

SEPARABLE RATIONAL CONNECTEDNESS AND RATIONAL SIMPLE CONNECTEDNESS

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ABSTRACT. In this note we characterize when a smooth, separably uniruled, projective variety of Picard number one is separably rationally connected. As an application, we prove weak approximation at all places of (strong) potentially good reduction for a Fano complete intersection in $\mathbb{P}^n$ of type $(d_1, \dots, d_c)$ in characteristic p such that $n > d_1 + \dots + d_c$ and $p > d_1, \dots, d_c$. We also prove that every smooth Fano complete intersection in $\mathbb{P}^n$ is *rationally simply connected* if $n > d_1^2 + \dots + d_c^2$. As an application, these satisfy weak approximation at all places.

In this note, we improve some results in [Tia15] and [STZ18] about separable rational connectedness and weak approximation for Fano complete intersections in positive characteristic. Using similar techniques, we also prove rational simple connectedness for all smooth Fano complete intersections in the broadest possible degree range, i.e., whenever the second graded piece of the Chern character (of the tangent bundle) is positive.

First let us recall some basic definitions.

Definition 1. For every field K , for every integral, n -dimensional, projective K -scheme X , for every $r > 0$, an r -**uniruling** of X over K ,

$$(h, \pi) : Y \rightarrow X \times_{\mathrm{Spec} K} M,$$

is a finite morphism of K -varieties such that π is proper and smooth with geometric fibers $\mathbb{P}^1$ and such that the K -morphism from the r -fold fiber product is dominant,

$$h^{(r)} : Y \times_M \cdots \times_M Y \rightarrow X \times_{\mathrm{Spec} K} \cdots \times_{\mathrm{Spec} K} X, \quad \mathrm{pr}_i \circ h^{(r)} = h \circ \mathrm{pr}_i.$$

Following [Kol96, Definition IV.1.7.3], the r -**uniruling index**, $u_r(K, X)$, is the greatest common divisor of $\deg(h^{(r)})$ for all r -unirulings with $h^{(r)}$ generically finite (0 if there are no r -unirulings).

We say that $X_{\overline{K}}$ is *uniruled* (resp. *rationally connected*) if $u_1(K, X) > 0$ (resp. $u_2(K, X) > 0$).

We say $X_{\overline{K}}$ is *separably uniruled* if there is a uniruling family such that $\mathrm{pr}_1 \circ h^{(1)} : Y \rightarrow X$ is dominant and separable.

Similarly, $X_{\overline{K}}$ is *separably rationally connected* if there is a 2-uniruling family such that

$$h^{(2)} : Y \times_M Y \rightarrow X \times_{\mathrm{Spec} K} X, \quad \mathrm{pr}_i \circ h^{(2)} = h \circ \mathrm{pr}_i.$$

is dominant and separable.

The following is a well-known open question:

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Question 2. Is every smooth Fano hypersurface separably rationally connected?

Some partial answers are given by Chen-Zhu[CZ14], the work of the second named author [Tia15], and in a recent joint work of the two authors with R. Zong [STZ18]. One key component of the joint work of authors with R. Zong is a result about closedness of the separable rational connectedness under some conditions, which we will generalize in this article.

We first strengthen a result of the second named author [Tia15, Theorem 5].

Theorem 3. *Let X be a smooth projective variety, whose Picard group is isomorphic to $\mathbb{Z}$, defined over an algebraically closed field of positive characteristic. Assume that X is separably uniruled. Then X is separably rationally connected if and only if $H^0(X, \Omega_X^i) = 0$ for $i = 1, \dots, \dim X$.*

Remark 4. One can prove that such X in Theorem 3 is *freely rationally connected* (FRC) as defined in [She10, Definition 1.2].

Proof of Theorem 3. The “only if” part is well-known. In the following we prove the “if” part.

Since X is separably uniruled, there is a free curve, i.e. a morphism $f : \mathbb{P}^1 \rightarrow X$ such that

$$\begin{aligned} f^*\Omega_X &\cong \mathcal{O}(-a_1) \oplus \dots \oplus \mathcal{O}(-a_r) \oplus \mathcal{O} \oplus \dots \oplus \mathcal{O}, \\ f^*T_X &\cong \mathcal{O}(a_1) \oplus \dots \oplus \mathcal{O}(a_r) \oplus \mathcal{O} \oplus \dots \oplus \mathcal{O}, a_1 \geq a_2 \geq \dots a_r > 0. \end{aligned}$$

Define the *positive rank* r of X (sometimes *positivity*) to be the maximum number of non-trivial summands in the above decomposition among all free curves. A free curve is called *maximally free* if the pull-back of the cotangent bundle has r negative summands.

