THE AMPLE CONE OF THE KONTSEVICH MODULI SPACE

IZZET COSKUN, JOE HARRIS, AND JASON STARR

ABSTRACT. We produce ample, respectively NEF, eventually free, divisors in the Kontsevich space $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ of n-pointed, genus 0, stable maps to $\mathbb{P}^r$, given such divisors in $\overline{\mathcal{M}}_{0,n+d}$. We prove this produces all ample, respectively NEF, eventually free, divisors in $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$. As a consequence, we construct a contraction of the boundary $\cup_{k=1}^{\lfloor d/2 \rfloor} \Delta_{k,d-k}$ in $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$ analogous to a contraction of the boundary $\cup_{k=3}^{\lfloor n/2 \rfloor} \tilde{\Delta}_{k,n-k}$ in $\overline{\mathrm{M}}_{0,n}$ first constructed by Keel and M$^{\underline{c}}$ Kernan.

1. Introduction

Positive-dimensional families of varieties *specialize* – non-general varieties in the family exhibit special properties. Given a parameter space, the subset parametrizing varieties with a special property is typically closed. Which special properties occur in codimension 1, respectively for every 1-parameter family of varieties? More precisely, when is the associated closed subset an effective divisor, respectively an ample divisor? These questions, among others, motivate the study of effective and ample divisors in parameter spaces of varieties.

The parameter space we study is the Kontsevich moduli space of n-pointed, genus 0, stable maps to projective space, denoted $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$. Here we study the ample cone, and more generally the NEF and eventually free cones. In a second article [CHS], using significantly different methods and under additional hypotheses, we study the effective cone.

Our goal is to study families of curves in a general target X. Fortunately, this largely reduces to the study for $\mathbb{P}^r$: As the Kontsevich space is functorial in the target, for every morphism $X \to \mathbb{P}^r$, NEF and base-point-free divisors on $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ give NEF and base-point-free divisors on $\overline{\mathcal{M}}_{0,n}(X, \beta)$ (this functoriality is one of many advantages of the Kontsevich space over the Hilbert scheme and the Chow variety).

Here is our main result.

Theorem 1.1. *Let r and d be positive integers, n a nonnegative integer such that $n + d \geq 3$. There is an injective linear map,*

$$v : Pic(\overline{\mathcal{M}}_{0,n+d})_{\mathbb{Q}}^{\mathfrak{S}_d} \to Pic(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d))_{\mathbb{Q}}.$$

2000 *Mathematics Subject Classification.* Primary:14D20, 14E99, 14H10.

During the preparation of this article the second author was partially supported by the NSF grant DMS-0200659. The third author was partially supported by the NSF grant DMS-0353692 and a Sloan Research Fellowship.

The NEF cone of $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d)$, respectively, the base-point-free cone, is the product of the cone generated by $\mathcal{H},\mathcal{T},\mathcal{L}_1,\dots,\mathcal{L}_n$ and the image under v of the NEF cone of $\overline{\mathcal{M}}_{0,n+d}//\mathfrak{S}_d$, respectively, the base-point-free cone.

The action of $\mathfrak{S}_d$ on $\overline{\mathcal{M}}_{0,n+d}$ permutes the last d marked points. The map v generates NEF and base-point-free divisors on the Kontsevich space from NEF and base-point-free divisors on $\overline{\mathcal{M}}_{0,n+d}//\mathfrak{S}_d$. In particular, it generates contractions of the Kontsevich space from contractions of $\overline{\mathcal{M}}_{0,n+d}//\mathfrak{S}_d$.

Theorem 1.2. *For every integer $r \geq 1$ and $d \geq 2$, there is a contraction,*

$$\mathit{cont} : \overline{M}_{0,0}(\mathbb{P}^r,d) \to Y,$$

restricting to an open immersion on the interior $M_{0,0}(\mathbb{P}^r,d)$ and whose restriction to the boundary divisor $\Delta_{k,d-k} \cong M_{0,1}(\mathbb{P}^r,k) \times_{\mathbb{P}^r} M_{0,1}(\mathbb{P}^r,d-k)$ factors through the projection to $\overline{M}_{0,1}(\mathbb{P}^r,d-k)$ for each $1 \leq k \leq \lfloor d/2 \rfloor$. The following divisor is the pull-back of an ample divisor on Y,

$$D_{r,d} = \mathcal{T} + \sum_{k=2}^{\lfloor d/2 \rfloor} k(k-1)\Delta_{k,d-k}.$$

Some connection between the ample cone of the Kontsevich space and the ample cone of $\overline{\mathcal{M}}_{0,n}$ is natural, and certainly not surprising to experts. A similar connection between the Fulton-MacPherson space and $\overline{\mathcal{M}}_{0,n}$ was proved in [Che]. The primary importance of Theorem 1.1 is the precise, simple description of v: with one exception, it maps each boundary divisor of $\overline{\mathcal{M}}_{0,n+d}$ to the corresponding boundary divisor of the Kontsevich space. This is used to construct the contraction in Theorem 1.2, which is analogous to the "democratic" contraction of the boundary of $\overline{\mathcal{M}}_{0,n}$ first constructed in an unpublished note of Keel and M$^{\underline{c}}$ Kernan.

Recently we were informed of different constructions of the contraction of Theorem 1.2 in [Par] and by Anca Mustaţă and Andrei Mustaţă. One advantage of our proof is that it uses only the existence of the map v, which is itself a formal consequence of the definition of the Kontsevich space. The proof of Theorem 1.2 also gives a new, very short construction of Keel-M$^{\underline{c}}$ Kernan's contraction of $\overline{\mathcal{M}}_{0,n}$.

Acknowledgments: We would like to thank B. Hassett and A. J. de Jong for suggesting helpful references and for useful discussions.

2. Statement of results

The Kontsevich moduli space $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d)$ compactifies the scheme parameterizing smooth, rational curves of degree d in $\mathbb{P}^r$. Precisely, it is the smooth, proper, Deligne-Mumford stack parameterizing families of data $(C,(p_1,...,p_n),f)$ of,

(i) a proper, connected, at-worst-nodal, genus 0 curve C,
(ii) an ordered sequence $p_1,\dots,p_n$ of distinct, smooth points of C,
(iii) and a degree-d morphism $f : C \to \mathbb{P}^r$ satisfying the following stability condition: every irreducible component of C mapped to a point under f contains at least 3 *special points*, i.e., marked points p_i and nodes of C.

In [Pa] R. Pandharipande gives generators of the Kontsevich space:

(1) the class $\mathcal{H}$ of the divisor of maps whose images intersect a fixed codimension two linear space in $\mathbb{P}^r$ (provided $r>1$ and $d>0$),

(2) the class $\mathcal{L}_i$ of the pull-back $\mathrm{ev}_i^*(\mathcal{O}_{\mathbb{P}^r}(1))$, for $1 \le i \le n$, associated to the i^{th} evaluation morphism, $\mathrm{ev}_i(C,(p_1,\dots,p_n),f) := f(p_i)$,
(3) and the classes $\Delta_{(A,d_A),(B,d_B)}$ of the boundary divisors consisting of maps with reducible domains. Here $A \sqcup B$ is any ordered partition of the marked points, and d_A and d_B are non-negative integers satisfying $d = d_A + d_B$. If $d_A = 0$ (respectively, if $d_B = 0$), we demand $\#A \ge 2$ (resp. $\#B \ge 2$).

