangle\langle s(M), \beta'\rangle)\Delta_{\beta',\beta''}.$$

Multiplying both sides by $\langle D,\beta\rangle$,

$$2\langle D,\beta\rangle s^*D + \langle D,\beta\rangle \pi_*(D\cdot C_1(\omega_\pi)) + \langle D,\beta\rangle^2 \pi_*(s(M)\cdot C_1(\omega_\pi)) =$$

$$\sum{}'(\langle D,\beta\rangle\langle D,\beta'\rangle\langle s(M),\beta''\rangle + \langle D,\beta\rangle\langle D,\beta''\rangle\langle s(M),\beta'\rangle)\Delta_{\beta',\beta''}.$$

First of all, by Lemma 5.4, $\langle D,\beta\rangle^2\pi_*(s(M)\cdot C_1(\omega_\pi)) = -\langle D,\beta\rangle^2\pi_*(s(M)\cdot s(M))$. Next, by Proposition 1.2(i),

$$\langle D,\beta\rangle\pi_*(D\cdot C_1(\omega_\pi)) = -\pi_*(D\cdot D) + \sum{}'\langle D,\beta'\rangle\langle D,\beta''\Delta_{\beta',\beta''}.$$

Finally,

$$\langle D,\beta\rangle\langle D,\beta'\rangle\langle s(M),\beta''\rangle + \langle D,\beta\rangle\langle D,\beta''\rangle\langle s(M),\beta'\rangle =$$

$$(\langle D,\beta'\rangle + \langle D,\beta''\rangle)\langle D,\beta'\rangle\langle s(M),\beta''\rangle + (\langle D,\beta'\rangle + \langle D,\beta''\rangle)\langle D,\beta''\rangle\langle s(M),\beta'\rangle =$$

$$\langle D,\beta'\rangle^2\langle s(M),\beta''\rangle + \langle D,\beta''\rangle^2\langle s(M),\beta'\rangle + \langle D,\beta'\rangle\langle D,\beta''\rangle(\langle s(M),\beta'\rangle + \langle s(M),\beta''\rangle) =$$

$$\langle D,\beta'\rangle^2\langle s(M),\beta''\rangle + \langle D,\beta''\rangle^2\langle s(M),\beta'\rangle + \langle D,\beta'\rangle\langle D,\beta''\rangle.$$

Plugging in these 3 identities and simplifying gives the relation. □

Proof of Proposition 1.2(ii). Let $\pi : \mathfrak{C} \to \mathfrak{M}_{0,0}$ denote the universal family. Let $\mathfrak{C}_{\text{smooth}}$ denote the smooth locus of π. The 2-fibered product $\text{pr}_1 : \mathfrak{C}_{\text{smooth}} \times_{\mathfrak{M}_{0,0}} \mathfrak{C} \to \mathfrak{C}_{\text{smooth}}$ together with the diagonal $\Delta : \mathfrak{C}_{\text{smooth}} \to \mathfrak{C}_{\text{smooth}} \times_{\mathfrak{M}_{0,0}} \mathfrak{C}$ determine a 1-morphism $\mathfrak{C}_{\text{smooth}} \to \mathfrak{M}_{1,0}$. This extends to a 1-morphism $\mathfrak{C} \to \mathfrak{M}_{1,0}$. The pullback of the universal curve is a 1-morphism $\pi' : \mathfrak{C}' \to \mathfrak{C}$ that factors through $\text{pr}_1 : \mathfrak{C} \times_{\mathfrak{M}_{0,0}} \mathfrak{C} \to \mathfrak{C}$. Denote the pullback of the universal section by $s : \mathfrak{C} \to \mathfrak{C}'$. Now $\mathfrak{C}$ is regular, and the complement of $\mathfrak{C}_{\text{smooth}}$ has codimension 2. Therefore $s^*\mathcal{O}_{\mathfrak{C}'}(s(\mathfrak{C}))$ can be computed on $\mathfrak{C}_{\text{smooth}}$. But the restriction to $\mathfrak{C}_{\text{smooth}}$ is clearly $\omega_\pi^\vee$. Therefore $s^*\mathcal{O}_{\mathfrak{C}'}(s(\mathfrak{C})) \cong \omega_\pi^\vee$ on all of $\mathfrak{C}$.