Given a general point $x \in X$, by [She10, Proposition 2.2], there is a well-defined subspace $D(x) \subset \Omega_X|_x$, as the subspace of the $\mathcal{O}$ -directions of a maximally free curve at x (i.e. $D(x)$ is independent of the choice of the maximally free curve). Furthermore, over an open subset U of X , which contains a general maximally free rational curve, the subspaces of $D(x)$ of $\Omega_X|_x$ glue together to a (locally free) coherent subsheaf of Ω_X (loc. cit. Proposition 2.5). Denote by $\mathcal{D}$ the saturated subsheaf of Ω_X which extends the locally free subsheaf given by $D(x), x \in U$.

Let $\phi : \mathbb{P}^1 \rightarrow X$ be a maximally free curve. If it is not very free, then we have

$$\begin{aligned} \phi^*\Omega_X &\cong \mathcal{O}(-a_1) \oplus \dots \oplus \mathcal{O}(-a_r) \oplus \mathcal{O} \oplus \dots \oplus \mathcal{O}, r < n, a_1 \geq a_2 \geq \dots a_r > 0, \\ \phi^*\mathcal{D} &\cong \mathcal{O} \oplus \dots \oplus \mathcal{O}, \end{aligned}$$

(c.f. the paragraph after Corollary 3.2, loc. cit.)

Thus $\mathcal{D}$ is locally free along a general maximally free curve. We have an inclusion

$$0 \rightarrow (\Lambda^{n-r}\mathcal{D})^{**} \rightarrow \Omega_X^{n-r}.$$

Since X is smooth, the reflexive rank 1 sheaf $(\Lambda^{n-r}\mathcal{Q})^{**}$ is locally free. Furthermore, X has Picard number one and $\deg \phi^*(\Lambda^{n-r}\mathcal{Q})^{**} = 0$, thus it is trivial. So we have a section of Ω_X^{n-r} . $\square$

Remark 5. Kollár (Exercise 5.19, Chap. V, [Kol96]) gives examples of separably uniruled, singular Fano varieties of Picard number one that are *not* separably rationally connected. The singularities are ordinary double points. Kollár proves that they are not separably rationally connected by writing down unexpected sections of Ω_X^{n-1} , where n is the dimension. We do not know smooth examples.

Before we state some applications of Theorem 3, we recall some definitions. For every positive integer N , an **N -decomposition of the diagonal** is a finite sequence $(e_i : D_i \rightarrow X, Z_i)_i$ of pairs of a proper morphism e_i with D_i integral of dimension $n - 1$ and an n -cycle $Z_i \in \text{CH}_n(X \times_{\text{Spec } K} D_i)$ such that the n -cycle

$$N \cdot [\Delta_X] - \sum_i (\text{Id} \times e_i)_* Z_i \in \text{CH}_n(X \times_{\text{Spec } K} X),$$

is in the image of the flat pullback map,

$$\text{pr}_1^* : \text{CH}_0(X) \rightarrow \text{CH}_n(X \times_{\text{Spec } K} X).$$

Following [CL16], the **torsion order** $\text{Tor}(K, X)$ is the greatest common divisor of all integers N such that there exists an N -decomposition of the diagonal. The torsion order equals 0 if there is no such decomposition.

Corollary 6. *Let X be a smooth projective Fano variety, whose Picard group is isomorphic to $\mathbb{Z}$, defined over an algebraically closed field $\kappa = \bar{\kappa}$ of characteristic $p > 0$. Assume that p is prime to the torsion order $\text{Tor}(\bar{\kappa}, X)$ and the uniruling index $u_1(\bar{\kappa}, X)$. Then X is separably rationally connected.*

Proof. First of all, X is separably uniruled since p is prime to $u_1(\bar{\kappa}, X)$ ([STZ18, Corollary 3.3]). Since $\text{Tor}(\bar{\kappa}, X)$ is prime to p , we have $H^0(X, \Omega_X^i) = 0$, by the argument of Totaro [Tot16, Proof of Lemma 2.2]. Thus this follows from Theorem 3. $\square$