The divisor classes $\mathcal{H}$ and $\mathcal{L}_i$ on $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d)$ are NEF and base-point-free. For $d \ge 2$, there is another NEF and base-point-free divisor class $\mathcal{T}$, the *tangency divisor*: Fixing a hyperplane $\Pi \subset \mathbb{P}^r$, $\mathcal{T}$ is the class of the divisor parametrizing stable maps $(C,p_1,\dots,p_i,f)$ for which $f^{-1}(\Pi)$ is not simply d reduced, smooth points of C. In terms of Pandharipande's generators, the class of $\mathcal{T}$ equals,

$$\mathcal{T} = \frac{d-1}{d}\mathcal{H} + \sum_{k=0}^{\lfloor d/2 \rfloor} \frac{k(d-k)}{d}\left(\sum_{A,B} \Delta_{(A,k),(B,d-k)}\right).$$

Finally, the map v from Theorem 1.1 is described in Section 3. Together, all nonnegative-linear combinations of these divisors give a cone in $\mathrm{Pic}(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d))_{\mathbb{Q}}$. We use the method of *test families* to prove this is the entire cone of NEF divisors, respectively, eventually free divisors. In other words, we find morphisms from *test varieties* to $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d)_{\mathbb{Q}}$. Since every NEF divisor, resp. eventually free divisor, pulls back to such a NEF divisor, resp. eventually free divisor, this constrains the NEF and eventually free divisors among all divisors. By producing sufficiently many test families, we prove every NEF, resp. eventually free divisor, is in our cone.

Hypothesis 2.1. For the rest of the paper assume that the triple (n,r,d) in $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d)$ satisfies $r \ge 1$, $d \ge 1$ and $n+d \ge 3$.

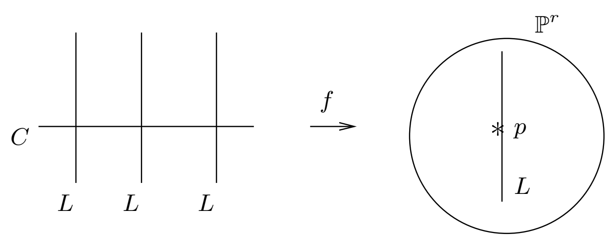


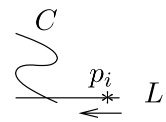
FIGURE 1. The morphism α.

The morphism α. There is a 1-morphism $\alpha : \overline{M}_{0,n+d} \times \mathbb{P}^{r-1} \to \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d)$ defined as follows. Fix a point $p \in \mathbb{P}^r$ and a line $L \subset \mathbb{P}^r$ containing p. To every curve C in $\overline{M}_{0,n+d}$ attach a copy of L at each of the last d marked points and denote the resulting curve by C'. Consider the morphism $f : C' \to \mathbb{P}^r$ that contracts C to p and maps the d rational tails isomorphically to L (see Figure 1). Since the space of lines in $\mathbb{P}^r$ passing through the point p is parameterized by $\mathbb{P}^{r-1}$, there is an induced 1-morphism $\alpha : \overline{M}_{0,n+d} \times \mathbb{P}^{r-1} \to \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d)$.

Since α is invariant for the action of $\mathfrak{S}_d$ permuting the last d marked points, the pull-back map determines a homomorphism

$$\alpha^* = (\alpha_1^*, \alpha_2^*) : \mathrm{Pic}(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d)) \to \mathrm{Pic}(\overline{\mathrm{M}}_{0,n+d})^{\mathfrak{S}_d} \times \mathrm{Pic}(\mathbb{P}^{r-1}).$$

We will denote the two projections of α^* by α_1^* and α_2^*.



slide p_i along L

FIGURE 2. The morphism β_i.

The morphisms β_i. For each $1 \leq i \leq n$, there is a 1-morphism $\beta_i : \mathbb{P}^1 \to \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ defined as follows. Fix a degree-$(d-1)$, $(n-1)$-pointed curve C containing all except the i^{th} marked point. At a general point of C, attach a line L. The resulting degree-d, reducible curve will be the domain of our map. The final, i^{th} marked point is in L. Varying p_i in L gives a 1-morphism $\beta_i : \mathbb{P}^1 \to \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ (see Figure 2). This definition has to be slightly modified in the cases $(n, d) = (1, 1)$ or $(2, 1)$. When $(n, d) = (1, 1)$, we assume that the line L with the varying marked point p_i constitutes the entire stable map. When $(n, d) = (2, 1)$, we assume that the map has L as the only component. One marked point is allowed to vary on L and the remaining marked point is held fixed at a point $p \in L$.

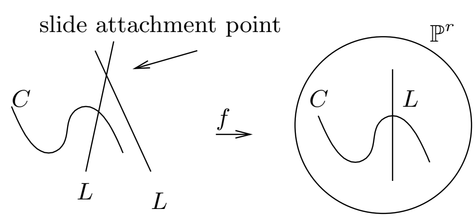


FIGURE 3. The morphism γ.

The morphism γ. If $d \geq 2$, there is a 1-morphism $\gamma : \mathbb{P}^1 \to \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ defined as follows. Take two copies of a fixed line L attached to each other at a variable point. Fix a point p in the second copy of L. Let C be a smooth, degree-$(d-2)$, genus 0, $(n+1)$-pointed stable map to $\mathbb{P}^r$ whose $(n+1)$-st point maps to p. Attach this to the second copy of L at p. Altogether, this gives a degree-d, n-pointed, genus 0 stable maps with three irreducible components. The n marked points are the first n marked points of C. The only varying aspect of this family of stable maps is the attachment point of the two copies of L. Varying the attachment point in $L \cong \mathbb{P}^1$ gives a stable map parameterized by $\mathbb{P}^1$, hence there is an induced 1-morphism $\gamma : \mathbb{P}^1 \to \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ (see Figure 3). When $(n, d) = (1, 2)$, we modify the definition by assuming that the map consists only of the two copies of the line L and the marked point is held fixed at the point p on the second copy of L.

Notation 2.2. If $d \geq 2$, denote by $P_{r,n,d}$ the Abelian group

$$P_{r,n,d} := \mathrm{Pic}(\overline{\mathrm{M}}_{0,n+d})^{\mathfrak{S}_d} \times \mathrm{Pic}(\mathbb{P}^{r-1}) \times \mathrm{Pic}(\mathbb{P}^1)^n \times \mathrm{Pic}(\mathbb{P}^1).$$

Denote by $u = u_{r,n,d} : \mathrm{Pic}(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)) \to P_{r,n,d}$ the pull-back map

$$u_{r,n,d} = (\alpha^*, (\beta_1^*, \ldots, \beta_n^*), \gamma^*).$$

If $d = 1$, denote by $P_{r,n,1}$ the Abelian group

$$P_{r,n,1} := \mathrm{Pic}(\overline{\mathrm{M}}_{0,n+d})^{\mathfrak{S}_d} \times \mathrm{Pic}(\mathbb{P}^{r-1}) \times \mathrm{Pic}(\mathbb{P}^1)^n$$

and denote by $u = u_{r,n,1} : \mathrm{Pic}(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, 1)) \to P_{r,n,1}$ the pull-back map

$$u_{r,n,1} = (\alpha^*, (\beta_1,^*, \dots, \beta_n^*))$$

Theorem 1.1 is equivalent to the following.

Theorem 2.3. *The map $u_{r,n,d} \otimes \mathbb{Q} : Pic(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d))_{\mathbb{Q}} \to P_{r,n,d} \otimes \mathbb{Q}$ is an isomorphism. The image under $u_{r,n,d} \otimes \mathbb{Q}$ of the ample cone, resp. NEF, eventually free cone of $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ equals the product of the ample cones, resp. NEF, eventually free cones of $Pic(\overline{\mathrm{M}}_{0,n+d})^{\mathfrak{S}_d}$, $Pic(\mathbb{P}^{r-1})$, and the factors $Pic(\mathbb{P}^1)$.*

This is equivalent to Theorem 1.1 because the linear map $u_{r,n,d}$ is simply the inverse of the product of the linear map v and the maps $\mathbb{Q} \to \mathrm{Pic}(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d))_{\mathbb{Q}}$ associated to each generator $\mathcal{H}$, $\mathcal{T}$ and $\mathcal{L}_1, \dots, \mathcal{L}_n$.