Pulling this back by $\zeta_C : C \to \mathfrak{C}$ gives a 1-morphism $\pi' : C' \to C$ that factors through $\text{pr}_1 : C \times_M C \to C$. Let D be a Cartier divisor class on C and consider the pullback to C' of pr_2^*D on $C \times_M C$. This is a Cartier divisor class D' on C'. Of course $s^*D' = D$. Moreover, by the projection formula the pushforward to $C \times_M C$ of $D'\cdot D'$ is $\text{pr}_2^*(D\cdot D)$. Therefore $(\pi')_*(D'\cdot D')$ is $(\text{pr}_1)_*\text{pr}_2^*(D\cdot D)$, i.e., $\pi^*\pi_*(D\cdot D)$. Finally, denote by,

$$\sum_{\beta',\beta''}\langle D,\beta''\rangle^2\widetilde{\Delta}_{\beta',\beta''},$$

the divisor class on C,

$$\sum_{\beta',\beta''}{}'(\langle D,\beta''\rangle^2\langle s,\beta'\rangle + \langle D,\beta'\rangle^2\langle s,\beta''\rangle)\Delta_{\beta',\beta''}.$$

The point is this: if π is smooth over every generic point of M, then the divisor class $\widetilde{\Delta}_{\beta',\beta''}$ is the irreducible component of $\pi^{-1}(\Delta_{\beta',\beta''})$ corresponding to the vertex v', i.e., the irreducible component with "curve class" β'. Therefore Proposition 1.2(ii) follows from Lemma 5.4. □

Remark 5.5. If $\langle D,\beta\rangle \neq 0$ then, at least up to torsion, Proposition 1.2(i) follows from Proposition 1.2(ii) by intersecting both sides of the relation by D and then applying π_*. This was pointed out by Pandharipande, who also proved Lemma 5.4 up to numerical equivalence in [8, Lem. 2.2.2] (by a very different method).

Lemma 5.6. *Let $s, s' : M \to C$ be sections with image in the smooth locus of π such that $s(M)$ and $s'(M)$ are disjoint. Then,*

$$\pi_*(s(M) \cdot s(M)) + \pi_*(s'(M) \cdot s'(M)) = -\sum_{\beta',\beta''} \langle s(M), \beta'\rangle\langle s'(M), \beta''\rangle\Delta_{\beta',\beta''}.$$

Proof. Apply Lemma 5.2 and use $s(M) \cdot s'(M) = 0$ and Lemma 5.3. □

Lemma 5.7. *Let $r \geq 2$ and $s_1, \ldots, s_r : M \to C$ be sections with image in the smooth locus of π and which are pairwise disjoint. Then,*

$$-\sum_{i=1}^{r} \pi_*(s_i(M)\cdot s_i(M)) = (r-2)\pi_*(s_1(M)\cdot s_1(M)) + \sum_{\beta',\beta''} \langle s_1(M), \beta'\rangle\langle s_2(M)+\cdots+s_r(M), \beta''\rangle\Delta_{\beta',\beta''}.$$

Proof. This follows from Lemma 5.6 by induction. □

Lemma 5.8. *Let $r \geq 2$ and let $s_1, \ldots, s_r : M \to C$ be sections with image in the smooth locus of π and which are pairwise disjoint. Then,*

$$-\sum_{i=1}^{r} \pi_*(s_i(M)\cdot s_i(M)) = r(r-2)\pi_*(s_1(M)\cdot s_1(M)) + \sum_{\beta',\beta''} \langle s_1(M), \beta'\rangle\langle s_2(M)+\cdots+s_r(M), \beta''\rangle^2\Delta_{\beta',\beta''}.$$

Combined with Lemma 5.7 this gives,

$$(r-1)(r-2)\pi_*(s_1(M) \cdot s_1(M)) =$$

$$-\textstyle\sum_{\beta',\beta''} \langle s_1(M), \beta'\rangle\langle s_2(M) + \cdots + s_r(M), \beta''\rangle(\langle s_2(M) + \cdots + s_r(M), \beta''\rangle - 1)\Delta_{\beta',\beta''},$$

which in turn gives,

$$-(r-1)\textstyle\sum_{i=1}^{r} \pi_*(s_i(M) \cdot s_i(M)) =$$

$$\textstyle\sum_{\beta',\beta''} \langle s_1(M), \beta'\rangle\langle s_2(M) + \cdots + s_r(M), \beta''\rangle(r - \langle s_2(M) + \cdots + s_r(M), \beta''\rangle)\Delta_{\beta',\beta''}.$$

In the notation of Example 2.2, this is,

$$-(r-1)(r-2)\pi_*(s_1(M) \cdot s_1(M)) = \sum_{(A,B),\ 1\in A} \#B(\#B-1)\Delta_{(A,B)},$$

and

$$-(r-1)\sum_{i=1}^{r} \pi_*(s_i(M) \cdot s_i(M)) = \sum_{(A,B),\ 1\in A} \#B(r-\#B)\Delta_{(A,B)}.$$