Corollary 7. *Let $(R, \mathfrak{m}, \kappa, K)$ be a DVR and $\mathcal{X} \rightarrow \text{Spec } R$ be a smooth projective family. Assume that the Picard group of the geometric generic fiber $\mathcal{X}_{\bar{K}}$ is isomorphic to $\mathbb{Z}$, and that the characteristic of κ is prime to the torsion order $\text{Tor}(\bar{K}, \mathcal{X}_{\bar{K}})$ and the uniruling index $u_1(\bar{K}, \mathcal{X}_{\bar{K}})$ of the geometric generic fiber $\mathcal{X}_{\bar{K}}$, then the central fiber is also separably rationally connected.*

Proof. The torsion order $\text{Tor}(\bar{\kappa}, \mathcal{X}_{\bar{\kappa}})$ (resp. the uniruling index $u_1(\bar{\kappa}, \mathcal{X}_{\bar{\kappa}})$) of the geometric closed fiber is a divisor of the torsion order $\text{Tor}(\bar{K}, \mathcal{X}_{\bar{K}})$ (resp. the uniruling index $u_1(\bar{K}, \mathcal{X}_{\bar{K}})$) of the geometric generic fiber. For the statement about torsion order, see, for example, [CL16, Proposition 3.1, 3.2]. For the statement about uniruling index, see [Kol96, Corollary IV. 1.7.4]. Thus by Corollary 6, we only need to show that the Picard group of the geometric closed fiber is isomorphic to $\mathbb{Z}$.

We claim that for any (Weil) divisor D in the geometric closed fiber $\mathcal{X}_{\kappa}$, $\text{Tor}(\bar{K}, \mathcal{X}_{\bar{K}}) \cdot D$ lies in the image of the specialization map $Sp : \text{Pic}(\mathcal{X}_{\bar{K}}) \rightarrow \text{Pic}(\mathcal{X}_{\bar{\kappa}})$. To see this, simply note that up to making extensions of R , there are prime effective divisors $D_i \subset \mathcal{X}$, a section $s : \text{Spec } R \rightarrow \mathcal{X}$, such that $\text{Tor}(\bar{K}, \mathcal{X}_{\bar{K}})[\Delta_{\mathcal{X}/R}]$ is rationally equivalent to $\text{Tor}(\bar{K}, \mathcal{X}_{\bar{K}}) \mathcal{X} \times_R s(\text{Spec } R) + Z$, where Z is a cycle supported in $\cup_i D_i \times_R \mathcal{X}$. Then the correspondence $\text{Tor}(\bar{K}, \mathcal{X}_{\bar{K}}) \Delta_{\mathcal{X}_{\bar{\kappa}}}$ acts on the Chow group of codimension 1 cycles of $\mathcal{X}_{\bar{\kappa}}$ and $\text{Tor}(\bar{K}, \mathcal{X}_{\bar{K}})$ times any such cycle is a linear combination of $D_i|_{\mathcal{X}_{\bar{\kappa}}}$, which of course lies in the image of the specialization map.

On the other hand, we know the cokernal of Sp is a finite p -group by [GJ18, Theorem 1.8]. Thus the specialization is surjective and the Picard group of the geometric closed fiber is $\mathbb{Z}$. $\square$

Theorem 8. *For every smooth, affine, connected curve B over a field $\kappa = \bar{\kappa}$, for every B -flat proper family $\mathcal{X}_B$. Assume that the geometric generic fiber $\mathcal{X}_{\kappa(B)}$ is Fano and $\text{Pic}(X) \cong \mathbb{Z}$, and that p is prime to the uniruling index and torsion order of the geometric generic fiber. Then weak approximation holds at every place of (strong) potentially good reduction.*

The proof is essentially the same as the proof of [STZ18, Corollary 1.18], except that now because of our stronger results about specializations of separable rational connectedness, we do not need the index one assumption anymore.

Note that for a complete intersection X of type $(d_1, \dots, d_c)$ such that $n > d_1 + \dots + d_c$, there is a uniruling family of lines, the degree of the evaluation morphism of which is $\Pi_i(d_i!)$. Thus the uniruling index of X is a divisor of $\Pi_i(d_i!)$. By [CL16, Proposition 4.2], the torsion order is also a divisor of $\Pi_j(d_j!)$. So we obtain the following.

Corollary 9. *Let $(R, \mathfrak{m}, \kappa, K)$ be a DVR and $\mathcal{X} \rightarrow \text{Spec } R$ be a smooth projective family. Assume that the generic fiber $\mathcal{X}_K$ is a smooth complete intersection in $\mathbb{P}^n$ of type $(d_1, \dots, d_c)$ such that $n > d_1 + \dots + d_c$. Furthermore, assume that the characteristic of the residue field κ is greater than $\max(d_1, \dots, d_c)$. Then the central fiber $\mathcal{X}_\kappa$ is separably rationally connected.*

Finally, we have the following version of weak approximation (cf. [STZ18, Theorem 1.1]).