Notation 2.4. Denote by $\underline{n}$ the set $\{1, \dots, n\}$. Denote by $\triangle = \triangle_{n,d}$ the set of 4-tuples $((A, d_A), (B, d_B))$ of an ordered partition $A \sqcup B$ of $\underline{n}$ and an ordered pair of nonnegative integers (d_A, d_B) such that $d_A + d_B = d$ and $\#A \geq 2$ ($\#B \geq 2$) if $d_A = 0$ ($d_B = 0$, respectively). Denote by $\triangle'$ the subset of $\triangle$ of data such that $\#A + d_A \geq 2$ and $\#B + d_B \geq 2$.

Recall that the group $\mathrm{Pic}(\overline{\mathrm{M}}_{0,n})$ is generated by boundary divisors $\tilde{\Delta}_{A,B}$, where $A \sqcup B$ is an ordered partition of $\underline{n}$ with $\#A \geq 2$ and $\#B \geq 2$. Let $\tilde{\Delta}_{k,n-k}$ denote the sum of the boundary divisors $\sum_{(A,B)} \tilde{\Delta}_{A,B}$, where the sum runs over pairs (A, B) such that $\#A = k$ and $\#B = n - k$. The group $\mathrm{Pic}(\overline{\mathrm{M}}_{0,n+d})^{\mathfrak{S}_d}$ is generated by boundary divisors $\tilde{\Delta}_{(A,d_A),(B,d_B)}$, where $((A, d_A), (B, d_B)) \in \Delta'$. The divisor $\tilde{\Delta}_{(A,d_A),(B,d_B)}$ denotes the $\mathfrak{S}_d$-invariant sum of boundary divisors $\sum_{(A',B')} \tilde{\Delta}_{(A,A'),(B,B')}$, where the sum runs over pairs (A', B') such that $A' \sqcup B'$ is a partition of the last d points and $\#A' = d_A$ and $\#B' = d_B$.

To apply Theorem 2.3, we need to express the images of the standard generators of $\mathrm{Pic}(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d))$ in terms of the standard generators for $\mathrm{Pic}(\overline{\mathrm{M}}_{0,n+d})^{\mathfrak{S}_d}$, $\mathrm{Pic}(\mathbb{P}^{r-1})$ and $\mathrm{Pic}(\mathbb{P}^1)$ factors.

Proposition 2.5. *(i) Assume $d \geq 2$ so that γ is defined. Then*

$$\gamma^*\mathcal{T} = \mathcal{O}_{\mathbb{P}^1}(2), \quad \gamma^*\mathcal{H} = 0, \quad \gamma^*\mathcal{L}_i = 0, \quad \textit{for} \quad 1 \leq i \leq n.$$

The pull-back $\gamma^*\Delta_{(A,d_A),(B,d_B)} = 0$ unless $(\#A, d_A)$ or $(\#B, d_B)$ is equal to $(0, 1)$ or $(0, 2)$. Moreover, if $(n, d) \neq (0, 3)$,

$$\gamma^*\Delta_{(\emptyset,1),(\underline{n},d-1)} = \mathcal{O}_{\mathbb{P}^1}(4) \quad \textit{and} \quad \gamma^*\Delta_{(\emptyset,2),(\underline{n},d-2)} = \mathcal{O}_{\mathbb{P}^1}(-1).$$

If $(n, d) = (0, 3)$, then $\gamma^*\Delta_{(\emptyset,1),(\underline{n},d-1)} = \mathcal{O}_{\mathbb{P}^1}(3)$.

(ii) Assume $n \geq 1$ so that $\beta_1, \dots, \beta_n$ are defined. Then

$$\beta_i^*\mathcal{H} = 0, \quad \beta_i^*\mathcal{L}_i = \mathcal{O}_{\mathbb{P}^1}(1), \quad \beta_i^*\mathcal{L}_j = 0 \quad \textit{if} \quad j \neq i, \quad \textit{and} \quad \beta_i^*\mathcal{T} = 0.$$

For every $1 \leq i \leq n$, the pull-back $\beta_i^*\Delta_{(\emptyset,1),(\underline{n},d-1)}$ equals $\mathcal{O}_{\mathbb{P}^1}(1)$ if $(n, d) \neq (1, 2)$, and equals $\mathcal{O}_{\mathbb{P}^1}(2)$ if $(n, d) = (1, 2)$. If $(n, d) \neq (1, 2)$, then $\beta_i^*\Delta_{(\{i\},1),(\{i\}^c,d-1)}$ equals $\mathcal{O}_{\mathbb{P}^1}(-1)$. And $\beta_i^*\Delta_{(A,d_A),(B,d_B)}$ equals 0 if neither (A, d_A) nor (B, d_B) equal $(\emptyset, 1)$ or $(\{i\}, 1)$.

(iii) $\alpha^*\mathcal{H} = (0, \mathcal{O}_{\mathbb{P}^{r-1}}(d))$, $\alpha^*\mathcal{L}_i = 0$, for $1 \leq i \leq n$, $\alpha^*\mathcal{T} = 0$.

Divisors in $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d)$	α_1^*	α_2^*	β_i^*	γ^*
$\mathcal{T}$	0	0	0	$\mathcal{O}_{\mathbb{P}^1}(2)$
$\mathcal{H}$	0	$\mathcal{O}_{\mathbb{P}^{r-1}}(d)$	0	0
$\mathcal{L}_i$	0	0	$\mathcal{O}_{\mathbb{P}^1}(1)$	0
$\mathcal{L}_{j\neq i}$	0	0	0	0
$\Delta_{(\emptyset,1),(\underline{n},d-1)}$	c	$\mathcal{O}_{\mathbb{P}^{r-1}}(-d)$	$\mathcal{O}_{\mathbb{P}^1}(-1)$	$\mathcal{O}_{\mathbb{P}^1}(4)$
$\Delta_{(\emptyset,2),(\underline{n},d-2)}$	$\tilde{\Delta}_{(\emptyset,2),(\underline{n},d-2)}$	0	0	$\mathcal{O}_{\mathbb{P}^1}(-1)$
$\Delta_{(\{i\},1),(\{i\}^c,d-1)}$	$\tilde{\Delta}_{(\{i\},1),(\{i\}^c,d-1)}$	0	$\mathcal{O}_{\mathbb{P}^1}(-1)$	0
$\Delta_{(A,d_A),(B,d_B)}$ all others	$\tilde{\Delta}_{(A,d_A),(B,d_B)}$	0	0	0

FIGURE 4. The pull-backs of the standard generators

If $\#A+d_A, \#B+d_B \geq 2$, then $\alpha^*\Delta_{(A,d_A),(B,d_B)}$ equals $\tilde{\Delta}_{(A,d_A),(B,d_B)}$. The pull-back $\alpha^*\Delta_{(\emptyset,1),(\underline{n},d-1)}$ equals $(c,\mathcal{O}_{\mathbb{P}^{r-1}}(-d))$, where c is the class

$$c = \frac{-1}{(n+d-1)(n+d-2)} \sum_{\substack{((A,d_A),(B,d_B))\\ \in\Delta'}} d_A(d_B+\#B)(d_B+\#B-1)\tilde{\Delta}_{(A,d_A),(B,d_B)}.$$

Proof. Items (i) and (ii) follow from Lemma 3.5 and Lemma 4.1. Item (iii), except for the computation of c, is straightforward. The class c equals $-\sum_{i=1}^{d}\psi_{n+i}$. To rewrite this as above, use [Pa, Lemma 2.2.1] (cf. also [dJS, Lemma 6.10]). $\square$

With the exceptions of $(n,d)=(0,3)$, $(1,2)$, and $(1,3)$, Proposition 2.5 is summarized by Figure 4. The phrase "all others" means, all pairs $((A,d_A),(B,d_B))$ such that neither $((A,d_A),(B,d_B))$ nor $((B,d_B),(A,d_A))$ already occur in the table. The lines γ^* and $\Delta_{(\emptyset,2),(\underline{n},d-2)}$ apply only if $d\geq 2$. The lines $\mathcal{L}_i$, $\mathcal{L}_j$ and $\Delta_{(\{i\},1),(\{i\}^c,d-1)}$ only apply if $n\geq 1$.