Proof. Denote $D = \sum_{i=2}^{r} s_i(M)$. Apply Lemma 5.4 to get,

$$2(r-1) \cdot 0 - \textstyle\sum_{i=2}^{r} \pi_*(s_i(M) \cdot s_i(M)) - (r-1)^2\pi_*(s_1(M) \cdot s_1(M)) =$$

$$\textstyle\sum_{\beta',\beta''} \langle s_1(M), \beta'\rangle\langle s_2(M) + \cdots + s_r(M), \beta''\rangle^2\Delta_{\beta',\beta''}.$$

Simplifying,

$$-\sum_{j=1}^{r} \pi_*(s_i(M)\cdot s_i(M)) = r(r-2)\pi_*(s_1(M)\cdot s_1(M)) + \sum \langle s_1(M), \beta'\rangle\langle s_2(M)+\cdots+s_r(M), \beta''\rangle^2\Delta_{\beta',\beta''}.$$

Subtracting from the relation in Lemma 5.7 gives the relation for $(r-1)(r-2)\pi_*(s_1(M)\cdot s_1(M))$. Multiplying the first relation by $(r-1)$, plugging in the second relation and simplifying gives the third relation. $\square$

Lemma 5.9. *Let $r \geq 2$ and let $s_1, \dots, s_r : M \to C$ be everywhere disjoint sections with image in the smooth locus. For every $1 \leq i < j \leq r$, using the notation from Example 2.2,*

$$\sum_{(A,B),\ i \in A} \#B(r-\#B)\Delta_{(A,B)} = \sum_{(A',B'), j \in A} \#B'(r-\#B')\Delta_{(A',B')}.$$

Proof. This follows from Lemma 5.8 by permuting the roles of 1 with i and j. $\square$

Lemma 5.10. *Let $r \geq 2$ and let $s_1, \dots, s_r : M \to C$ be everywhere disjoint sections with image in the smooth locus of π. For every Cartier divisor class D on C of relative degree $\langle D, \beta \rangle$,*

$$2(r-1)(r-2)\langle D,\beta\rangle s_1^* D = (r-1)(r-2)\pi_*(D\cdot D) + \sum_{\beta',\beta''} \langle s_1(M), \beta'\rangle a(D,\beta'')\Delta_{\beta',\beta''},$$

where,

$$a(D,\beta'') = (r-1)(r-2)\langle D,\beta''\rangle^2 - \langle D,\beta\rangle^2 \langle s_2(M)+\dots+s_r(M), \beta''\rangle(\langle s_2(M)+\dots+s_r(M),\beta''\rangle - 1).$$

In particular, if $r \geq 3$, then modulo torsion $s_i^* D$ is in the span of $\pi_*(D\cdot D)$ and boundary divisors for every $i = 1, \dots, r$.

Proof. This follows from Lemma 5.4 and Lemma 5.8. $\square$

Lemma 5.11. *Let $r \geq 2$ and let $s_1, \dots, s_r : M \to C$ be everywhere disjoint sections with image in the smooth locus of π. Consider the sheaf $\mathcal{E} = \Omega_\pi(s_1(M) + \dots + s_r(M))$. The perfect complex $R\pi_* RHom_{\mathcal{O}_C}(\mathcal{E}, \mathcal{O}_C)$ has rank $3-r$ and the first Chern class of the determinant is $-2\Delta - \sum_{i=1}^r (s_i(M)\cdot s_i(M))$. In particular, if $r \geq 2$, up to torsion,*

$$C_1(detR\pi_* RHom_{\mathcal{O}_C}(\Omega_\pi(s_1(M)+\dots+s_r(M)), \mathcal{O}_C)) =$$

$$-2\Delta + \tfrac{1}{r-1}\textstyle\sum_{(A,B),\ 1\in A} \#B(r-\#B)\Delta_{(A,B)}.$$

Proof. There is a short exact sequence,

$$0 \longrightarrow \Omega_\pi \longrightarrow \Omega_\pi(s_1(M)+\dots+s_r(M)) \longrightarrow \oplus_{i=1}^r (s_i)_*\mathcal{O}_M \longrightarrow 0.$$

Combining this with Lemma 4.2, Lemma 5.8, and chasing through exact sequences gives the lemma. $\square$