Theorem 10. *For every smooth, affine, connected curve B over a field $\kappa = \bar{\kappa}$, for every B -flat complete intersection $\mathcal{X}_B \subset B \times_{\text{Spec } \kappa} \mathbb{P}_\kappa^n$ of c hypersurfaces of degrees $(d_1, \dots, d_c)$, weak approximation holds at every place of (strong) potentially good reduction provided that the Fano index is ≥ 2 , and $p := \text{char}(\kappa)$ satisfies $p > \max(d_1, \dots, d_c)$.*

Here we replace the somewhat unnatural assumption “(strong) potentially good reduction as complete intersections” in [STZ18, Theorem 1.1] with the *usual* (strong) potentially good reduction assumption. This is possible since we know that for a place of (strong) potentially good reduction, the central closed fiber is still separably rationally connected, thanks to Corollary 9. Other than that, the proof goes exactly the same as that of [STZ18, Theorem 1.1]. We refer the reader to [STZ18] for details of the proof.

Remark 11. It is an interesting question to study (smooth) deformations of complete intersections. A construction of Mori gives a deformation of hypersurfaces of degree mn to a degree m cyclic cover of a hypersurface of degree n . In particular, this means that Theorem 10 is indeed an improvement of [STZ18, Theorem 1.1].

We finish with a brief discussion of the higher Picard number case.

It is very easy to show that on a smooth projective separably rationally connected variety X , the group of rational one cycles modulo numerical equivalence $N_1(X)_\mathbb{Q}$ is generated by (very) free rational curves. Thus if this group is not generated by

free rational curves, then the variety X is not separably rationally connected. The same argument as in the proof of Theorem 3 gives the following.

Theorem 12. *Let X be a smooth projective variety over an algebraically closed field of positive characteristic p . Assume that X is separably uniruled and the group $N_1(X)_{\mathbb{Q}}$ of rational one cycles modulo numerical equivalence is generated by free rational curves. Then X is separably rationally connected if and only if $H^0(X, \Omega_X^i) = 0, i = 1, \dots, \dim X$.*

1. TWISTING AND VERY TWISTING SURFACES

One technical difficulty of the theory of rational simple connectedness is to check that a given smooth projective variety is rationally simply connected. At the heart of the definition of rational simple connectedness is the existence of twisting and very twisting surfaces. Essentially, these are free and very free curves in a moduli space associated to the original variety

Here we give a general construction of twisting and very twisting surfaces via *bootstrapping*. Namely, assuming that there is a family of twisting surfaces, we give a general method of producing a very twisting surface.

Situation 13. Let X be smooth connected projective variety over the *complex numbers*. Assume that the Picard group of X is generated by a very ample divisor H . We also write $\mathcal{O}_X(1)$ for $\mathcal{O}_X(H)$. Denote the Fano scheme of lines of X by F , the universal family of lines by $\pi : U \rightarrow F$, and the evaluation morphism by $f : U \rightarrow X$. Assume that X is uniruled by lines, or equivalently, that f is dominant. Assume that for a general $x \in X$, the fiber $f^{-1}(x)$ has Picard number 1 (in particular, it is connected). Suppose that there is a morphism $\phi : \mathbb{P}^1 \rightarrow U$ such that all of the following hold:

- (A) The bundle $\phi^* f^* T_X$ is globally generated.
- (B) The sheaf $\phi^* T_f$ is locally free, globally generated, and has positive degree.
- (C) The line bundle $\phi^* T_\pi$ has non-negative degree.

Remark 14. Since X as above is uniruled by lines, for a general point x in X , the fiber $f^{-1}(x)$ is smooth.

Definition 15. A rational curve $\phi : \mathbb{P}^1 \rightarrow U$ is *relatively free* if it satisfies all the conditions (A), (B), (C) as in Situation 13.

A twisting surface (Σ, D) consists of a ruled surface Σ over $\mathbb{P}^1$ and a divisor $D \subset \Sigma$, which is the image of a section of $\Sigma \rightarrow \mathbb{P}^1$, together with a morphism $\Sigma \rightarrow X$, such that the rulings are mapped to lines in X , and such that the induced morphism $\phi : \mathbb{P}^1 \rightarrow U$ is relatively free.