In [Kaw], Kawamata associated an effective, NEF $\mathbb{Q}$-Cartier divisor $\mathcal{L}$ on $\overline{\mathrm{M}}_{0,n}$ to every n-tuple of rational numbers, $(d_1,\dots,d_n)$ satisfying $0<d_i\leq 1$ and $d_1+\dots+d_n=2$. In an unpublished note Keel and M$^{\underline{c}}$ Kernan proved the following.

Theorem 2.6 (Keel-M$^{\underline{c}}$ Kernan)**.** *The $\mathbb{Q}$-Cartier divisor $\mathcal{L}$ is eventually free.*

In particular, when $d_1=\dots=d_n=2/n$, the divisor class of $\mathcal{L}$ equals $(1/n(n-1))D_n$, where

$$D_n = \sum_{k=2}^{\lfloor n/2\rfloor} k(k-1)\tilde{\Delta}_{k,n-k}.$$

This is the divisor class giving the "democratic" contraction of the boundary of $\overline{\mathcal{M}}_{0,n}$, cf. [Has, §2.1.2]. One application of Theorem 2.3 is the construction of the analogous contraction in Theorem 1.2 as well as a new, short construction of the democratic contraction.

Theorem 2.7. *For every integer $n \geq 4$, there is a contraction*

$$cont : \overline{\mathrm{M}}_{0,n} \to Y$$

restricting to an open immersion on the interior $M_{0,n}$ and whose restriction to the boundary divisor $\Delta_{k,n-k} = M_{0,k+1} \times M_{0,n+1-k}$ factors through projection to $M_{0,n+1-k}$ for each $3 \leq k \leq \lfloor n/2 \rfloor$. The divisor $\mathcal{D}_n$ is the pull-back of an ample divisor on Y.

It follows easily that for every rational number b satisfying

$$\frac{2}{(n-1)} < b < \frac{2}{\lfloor n/2 \rfloor} \quad \text{resp.} \quad b = \frac{2}{\lfloor n/2 \rfloor},$$

setting $B = b\,(\mathrm{cont})_*(\tilde{\Delta}_{2,n-2})$, $K_Y + B$ is an ample $\mathbb{Q}$-Cartier divisor, and (Y, B) is Kawamata log terminal, resp. log canonical (for this one only needs the existence of the contraction and the formula for $\mathcal{L}$).

3. The splitting homomorphism

In this section we define a map $v : \mathrm{Pic}(\overline{\mathrm{M}}_{0,n+d})^{\mathfrak{S}_d} \to \mathrm{Pic}(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)) \otimes \mathbb{Q}$ that maps the NEF divisors in $\mathrm{Pic}(\overline{\mathrm{M}}_{0,n+d})^{\mathfrak{S}_d}$ to NEF divisors in $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$. The map v gives a splitting of the map α_1^* defined in the introduction and is essential for the proof of Theorem 2.3.

Let $\Pi \subset \mathbb{P}^r$ be a hyperplane not containing the point p used to define the morphisms α and γ. Assume that the degree $d-1$ curve used to define the morphisms β_i is not tangent to Π, and none of the marked points on this curve are contained in Π. Finally, assume that the degree $d-2$ curve used to define the morphism γ is not tangent to Π and none of the marked points are contained in Π.

Denote by $\mathcal{M}_{0,n+d}(\mathbb{P}^r, d)$ the open substack of $\overline{\mathcal{M}}_{0,n+d}(\mathbb{P}^r, d)$ parameterizing stable maps with irreducible domain. Let

$$\mathrm{ev}_{n+1,\ldots,n+d} : \mathcal{M}_{0,n+d}(\mathbb{P}^r, d) \to (\mathbb{P}^r)^d$$

be the evaluation morphism associated to the last d marked point. Denote by $\mathcal{M}_{0,n+d}(\mathbb{P}^r, d)_\Pi$ the inverse image of Π^d and by $\overline{\mathcal{M}}_{0,n+d}(\mathbb{P}^r, d)_\Pi$ the closure of $\mathcal{M}_{0,n+d}(\mathbb{P}^r, d)_\Pi$ in $\overline{\mathcal{M}}_{0,n+d}(\mathbb{P}^r, d)$.

$\overline{\mathcal{M}}_{0,n+d}(\mathbb{P}^r, d)_\Pi$ is $\mathfrak{S}_d$-invariant under the action of $\mathfrak{S}_d$ on $\overline{\mathcal{M}}_{0,n+d}(\mathbb{P}^r, d)$ permuting the last d marked points. Denote by

$$\pi : \overline{\mathcal{M}}_{0,n+d}(\mathbb{P}^r, d) \to \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$$

the forgetful 1-morphism that forgets the last d marked points and stabilizes the resulting family of prestable maps. This is $\mathfrak{S}_d$-invariant. Denote by

$$\rho : \overline{\mathcal{M}}_{0,n+d}(\mathbb{P}^r, d) \to \overline{\mathrm{M}}_{0,n+d}$$

the 1-morphism that stabilizes the universal family of marked prestable curves over $\overline{\mathcal{M}}_{0,n+d}(\mathbb{P}^r, d)$. This is $\mathfrak{S}_d$-equivariant.

Lemma 3.1. *The* 1*-morphism* $\pi : \mathcal{M}_{0,n+d}(\mathbb{P}^r, d)_\Pi \to \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ *is étale. Denoting the image by* O_Π*, the morphism* $\pi : \mathcal{M}_{0,n+d}(\mathbb{P}^r, d)_\Pi \to O_\Pi$ *is an* $\mathfrak{S}_d$*-torsor.*

Proof. Let $(C, (p_1, \dots, p_n, q_1, \dots, q_d), f)$ be a stable map in $\mathcal{M}_{0,n+d}(\mathbb{P}^r, d)_\Pi$. Then $(C, (p_1, \dots, p_n), f)$ satisfies,

(i) C is irreducible,
(ii) $f^{-1}(\Pi)$ is a reduced Cartier divisor, and
(iii) none of the marked points p_i is contained in $f^{-1}(\Pi)$.

Conversely, for every stable map $(C, (p_1, \dots, p_n), f)$ satisfying (i)–(iii), for every labeling of $f^{-1}(\Pi)$ as $q_1, \dots, q_d$, $(C, (p_1, \dots, p_n, q_1, \dots, q_d), f)$ is a stable map in $\mathcal{M}_{0,n+d}(\mathbb{P}^r, d)_\Pi$. Thus O_Π is the open substack of stable maps satisfying (i)–(iii) and $\mathcal{M}_{0,n+d}(\mathbb{P}^r, d)_\Pi$ is the $\mathfrak{S}_d$-torsor over O_Π parameterizing labelings of the fibers of $f^{-1}(\Pi)$. ☐

Denote by $q : \overline{\mathrm{M}}_{0,n+d} \to \overline{\mathrm{M}}_{0,n+d}/\mathfrak{S}_d$ the geometric quotient. The composition $q \circ \rho : \overline{\mathcal{M}}_{0,n+d}(\mathbb{P}^r, d)_\Pi \to \overline{\mathrm{M}}_{0,n+d}/\mathfrak{S}_d$ is $\mathfrak{S}_d$-equivariant. Because $\mathcal{M}_{0,n+d}(\mathbb{P}^r, d)_\Pi$ is an $\mathfrak{S}_d$-torsor over O_Π, there is a unique 1-morphism $\phi'_\Pi : O_\Pi \to \overline{\mathrm{M}}_{0,n+d}/\mathfrak{S}_d$ such that $\phi' \circ \pi = q \circ \rho$.

Definition 3.2. Define U_Π to be the maximal open substack of $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ over which ϕ'_Π extends to a 1-morphism, denoted

$$\phi_\Pi : U_\Pi \to \overline{\mathrm{M}}_{0,n+d}/\mathfrak{S}_d.$$

Define I_Π to be the normalization of the closure in $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d) \times \overline{\mathrm{M}}_{0,n+d}/\mathfrak{S}_d$ of the image of the graph of ϕ'_Π, i.e., I_Π is the normalization of the image of $(\pi, q \circ \rho)$. Define $\widetilde{I}_\Pi$ to be the normalization of the image of (π, ρ) in $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d) \times \overline{\mathrm{M}}_{0,n+d}$. Finally, define $\widetilde{U}_\Pi$ to be the inverse image of U_Π in $\widetilde{I}_\Pi$.