6. Proof of Theorem 1.1

Let k be a field, let X be a connected, smooth algebraic space over k of dimension n, let M be an Artin stack over k, let $\pi : C \to M$ be a flat 1-morphism, representable by proper algebraic spaces whose geometric fibers are connected, at-worst-nodal curves of arithmetic genus 0, let $s_1, \dots, s_r : M \to C$ be pairwise disjoint sections with image contained in the smooth locus of π (possibly $r = 0$, i.e., there are no sections), and let $f : C \to X$ be a 1-morphism of k-stacks. In this setting, Behrend and Fantechi introduced a perfect complex $E^\bullet$ on M of amplitude $[-1,1]$ and a morphism to the cotangent complex, $\phi : E^\bullet \to L_M^\bullet$, [3]. If $\mathrm{char}(k) = 0$ and M is the Deligne-Mumford stack of stable maps to X, Behrend and Fantechi prove $E^\bullet$

has amplitude $[-1,0]$, $h^0(\phi)$ is an isomorphism and $h^{-1}(\phi)$ is surjective. In many interesting cases, ϕ is a quasi-isomorphism. Then $\det(E^\bullet)$ is an invertible dualizing sheaf for M. Because of this, $\det(E^\bullet)$ is called the *virtual canonical bundle*. In this section the relations from Section 5 are used to give a formula for the divisor class of the virtual canonical bundle. Assume that Hypothesis 5.1 holds for π.

Denote by $L_{(\pi,f)}$ the cotangent complex of the morphism $(\pi,f): C \to M \times X$. This is a perfect complex of amplitude $[-1,0]$. There is a distinguished triangle,

$$L_\pi \longrightarrow L_{(\pi,f)} \longrightarrow f^*\Omega_X[1] \longrightarrow L_\pi[1].$$

There is a slight variation $L_{(\pi,f,s)}$ taking into account the sections which fits into a distinguished triangle,

$$L_\pi(s_1(M)+\cdots+s_r(M)) \longrightarrow L_{(\pi,f,s)} \longrightarrow f^*\Omega_X[1] \longrightarrow L_\pi(s_1(M)+\cdots+s_r(M))[1].$$

The complex $E^\bullet$ is defined to be $(R\pi_*(L^\vee_{(\pi,f,s)})[1])^\vee$, where $(F^\bullet)^\vee$ is $RHom(F^\bullet,\mathcal{O})$. In particular, $\det(E^\bullet)$ is the determinant of $R\pi_*(L^\vee_{(\pi,f,s)})$. From the distinguished triangle, $\det(E^\bullet)$ is

$$\det(R\pi_* RHom_{\mathcal{O}_C}(\Omega_\pi(s_1(M)+\cdots+s_r(M)),\mathcal{O}_C)) \otimes \det(R\pi_* f^*T_X)^\vee.$$

By Lemma 5.11, the first term is known. The second term follows easily from Grothendieck-Riemann-Roch.

Lemma 6.1. *Assume that the relative degree of $f^*C_1(\Omega_X)$ is nonzero. Then $R\pi_* f^*T_X[-1]$ has rank $\langle -f^*C_1(\Omega_X),\beta\rangle + n$, and up to torsion the first Chern class of the determinant is,*

$$\tfrac{1}{2\langle -f^*C_1(\Omega_X),\beta\rangle}\left[2\langle -f^*C_1(\Omega_X),\beta\rangle \pi_* f^*C_2(\Omega_X)\right.$$

$$-(\langle -f^*C_1(\Omega_X),\beta\rangle+1)\pi_* f^*C_1(\Omega_X)^2+$$

$$\left.\sum{}' \langle -f^*C_1(\Omega_X),\beta'\rangle\langle -f^*C_1(\Omega_X),\beta''\rangle \Delta_{\beta',\beta''}\right].$$

Proof. The Todd class τ_π of π is $1-\frac{1}{2}C_1(\omega_\pi)+\tau_2+\ldots$, where $\pi_*\tau_2 = 0$. The Chern character of f^*T_X is,

$$n - f^*C_1(\Omega_X) + \frac{1}{2}(f^*C_1(\Omega_X)^2 - 2f^*C_2(\Omega_X)) + \ldots$$

Therefore $\mathrm{ch}(f^*T_X)\cdot\tau_\pi$ equals,

$$n-\left[f^*C_1(\Omega_X)+\frac{n}{2}C_1(\Omega_\pi)\right]+\frac{1}{2}\left[f^*C_1(\Omega_X)^2 - 2f^*C_2(\Omega_X) + f^*C_1(\Omega_X)\cdot C_1(\omega_\pi)\right]+n\tau_2+\ldots$$

Applying π_* and using that $\pi_*\tau_2 = 0$, the rank is $n+\langle -f^*C_1(\Omega_X),\beta\rangle$, and the determinant has first Chern class,

$$\frac{1}{2}\pi_*\left[f^*C_1(\Omega_X)^2 - 2f^*C_2(\Omega_X)\right] + \frac{1}{2}\pi_*(f^*C_1(\Omega_X)\cdot C_1(\omega_\pi)).$$

Applying Proposition 1.2 and simplifying gives the relation. □

Putting the two terms together gives the formulas in Theorem 1.1.

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