Remark 16. This is essentially the same situation considered by M. Deland [?]. He considered a seemingly more general situation of $f : W \rightarrow X$, without assuming W is the universal family of lines and, hence, no hypothesis (C) involved (but W is assumed to be smooth). However, his main theorem, when formulated in this situation, is just a direct consequence of well-known results of smoothing of combs (see, for example, [Kol96, Chapter II, Proposition 7.10, Lemma 7.10.1]). Since general fibers of f are rationally connected, thus it is very easy to find curves satisfying conditions (A) and (B) simultaneously, without controlling the T_π -degree,

(e.g., a very free curve in a general fiber works). It is the tension between positivity of ϕ^*T_f and ϕ^*T_π that makes very twisting surfaces difficult to find.

The Fano scheme of X naturally embeds into the Grassmannian of lines in a projective space, coming from the embedding of X by $\mathcal{O}_X(H)$. Thus there is an ample line bundle $\mathcal{O}_F(1)$, which is the pull-back of the ample generator of the Picard group of the Grassmannian. For a fiber U_x of $f : U \rightarrow X$, the restriction of $\pi^*\mathcal{O}_F(1)$ may not be a generator of the Picard group, although this is the case in our main application later. Since we assume the Picard group of U_x is cyclic for x general, to ease the notations, we use rational numbers to index all the line bundles. The normalization is taken in such a way that $\pi^*\mathcal{O}_F(1)|_{U_x}$ is indexed as 1. This convention applies in the statement of the following theorem.

Theorem 17. *In Situation 13, either a general fiber U_x of $f : U \rightarrow X$ satisfies $H^0(U_x, \Lambda^r \Omega_{U_x}(e)) \neq 0, e \leq \frac{r}{2}$ for some r and e , or there is a map $\phi : \mathbb{P}^1 \rightarrow U$ such that the following hold:*

- (A) *The bundle $\phi^*f^*T_X$ is globally generated.*
- (B) *The sheaf ϕ^*T_f is locally free and ample.*
- (C) *The line bundle ϕ^*T_π has positive degree.*

Definition 18. A rational curve $\phi : \mathbb{P}^1 \rightarrow U$ is *relatively very free* if it satisfies all the conditions (A), (B), (C) as in Theorem 17.

A very twisting surface (Σ, D) consists of a ruled surface Σ over $\mathbb{P}^1$ and a divisor $D \subset \Sigma$, which is the image of a section of $\Sigma \rightarrow \mathbb{P}^1$, together with a morphism $\Sigma \rightarrow X$, such that the rulings are mapped to lines in X , and such that the induced morphism $\phi : \mathbb{P}^1 \rightarrow U$ is relatively very free.

Remark 19. We replace the geometric condition in the statement of the main theorem in [?] by a cohomological one. We also allow the Fano scheme to be singular.

There are several cases where one can easily verify that the global sections do not exist, for example, when X is a complete intersection in a projective space.

Theorem 20. *Let X be a smooth complete intersection of type $(d_1, \dots, d_c)$ in $\mathbb{P}^n$ such that $\sum d_i^2 \leq n$. Then X admits a very twisting surface, and is rationally simply connected.*

Remark 21. For the precise definition of rational simple connectedness and its application to function field arithmetic, we refer the readers to [dJHS11].

To begin the proof, we first make one simple observation. The general fiber of $f : U \rightarrow X$ is smooth and connected, thus irreducible. Therefore, there is a unique irreducible component F^0 of F and a family of lines $U^0 \rightarrow F^0$ such that $f : U^0 \rightarrow X$ is dominant. We choose a strong resolution of singularities $\tilde{F} \rightarrow F^0$ (that is, $\tilde{F} \rightarrow F^0$ is an isomorphism over the smooth locus of F^0), and we denote the pullback of the universal family of lines by $\tilde{U} \rightarrow \tilde{F}$. The relative free curves lie in the smooth locus of the morphism $U \rightarrow X$, and thus they also lie in the smooth locus of U . Therefore they lie in the smooth locus of F^0 . This means that in the course of the proof we can (and will) assume the following by replacing F with $\tilde{F}$ (and U with $\tilde{U}$).