There is a pull-back map of $\mathfrak{S}_d$-invariant invertible sheaves,

$$\rho^* : \mathrm{Pic}(\overline{\mathrm{M}}_{0,n+d})^{\mathfrak{S}_d} \to \mathrm{Pic}(\widetilde{I}_\Pi)^{\mathfrak{S}_d},$$

which further restricts to $\mathrm{Pic}(\widetilde{U}_\Pi)^{\mathfrak{S}_d}$. After étale base-change from U_Π to a scheme, the morphism $\widetilde{U}_\Pi \to U_\Pi$ is the geometric quotient of $\widetilde{U}_\Pi$ by the action of $\mathfrak{S}_d$. Therefore the pull-back map $\mathrm{Pic}(U_\Pi) \to \mathrm{Pic}(\widetilde{U}_\Pi)^{\mathfrak{S}_d}$ is an isomorphism after tensoring with $\mathbb{Q}$; in fact, both the kernel and cokernel are annihilated by $d!$. Because $\overline{\mathrm{M}}_{0,n+d}/\mathfrak{S}_d$ is a proper scheme and because $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ is separated and normal, by the valuative criterion of properness the complement of U_Π has codimension ≥ 2. The smoothness of $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ and [Ha, Prop. 6.5(c)] imply that the restriction map $\mathrm{Pic}(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)) \to \mathrm{Pic}(U_\Pi)$ is an isomorphism.

Definition 3.3. Define $v : \mathrm{Pic}(\overline{\mathrm{M}}_{0,n+d})^{\mathfrak{S}_d} \to \mathrm{Pic}(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)) \otimes \mathbb{Q}$ to be the unique homomorphism commuting with ρ^* via the isomorphisms above.

The map v is independent of the choice of Π, hence it sends NEF divisors to NEF divisors.

Lemma 3.4. *For every base-point-free invertible sheaf* $\mathcal{L}$ *in* $Pic(\overline{M}_{0,n+d})^{\mathfrak{S}_d}$*,* $v(\mathcal{L})$ *is base-point-free. In particular, for every ample invertible sheaf* $\mathcal{L}$*,* $v(\mathcal{L})$ *is NEF. Thus, by Kleiman's criterion, for every NEF invertible sheaf* $\mathcal{L}$*,* $v(\mathcal{L})$ *is NEF.*

Proof. For every $[(C,(p_1,\dots,p_n),f)]$ in $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d)$, there exists a hyperplane Π satisfying the conditions above and such that $f^{-1}(\Pi)$ is a reduced Cartier divisor containing none of $p_1,\dots,p_n$. By Lemma 3.1, $(C,(p_1,\dots,p_n),f)$ is contained in U_Π. Since $\mathcal{L}$ is base-point-free, there exists a divisor D in the linear system $|\mathcal{L}|$ not containing $\phi_\Pi[(C,(p_1,\dots,p_n),f)]$. By the proof of [Ha, Prop. 6.5(c)], the closure of $\phi_\Pi^{-1}(D)$ is in the linear system $|v(\mathcal{L})|$; and it does not contain $[(C,(p_1,\dots,p_n),f)]$. □

Lemma 3.5. (i) *The images of* α, β_i *and* γ *are contained in* U_Π.

(ii) *The morphisms* $\phi_\Pi\circ\beta_i$ *and* $\phi_\Pi\circ\gamma$ *are constant morphisms. Therefore* $\beta_i^*\circ v$ *and* $\gamma^*\circ v$ *are the zero homomorphism.*

(iii) *The composition of* α *with* ϕ_Π *equals* $q\circ pr_{\overline{M}_{0,n+d}}$. *Therefore*

$$\alpha^*\circ v : Pic(\overline{M}_{0,n+d})^{\mathfrak{S}_d} \to Pic(\overline{M}_{0,n+d})^{\mathfrak{S}_d}\times Pic(\mathbb{P}^{r-1})$$

is the homomorphism whose projection on the first factor is the identity, and whose projection on the second factor is 0.

Proof. **(i):** The image of α is contained in O_Π. Denote by q the intersection point of L and Π.

The image $\beta_i(L-\{q\})$ is contained in O_Π. The stable map $\beta_i(q)$ sends the i^{th} marked point into Π. Up to labeling the d points of the inverse image of Π, there is only one $(n+d)$-pointed stable map in $\overline{\mathcal{M}}_{0,n+d}(\mathbb{P}^r,d)_\Pi$ that stabilizes to this stable map. It is obtained from $\beta_i(q)$ by removing the i^{th} marked point from L, attaching a contracted component C' to L at q, containing the i^{th} marked point and exactly one of the last d marked points, and labeling the $d-1$ points in $C\cap\Pi$ with the remaining $d-1$ marked points.

Similarly, $\gamma(L-\{q\})$ is contained in O_Π. The stable map $\gamma(q)$ has two copies of L attached to each other at q. This appears to be a problem, because the inverse image of $\gamma(q)$ in $\overline{\mathcal{M}}_{0,n+d}(\mathbb{P}^r,d)_\Pi$ is 1-dimensional, isomorphic to $\overline{\mathrm{M}}_{0,4}$. The stable maps have a contracted component C' such that both copies of L are attached to C' and 2 of the d new marked points are attached to C'. The remaining $d-2$ marked points are the points of $C\cap\Pi$. However, the map ρ that stabilizes the resulting prestable $(n+d)$-marked curve is constant on this $\overline{\mathrm{M}}_{0,4}$. Indeed, the first copy of L has no marked points and is attached to C' at one point. So the first step in stabilization will prune L reducing the number of special points on C' from 4 to 3.

(ii): In the family defining β_i, only the i^{th} marked point on L varies. After adding the d new marked points, L is a 3-pointed prestable curve; marked by the node p, the i^{th} marked point, and the point q. For every base the only family of genus 0, 3-pointed, stable curves is the constant family. So upon stabilization, this family of genus 0, 3-pointed, stable curves becomes the constant family.

In the family defining γ, only the attachment point of the two copies of L varies. The first copy of L gives a family of 2-pointed, prestable curves; marked by q and the attachment point of the two copies of L. This is unstable. Upon stabilization, the first copy of L is pruned and the marked point q on the first copy is replaced by a marked point on the second copy at the original attachment point. Now the second copy of L gives a family of 3-pointed, prestable curves; marked by the attachment point p of the second and third irreducible components, the attachment point of the first and second components, and q. For the same reason as in the last paragraph, this becomes a constant family.

(iii): Each stable map in $\alpha(\overline{\mathrm{M}}_{0,n+d} \times \mathbb{P}^{r-1})$ is obtained from a genus 0, $(n+d)$-pointed, stable curve $(C_0, (p_1, \ldots, p_n, q_1, \ldots, q_d))$ and a line L in $\mathbb{P}^r$ containing p by attaching for each $1 \leq i \leq n$, a copy C_i of L to C_0 where p in C_i is identified with q_i in C_0. The map to $\mathbb{P}^r$ contracts C_0 to p, and sends each curve C to L via the identity morphism. Denoting by r the intersection point of L and Π, the inverse image of Π consists of the d points $r_1, \ldots, r_d$, where r_i is the copy of r in C_i.