Situation 22. Let X be smooth connected projective variety over the complex numbers. Assume that the Picard group of X is generated by a very ample divisor H . We also write $\mathcal{O}_X(1)$ for $\mathcal{O}_X(H)$. There is a family of lines by $\pi : U \rightarrow F$ over a smooth projective base F , and the evaluation morphism by $f : U \rightarrow X$. Assume that f is dominant and F is birational to one irreducible component of the Fano scheme of lines of X . We assume that for a general $x \in X$, $f^{-1}(x)$ has Picard number 1 (in particular, it is connected). Suppose that there is a morphism $\phi : \mathbb{P}^1 \rightarrow U$ such that all of the following hold:

- (A) The bundle $\phi^* f^* T_X$ is globally generated.
- (B) The image of ϕ lie in the locus of free lines U^{free} . The sheaf $\phi^* T_f$ is locally free, globally generated, and has positive degree.
- (C) The line bundle $\phi^* T_\pi$ has non-negative degree.

The following is essentially due to Deland [?], with similar results proved by Shen [She10]. In fact, Deland uses the same argument as Shen. Even though Deland assumes that U is smooth in his setup, the argument of Shen works in the normal setting as well.

Definition 23. Given a relatively free curve $\psi : C \rightarrow U$, we write

$$\psi^* T_f \cong \bigoplus_{i=1}^p \mathcal{O}(a_i) \oplus (\bigoplus \mathcal{O}), a_i > 0.$$

The *positive rank* of ψ (sometimes *positivity*) is the number p .

A relatively free curve $\psi : C \rightarrow U$ is *maximal* if its positive rank is maximal among all relative free curves.

Lemma 24. *Given two relatively free curves glued at a point, a general deformation of the union is again relatively free by upper-semi-continuity of cohomology. If one of them is maximal, a general deformation is also maximal.*

Lemma 25. [?, 3.10-3.12] [She10, Proposition 2.5] *There is an algebraically integrable foliation $D \subset T_f \subset T_U$ such that, for a general maximal relatively free curve $\psi : C \rightarrow U$, the foliation D is locally free along $\psi(C)$, and, if we write*

$$\psi^* T_f \cong \bigoplus_{i=1}^p \mathcal{O}(a_i) \oplus \bigoplus \mathcal{O}, a_i > 0,$$

then $\psi^ D$ is isomorphic to $\bigoplus_{i=1}^p \mathcal{O}(a_i)$.*

Remark 26. One can show that $f^{-1}(x)$ is uniruled ([?, Remark 3.14, Theorem 3.15]). Since it has Picard number one, it is Fano and rationally connected. It then follows that the Picard group of $f^{-1}(x)$ is isomorphic to $\mathbb{Z}$.

Construction 27. Denote by Q the quotient of $D \subset T_f$ in Lemma 25. Assume the rank of Q is $r > 1$ (otherwise we are done), then $\mathcal{L} = (\Lambda^r Q)^*$ is a reflexive sheaf of rank 1, and thus a line bundle on U since U is smooth. We have an inclusion

$$\mathcal{L} \subset (\Lambda^r \Omega_f)^{**}.$$

Corollary 28. *There is a $\mathbb{Q}$ -Weil divisor E on U , such that*

$$c_1(\mathcal{L}) \sim_{\mathbb{Q}} E + k c_1(\pi^* \mathcal{O}_F(1)) + k' c_1(f^* \mathcal{O}_X(1)), k, k' \in \mathbb{Q}.$$

Here, for every invertible sheaf $\mathcal{A}$, we use $c_1(\mathcal{A})$ to denote the underlying Weil divisor of any Cartier divisor on U whose associated invertible sheaf is isomorphic to $\mathcal{A}$. The morphism f maps the support of the divisor E to a closed subset of X whose codimension is at least 2. Furthermore, a relative free curve can be deformed away from the support of E .

Proof. The line bundle $\mathcal{L}$ corresponds to a Weil divisor $D_{\mathcal{L}}$ on U . Since U_x has cyclic Picard group for a general $x \in X$, we have $\pi^*(E')|_{U_x} \sim_{\mathbb{Q}} k\pi^*\mathcal{O}_F(1)$. Then

$$D_{\mathcal{L}} - kc_1(\pi^*\mathcal{O}_F(1)) \sim_{\mathbb{Q}} \sum a_{ij}D_{ij} + \sum b_k E_k, a_{ij} \in \mathbb{Q},$$

where D_{ij}, E_k are prime divisors in U such that $D_i = f(D_{ij})$ is a divisor in X and $f(E_k)$ has codimension at least 2.

We restrict to an open subset V of U such that $f : V \rightarrow X$ is flat. The open subset V may be chosen such that $f(V)$ is open in X and the complement of $f(V)$ has codimension at least 2. In particular, all the D_i intersects $f(V)$ non-trivially. Also note that $E_k \cap V = \emptyset$. It then suffices to show that for a fix i , $\sum_j a_{ij}D_{ij}$ is linearly equivalent to the a multiple of $c_1(f^*(\mathcal{O}_X(1)))$ on V .

Note that $f^*D_i|_V = f^{-1}(D_i)|_V$. For each D_i , there is a unique rational number k_i such that $D'_i = \sum a_{ij}D_{ij}|_V + t_i f^{-1}D_i|_V$ is an effective Weil divisor and such that some irreducible component of $f^{-1}(D_i)$ does not appear in the expression. By construction $D'_i|_{U_x} \sim_{\mathbb{Q}} 0$ for every $x \in X$. In particular, we may choose x to lie in $D_i \cap f(V)$. We can find a curve C_x in U_x that is not contained in the support of D'_i and that intersects the support of D'_i . So, by contradiction, $D'_i|_V$ has to be the zero divisor. Set k to be $\sum k_i$. Then we have

$$c_1(\mathcal{L}) \sim_{\mathbb{Q}} c_1(\mathcal{O}_U(E)) + kc_1(\pi^*\mathcal{O}_F(1)) + k'c_1(f^*\mathcal{O}_X(1)), \quad k, k' \in \mathbb{Q}.$$

Moreover, the support of E maps to a closed subset in X whose codimension is at least 2.

For a relatively free curve $\phi : \mathbb{P}^1 \rightarrow U$, since T_f is globally generated, the deformation space of ϕ dominates the deformation space of $f \circ \phi$. Since $f \circ \phi : \mathbb{P}^1 \rightarrow X$ is a free curve, a general deformation of $f \circ \phi$ is disjoint from any specified codimension 2 closed subset. Therefore, a general deformation of a relatively free curve is disjoint from the support of E . $\square$

Lemma 29. *In Situation 22, if either the $\mathbb{Q}$ -span of the numerical equivalence classes of relatively free curves is at least 2-dimensional or $\mathcal{L} \sim_{\mathbb{Q}} \mathcal{O}_U(D)$ in Corollary 28, then existence of relatively very free curves holds in Theorem 17.*

Proof. If the $\mathbb{Q}$ -span of the numerical equivalence classes of relatively free curves is at least 2-dimensional, the same is true for the class spanned by maximal relatively free curves by Lemma 24. Since $\mathcal{L}$ has degree 0 with respect to maximal relatively free curves, $\mathcal{L}$ is $\mathbb{Q}$ -linearly equivalent to $\mathcal{O}_U(E)$ in Corollary 28. Thus it suffices to consider the case $\mathcal{L} \sim_{\mathbb{Q}} \mathcal{O}_U(E)$. Then $\mathcal{L}|_{U_x}$ is trivial. So there is a global section of $(\Lambda^r \Omega_f|_{U_x})^{**} = \Lambda^r \Omega_{U_x}$ given by the inclusion $\mathcal{L} \subset (\Lambda^r \Omega_f|_{U_x})^{**}$. But this is impossible, since a general fiber $f^{-1}(x)$ is Fano (Remark 26). $\square$

Lemma 30. *In Situation 22, if ϕ^*T_{π} has positive degree, then existence of relatively very free curve part of Theorem 17 holds.*

Proof. First recall that $f^{-1}(x)$ is Fano and rationally connected (Remark 26). A very free curve of a general fiber $\psi : C \rightarrow f^{-1}(x) \subset U$ is a free curve in U , and the sheaf ψ^*T_f is locally free and ample. We also know that $\deg \psi^*T_{\pi}$ is negative.

By composing ϕ with high degree endomorphisms of $\mathbb{P}^1$, we may assume that $\deg \phi^*T_{\pi} + \deg \psi^*T_{\pi}$ is positive. Thus a general deformation of a union of a relatively free curve with sufficiently positive T_{π} -degree and a general very free curve

in a general fiber of f is again a relatively free curve. This shows that the subspace in $H_2(U, \mathbb{Q})$ spanned by classes of relatively free curves is at least 2-dimensional. Thus the statement follows from Lemma 29. $\square$

Lemma 31. *Let L be a general fiber of π . Then $T_f|_L$ is isomorphic to $\oplus \mathcal{O}(-1)$.*