The component C_i is a 2-pointed, prestable curve: marked by the attachment point p of C_i and by r_i. This is unstable. So, upon stabilization, C_i is pruned and the marked point r_i is replaced by a marking on C_0 at the point of attachment of C_0 and C_i, namely q_i. Therefore, up to relabeling of the last d marked points, the result is the genus 0, $(n+d)$-pointed, stable curve we started with, $(C_0, (p_1, \ldots, p_n, q_1, \ldots, q_d))$. $\square$

4. More divisors

In the previous section we constructed a map (see Definition 3.3)

$$v : \mathrm{Pic}(\overline{\mathrm{M}}_{0,n+d})^{\mathfrak{S}_d} \to \mathrm{Pic}(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)) \otimes \mathbb{Q}.$$

In this section we prove that the image of v, the divisor classes $\mathcal{H}$, $\mathcal{T}$ and the tautological divisors $\mathcal{L}_i$, generate $\mathrm{Pic}(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)) \otimes \mathbb{Q}$.

The divisor class $\mathcal{H}_\Lambda$, [Pa, Prop. 1] is the class of stable maps whose image intersects a fixed codimension 2 linear space Λ of $\mathbb{P}^r$. This is defined to be the empty divisor if $r = 1$. For convenience, assume Λ is contained in Π and does not intersect L or the curves C used to define β_i and γ. If $n \geq 1$, the divisors $\mathcal{L}_{i,\Pi}$, $i = 1, \ldots, n$, [Pa, Prop. 1] are the pull-back by ev_i of the Cartier divisor Π. If $d \geq 1$, the last divisor is $\mathcal{T}_\Pi$, [Pa, §2.3]; the divisor of stable maps $(C, (p_1, \ldots, p_n), f)$ such that $f^{-1}(\Pi)$ is not a reduced, finite set of degree d. This is defined to be the empty divisor if $d = 1$. In [Pa] Pandharipande proves that $\mathcal{H}_\Lambda$, $\mathcal{L}_{i,\Pi}$ and $\mathcal{T}_\Pi$ are irreducible Cartier divisors (when they are nonempty).

Lemma 4.1. (i) *The Cartier divisors $\mathcal{T}_\Pi$, $\mathcal{L}_{i,\Pi}$ and $\mathcal{H}_\Lambda$ are NEF.*

(ii) *The pull-backs $\alpha^*(\mathcal{T}_\Pi)$ and $\alpha^*(\mathcal{L}_{i,\Pi})$ are zero. The pull-back $\alpha^*(\mathcal{H}_\Lambda)$ equals $(0, \mathcal{O}_{\mathbb{P}^{r-1}}(d))$ in $Pic(\overline{M}_{0,n+d})^{\mathfrak{S}_d} \times Pic(\mathbb{P}^{r-1})$; if $r = 1$, then $\mathcal{O}_{\mathbb{P}^{r-1}}(1)$ is the trivial invertible sheaf.*

(iii) *Assume $n \geq 1$ so that β_i is defined for $1 \leq i \leq n$. The pull-backs $\beta_i^*(\mathcal{T}_\Pi)$ and $\beta_i^*(\mathcal{H}_\Pi)$ are zero. For $1 \leq j \leq n$ different from i, $\beta_i^*(\mathcal{L}_{j,\Pi})$ is zero. Finally, $\beta_i^*(\mathcal{L}_{i,\Pi})$ is $\mathcal{O}_{\mathbb{P}^1}(1)$.*

(iv) *Assume $d \geq 2$ so that γ is defined. The pull-backs $\gamma^*(\mathcal{H}_\Lambda)$ and $\gamma^*(\mathcal{L}_{i,\Pi})$ are zero, and $\gamma^*(\mathcal{T}_\Pi)$ is $\mathcal{O}_{\mathbb{P}^1}(2)$ in $Pic(\mathbb{P}^1)$.*

Proof. **(i):** By an argument similar to the one in Lemma 3.4, these divisors are base-point-free (whenever they are non-empty). The divisor $\mathcal{H}_\Lambda$ is big if $r \geq 2$, and $\mathcal{T}_\Pi$ is big if $d \geq 2$. The divisors $\mathcal{L}_i$ are not big.

(ii): By the proof of Lemma 3.5, the image of α is in O_Π, which is disjoint from $\mathcal{T}_\Pi$. Also, $\mathrm{ev}_i \circ \alpha$ is the constant morphism with image p, so the inverse image of $\mathcal{L}_i$ is empty. Finally, the pull-back of $\mathcal{H}_\Pi$ equals the pull-back under the diagonal Δ of the Cartier divisor $\sum_{j=1}^d \mathrm{pr}_j^{-1}(\Lambda)$ in $(\mathbb{P}^{r-1})$, where Λ is considered as a divisor in $\mathbb{P}^{r-1}$ via projection from p.

(iii): Since the image of β_i is disjoint from $\mathcal{H}_\Pi$, $\mathcal{T}_\Pi$ and $\mathcal{L}_{j,\Pi}$ for $j \neq i$, the corresponding pull-backs are zero. The map $ev_i \circ \beta_i : \mathbb{P}^1 \to \mathbb{P}^r$ embeds $\mathbb{P}^1$ as the line L in $\mathbb{P}^r$, hence $\beta_i^*(\mathcal{L}_{i,\Pi}) = \mathcal{O}_{\mathbb{P}^1}(1)$.

(iv): Since neither the image curve nor the marked points vary under γ, clearly $\gamma^*\mathcal{H}_\Lambda$ and $\gamma^*\mathcal{L}_{i,\Pi}$ are zero. To compute $\gamma^*\mathcal{T}_\Pi$, use [Pa, Lem 2.3.1]. □

The main observation of this section is the following.

Proposition 4.2. *The $\mathbb{Q}$-vector space Pic$(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)) \otimes \mathbb{Q}$ is generated by $\mathcal{T}_\Pi$, $\mathcal{H}_\Lambda$, $\mathcal{L}_{i,\Pi}$ for $1 \leq i \leq n$, and the image of v.*

Proof. When $r \geq 2$, Pandharipande proves that the classes of the divisors $\mathcal{H}_\Lambda$, $\mathcal{L}_{i,\Pi}$ for $1 \leq i \leq n$, and the boundary divisors $\Delta_{(A,d_A),(B,d_B)}$ for $((A, d_A), (B, d_B)) \in \Delta$ generate the $\mathbb{Q}$-vector space $\text{Pic}(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)) \otimes \mathbb{Q}$, cf. [Pa, Prop. 1]. The tangency divisor $\mathcal{T}$ can be expressed in terms of $\mathcal{H}$ and the boundary divisors as follows [Pa, Lem 2.3.1]:

$$\mathcal{T} = \frac{d-1}{d}\mathcal{H} + \sum_{j=0}^{\lfloor \frac{d}{2} \rfloor} \frac{j(d-j)}{d} \sum_{((A,d_A),(B,d_B)),d_A=j} \Delta_{(A,d_A),(B,d_B)}.$$

From Lemmas 4.1 and 3.5 and by pairing with one-parameter families, we see that

$$v(\tilde{\Delta}_{(A,d_A),(B,d_B)}) = \Delta_{(A,d_A),(B,d_B)}$$

unless $(\#A, d_A)$ or $(\#B, d_B)$ equals one of $(0, 2)$ or $(1, 1)$.

$$v(\tilde{\Delta}_{(A,d_A),(B,d_B)}) = \frac{1}{2}\ \mathcal{T} + \Delta_{(A,d_A),(B,d_B)}$$

if $(\#A, d_A)$ or $(\#B, d_B)$ equals $(0, 2)$. Finally,

$$v(\tilde{\Delta}_{(\{i\},1),(\{i\}^c,d-1)}) = \Delta_{(\{i\},1),(\{i\}^c,d-1)} + \mathcal{L}_{i,\Pi}.$$

Consequently, it follows that the classes of the divisors $\mathcal{H}, \mathcal{T}$, $\mathcal{L}_{i,\Pi}$ and the image of v generate the classes of all the boundary divisors in the Kontsevich moduli space. Hence, they generate $\text{Pic}(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)) \otimes \mathbb{Q}$.

We can reduce the case $r = 1$ to the case $r \geq 2$. Because L is disjoint from Λ, there is a unique linear projection

$$\text{pr}_\Lambda : (\mathbb{P}^r - \Lambda) \to L$$

whose restriction to L is the identity. This is a vector bundle over L whose associated sheaf of sections is $\mathcal{O}_L(1)^{\oplus(r-1)}$. Composing a stable map to $(\mathbb{P}^r - \Lambda)$ with pr_Λ gives a stable map to L. This defines a 1-morphism,

$$\overline{\mathcal{M}}_{0,n}(\text{pr}_\Lambda, d) : (\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d) - \mathcal{H}_\Lambda) \to \overline{\mathcal{M}}_{0,n}(L, d).$$

This is a vector bundle over $\overline{\mathcal{M}}_{0,n}(L, d)$ whose associated sheaf of sections is the sheaf whose fiber at $(C, (p_1, \ldots, p_n), f)$ equals $H^0(C, f^*\mathcal{O}_L(1)^{\oplus(r-1)})$. Thus the pull-back homomorphism,

$$\overline{\mathcal{M}}_{0,n}(\text{pr}_\Lambda, d)^* : \text{Pic}(\overline{\mathcal{M}}_{0,n}(L, d)) \to \text{Pic}(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d) - \mathcal{H}_\Lambda),$$

is an isomorphism, cf. [Ful, Thm. 3.3(a)].

The hyperplane Π is the closure of $\text{pr}_\Lambda^{-1}(L \cap \Pi)$. Thus $U_\Pi - \mathcal{H}_\Lambda \cap U_\Pi$ (see Definition 3.2) is the inverse image of the corresponding open substack of $\overline{\mathcal{M}}_{0,n}(L, d)$ for $L \cap \Pi$ inside L. The inverse image of $\mathcal{T}_{L\cap\Pi}$, resp. $\mathcal{L}_{i,L\cap\Pi}$, equals the restriction of $\mathcal{T}_\Pi$, resp. $\mathcal{L}_{i,\Pi}$. And $\phi_{L\cap\Pi} \circ \overline{\mathcal{M}}_{0,n}(\text{pr}_\Lambda, d)$ equals the restriction of ϕ_Π. Thus

$\mathrm{Pic}(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d)-\mathcal{H}_\Lambda)\otimes\mathbb{Q}$ is generated by $\mathcal{T}_\Pi$, $\mathcal{L}_{i,\Pi}$ for $1\le i\le n$, and the image of v if and only if the same is true for $\mathrm{Pic}(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^1,d))\otimes\mathbb{Q}$.

□

5. Proof of the main theorem

In this section we complete the proof of Theorem 2.3. Recall that Theorem 2.3 asserts that the NEF cone of the Kontsevich moduli space $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d)$ equals the NEF cone in $P_{r,n,d}\otimes\mathbb{Q}$, where $P_{r,n,d}$ is the Abelian group

$$P_{r,n,d}:=\mathrm{Pic}(\overline{\mathrm{M}}_{0,n+d})^{\mathfrak{S}_d}\times\mathrm{Pic}(\mathbb{P}^{r-1})\times\mathrm{Pic}(\mathbb{P}^1)^n\times\mathrm{Pic}(\mathbb{P}^1).$$

The identification of $\mathrm{Pic}(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d))\otimes\mathbb{Q}$ with $P_{r,n,d}\otimes\mathbb{Q}$ is given by the map

$$u=u_{r,n,d}:=(\alpha^*,(\beta_1^*,\ldots,\beta_n^*),\gamma^*)$$

(see §1).

Denote by

$$\widetilde{v}:P_{r,n,d}\otimes\mathbb{Q}\to\mathrm{Pic}(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d))\otimes\mathbb{Q}$$

the unique homomorphism whose restriction to $\mathrm{Pic}(\overline{\mathrm{M}}_{0,n+d})^{\mathfrak{S}_d}$ is v (see Definition 3.3) , whose restriction to $\mathrm{Pic}(\mathbb{P}^{r-1})$ sends $\mathcal{O}_{\mathbb{P}^{r-1}}(1)$ to $[\mathcal{H}_\Lambda]$, whose restriction to the i^{th} factor of $\mathrm{Pic}(\mathbb{P}^1)^n$ sends $\mathcal{O}_{\mathbb{P}^1}(1)$ to $[\mathcal{L}_i]$ if $n\ge 1$, and whose restriction to the last factor $\mathrm{Pic}(\mathbb{P}^1)$ (assuming $d\ge 2$) sends $\mathcal{O}_{\mathbb{P}^1}(1)$ to $1/2\ [\mathcal{T}_\Pi]$. By Lemma 3.5 (ii), (iii) and by Lemma 4.1, $u\otimes\mathbb{Q}\circ\widetilde{v}$ is the identity map. In particular, $\widetilde{v}$ is injective. By Proposition 4.2, $\widetilde{v}$ is surjective. Thus $\widetilde{v}$ and $u\otimes\mathbb{Q}$ are isomorphisms.

Because α, β_i and γ are morphisms, for every NEF, resp. eventually free, divisor D in $\mathrm{Pic}(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d))\otimes\mathbb{Q}$, $\alpha^*(D)$, $\beta_i^*(D)$, and $\gamma^*(D)$ are NEF, resp. eventually free. Denote,

$$D_1=\alpha_1^*(D),\quad a\ [\mathcal{O}_{\mathbb{P}^{r-1}}(1)]=\alpha_2^*(D),\quad b_i\ [\mathcal{O}_{\mathbb{P}^1}(1)]=\beta_i^*(D),\quad c\ [\mathcal{O}_{\mathbb{P}^1}(1)]=\gamma^*(D),$$

where by convention a is defined to be 0 if $r=1$ and c is defined to be 0 if $d=1$. If D is NEF, resp. eventually free, D_1 is NEF, resp. eventually free, in $\mathrm{Pic}(\overline{\mathrm{M}}_{0,n+d})^{\mathfrak{S}_d}$, and $a,b_i,c\ge 0$.

Conversely, by Lemma 3.4, for every NEF, resp. eventually free, divisor D_1 in $\mathrm{Pic}(\overline{\mathrm{M}}_{0,n+d})^{\mathfrak{S}_d}$, $v(D_1)$ is NEF, resp. eventually free. By Lemma 4.1(i), for $a,b_i,c\ge 0$, $a[\mathcal{H}_\Lambda]$, $b_i[\mathcal{L}_{i,\Pi}]$ and $c/2\ [\mathcal{T}_\Pi]$ are NEF and eventually free. Since a sum of NEF, resp. eventually free, divisors is NEF, resp. eventually free, $D=v(D_1)+a\ [\mathcal{H}_\Lambda]+b_i\ [\mathcal{L}_i]+c/2\ [\mathcal{T}_\Pi]$ is NEF, resp. eventually free. Therefore D is NEF if and only if $u\otimes\mathbb{Q}(D)$ is in the product of the NEF cones of the factors. This argument needs to be modified in the obvious way when $(n,d)=(0,3)$ and $(1,2)$ to account for the slight variations in the formulae.

Because the interior of a product of cones equals the product of the interiors of the cones, by Kleiman's criterion, D is ample iff $u\otimes\mathbb{Q}(D)$ is contained in the product of the ample cones of the factors. □

Remark 5.1. Since the analogue of the F-conjecture is known for $\overline{\mathrm{M}}_{0,d}/\mathfrak{S}_d$ when $d\le 11$ by [KM], Theorem 2.3 provides an explicit description of the NEF cone of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r,d)$ for $r\ge 2$ and $d\le 11$. For example, when $d=2,3$, the NEF cone is bounded by the rays $\mathcal{H}$ and $\mathcal{T}$. When $d=4,5$, the NEF cone is generated by the rays $\mathcal{H}$, $\mathcal{T}$ and $\mathcal{H}+\Delta_{1,d-1}+4\ \Delta_{2,d-2}$.

6. The contractions

Theorem 2.7(i)–(v) and Theorem 1.2 are proved simultaneously by induction on n, resp. d, in the following 2 lemmas. Since the divisor $D_n = \sum_{k=2}^{\lfloor n/2 \rfloor} k(k-1)\tilde{\Delta}_{k,n-k}$ is ample on $\overline{\mathrm{M}}_{0,n}$ for $n = 4, 5$, the base cases $n = 4, 5$ for Theorem 2.7 are immediate. The base cases $d = 2, 3$ for Theorem 1.2 are also straightforward.

Lemma 6.1. *Let $d \geq 4$ be an integer. The existence of a contraction as in Theorem 2.7 for $n = d$ implies the existence of a contraction as in Theorem 1.2 for d.*

Proof. The divisor $D_{r,d} = \mathcal{T} + \sum_{k=2}^{\lfloor d/2 \rfloor} k(k-1)\Delta_{k,d-k}$ equals $v(D_d)$. By hypothesis, $v(D_d)$ is eventually free, thus $D_{r,d}$ is eventually free by Lemma 3.4. Define $\mathrm{cont}_{r,d} : \overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d) \to Y_{r,d}$ to be the associated morphism with connected fibers and normal target.

Denote by $O_{r,d}$ the maximal open subscheme of $Y_{r,d}$ over which $\mathrm{cont}_{r,d}$ is finite. The claim is that $\mathrm{cont}_{r,d}^{-1}(O_{r,d})$ contains $\mathcal{M}_{0,0}(\mathbb{P}^r, d)$. Every proper, irreducible curve B in $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$ contracted by $\mathrm{cont}_{r,d}$ has intersection number 0 with $D_{r,d}$. If B intersects $\mathcal{M}_{0,0}(\mathbb{P}^r, d)$, the intersection number with every $\Delta_{k,d-k}$, $k = 1, \dots, \lfloor d/2 \rfloor$ is nonnegative. Since $\mathcal{T}$ is NEF, the intersection number of B with $\mathcal{T}$ is nonnegative. Since $D_{r,d} \cdot B = 0$, B has intersection number zero with $\mathcal{T}$ and $\Delta_{k,d-k}$, $k = 2, \dots, \lfloor d/2 \rfloor$. From the expression of $\mathcal{T}$ and the fact that $\mathcal{H}$ and $\Delta_{1,d-1}$ have nonnegative intersection with B, it follows that B has intersection number zero with $\mathcal{H}$ and $\Delta_{1,d-1}$, as well. Since there exists an ample linear combination of these divisors, we obtain a contradiction. Thus B is contained in the complement of $\mathcal{M}_{0,0}(\mathbb{P}^r, d)$, proving the claim.

By Zariski's Main Theorem, $\mathrm{cont}_{r,d} : \mathrm{cont}_{r,d}^{-1}(O_{r,d}) \to O_{r,d}$ is an isomorphism. In particular, $\mathrm{cont}_{r,d} : \mathcal{M}_{0,0}(\mathbb{P}^r, d) \to O_{r,d}$ is an open immersion.

The 1-morphism ϕ_Π from Definition 3.2 maps $\Delta_{k,d-k}$ to $\tilde{\Delta}_{k,d-k}$ compatible with the boundary maps. Thus $\mathrm{cont}_{r,d}$ satisfies the conclusion of Theorem 1.2. $\square$

Lemma 6.2. *Let $n \geq 6$ be an integer. The existence of a contraction as in Theorem 1.2 for $d = n - 2$ implies the existence of a contraction as in Theorem 2.7 for n.*

Proof. Denote by $\mathrm{ev} : \overline{\mathcal{M}}_{0,n}(\mathbb{P}^{n-2}, n-2) \to (\mathbb{P}^{n-2})^n$ the evaluation 1-morphism. Denote by $\Phi : \overline{\mathcal{M}}_{0,n}(\mathbb{P}^{n-2}, n-2) \to \overline{\mathrm{M}}_{0,n}$ the forgetful 1-morphism. Denote by $U \subset (\mathbb{P}^{n-2})^n$ the open subset parameterizing n-tuples of points in linear general position, i.e., the span of every $(n-1)$-tuple equals $\mathbb{P}^{n-2}$. Kapranov proves that the 1-morphism,

$$(\mathrm{ev}, \Phi) : \overline{\mathcal{M}}_{0,n}(\mathbb{P}^{n-2}, n-2) \to (\mathbb{P}^{n-2})^n \times \overline{\mathrm{M}}_{0,n},$$

is an isomorphism over $U \times \overline{\mathrm{M}}_{0,n}$, [Kap]. Fix a general point q in U, and identify $\overline{\mathrm{M}}_{0,n}$ with the fiber over $\{q\} \times \overline{\mathrm{M}}_{0,n}$.

The forgetful 1-morphism $\pi : \overline{\mathcal{M}}_{0,n}(\mathbb{P}^{n-2}, n-2) \to \overline{\mathcal{M}}_{0,0}(\mathbb{P}^{n-2}, n-2)$ restricts to a 1-morphism $p : \overline{\mathrm{M}}_{0,n} \to \overline{\mathcal{M}}_{0,0}(\mathbb{P}^{n-2}, n-2)$. Denote by $\mathrm{cont} : \overline{\mathrm{M}}_{0,n} \to Y$ the Stein factorization of,

$$\mathrm{cont}_{n-2,n-2} \circ p : \overline{\mathrm{M}}_{0,n} \to \mathrm{cont}_{n-2,n-2}(p(\overline{\mathrm{M}}_{0,n})).$$

It is straightforward that $p^{-1}(\Delta_{k-1,n-1-k}) = \tilde{\Delta}_{k,n-k}$ for every $2 \leq k \leq \lfloor n/2 \rfloor$ compatibly with the boundary maps. Thus cont satisfies the conclusion of Theorem 2.7. $\square$

Remark 6.3. Pairing with test curves gives that

$$v\left(\frac{1}{n-1}D_n\right) = \frac{1}{n-1}D_{r,n} \quad \text{and} \quad p^*\left(\frac{1}{n-3}D_{n-2,n-2}\right) = \frac{1}{n-1}D_n.$$

The image of the ample cone of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^n, n)$ under p^* is not all of the ample cone of $\overline{\mathrm{M}}_{0,n+2}$, already for $n = 6$.

References

[Che] L. Chen. The nef cone of the moduli space of curves and of a Fulton-MacPherson space.

[CHS] I. Coskun, J. Harris, and J. Starr. The effective cone of the Kontsevich moduli space. *submitted*.

[dJS] A.J. de Jong and J. Starr. Divisor classes and the virtual canonical bundle. *preprint* (2005).

[Ful] W. Fulton. *Intersection theory*, volume 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics*. Springer-Verlag, Berlin, second edition, 1998.

[Ha] R. Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[Has] B. Hassett. Moduli spaces of weighted pointed stable curves. *Adv. Math.* **173**(2003), 316–352.

[Kap] M. M. Kapranov. Veronese curves and Grothendieck-Knudsen moduli space $\overline{M}_{0,n}$. *J. Algebraic Geom.* **2**(1993), 239–262.

[Kaw] Y. Kawamata. Subadjunction of log canonical divisors for a subvariety of codimension 2. In *Birational algebraic geometry (Baltimore, MD, 1996)*, volume 207 of *Contemp. Math.*, pages 79–88. Amer. Math. Soc., Providence, RI, 1997.

[KM] S. Keel and J. McKernan. Contractible extremal rays on $\overline{M}_{0,n}$. preprint, 1996.

[Pa] R. Pandharipande. Intersections of $\mathbf{Q}$-divisors on Kontsevich's moduli space $\overline{M}_{0,n}(\mathbf{P}^r, d)$ and enumerative geometry. *Trans. Amer. Math. Soc.* **351**(1999), 1481–1505.

[Par] A. Parker. *An elementary GIT construction of the moduli space of stable maps*. PhD thesis, University of Texas at Austin, 2005.

Department of Mathematics, M.I.T., Cambridge, MA 02139
E-mail address: `coskun@math.mit.edu`

Department of Mathematics, Harvard University, Cambridge, MA 02138
E-mail address: `harris@math.harvard.edu`

Department of Mathematics, M.I.T., Cambridge, MA 02139
E-mail address: `jstarr@math.mit.edu`