Proof. Note that $T_f|_L$ is the kernel of the map

$$T_U|_L \rightarrow f^*T_X|_L \rightarrow 0,$$

which, by deformation theory, can be identified with the kernel of the evaluation map

$$H^0(L, N_{L/X}) \otimes \mathcal{O}_L \rightarrow N_{L/X}.$$

For a general L ,

$$N_{L/X} \cong \mathcal{O}(1) \oplus \dots \oplus \mathcal{O}(1) \oplus \mathcal{O} \oplus \dots \oplus \mathcal{O}.$$

The statement follows from this description. $\square$

Proof of Theorem 17. We write $\mathcal{L} \sim_{\mathbb{Q}} \mathcal{O}_U(E + k[\pi^*\mathcal{O}_F(1)] + k'[f^*\mathcal{O}_X(1)])$ as in Corollary 28 (recall that k, k' could be rational numbers).

In the following we will restrict $\mathcal{L}$ and $(\Lambda^r \Omega_f)^{**}$ (see Construction 27) to a general fiber of $\pi : U \rightarrow F$ (resp. $f : U \rightarrow X$). The sheaf $(\Lambda^r \Omega_f)^{**}$ are locally free near general fibers of π (resp. f by assumption). and E does not intersect a general fiber of $\pi : U \rightarrow F$ (resp. f , by Corollary 28).

We restrict the inclusion $\mathcal{L} \subset (\Lambda^r \Omega_f)^{**}$ to a general fiber of $\pi : U \rightarrow F$. The map of sheaves is still injective. By Lemma 31, $(\Lambda^r \Omega_f)^{**} \cong \oplus \mathcal{O}_{\mathbb{P}^1}(r)$. Therefore, we must have $k' \leq r$.

Let $\phi : C \rightarrow U$ be a relatively free curve. Denote by S^* the dual of the rank 2 universal subbundle S on F . Since S^* is a nef vector bundle and $\Lambda^2 S^*$ is ample, we may write

$$\phi^* \pi^* S^* \cong \mathcal{O}(a) \oplus \mathcal{O}(a+b), a > 0, b \geq 0.$$

The family $U \rightarrow F$ is just $\mathbb{P}_F(S^*)$, where the relative projective space bundle consists of one-dimensional quotients of fibers of S^* . So the morphism $\phi : C \rightarrow U$ is determined by a short exact sequence

$$0 \rightarrow K \rightarrow \phi^* \pi^* S^* \cong \mathcal{O}(a) \oplus \mathcal{O}(a+b) \rightarrow M \rightarrow 0$$

where K and M are invertible sheaves. Then $\phi^* T_\pi$ is isomorphic to $K^* \otimes M$. Since $\phi^* T_\pi$ has non-negative degree, $\deg M \geq \deg K$. As a result, K cannot be $\mathcal{O}(a+b)$. Thus $\deg M \geq a+b$ and $\deg \phi^* T_\pi \geq b$. We may assume that $\deg \phi^* T_\pi = b = 0$, otherwise we have a relatively very free curve by Lemma 30. Since we have an isomorphism $M \cong \phi^* f^* \mathcal{O}_X(1)$, we have $\deg \phi^* f^* \mathcal{O}_X(1)$ equals a .

Then $\deg \phi^* \pi^* \mathcal{O}_F(k)$ equals $k \deg \Lambda^2 S^* = 2ka > 0$, and $0 = \deg \phi^* \mathcal{L} = (2k + k')a$. Thus k' equals $-2k$.

We restrict the inclusion $\mathcal{L} \subset (\Lambda^r \Omega_f)^{**}$ to a general smooth fiber U_x of $f : U \rightarrow X$. This gives an inclusion $\mathcal{O}_F(k)|_{U_x} \subset (\Lambda^r \Omega_f)^{**}|_{U_x} \cong \Lambda^r \Omega_{U_x}$, or equivalently, a global section of $\Lambda^r \Omega_{U_x}(-k)$, $1 \leq -k \leq \frac{r}{2}$. $\square$

Proof of Theorem 20. Previous work of de Jong-Starr has reduced the proof to the existence of relative very free curves. They have also proven the existence of relative free curves [dJS06, Corollary 6.11 (take $m = 2$) and Lemmas 7.8 and 7.10]. It is well-known that in this case U_x is a smooth complete intersection in a projective space. Thus $\mathcal{O}_{U_x}(1) = \pi^*\mathcal{O}_F(1)|_{U_x}$ generates the Picard group of U_x . Thus by Theorem 17, it suffices to check that $H^0(U_x, \Omega_{U_x}^i(e)) = 0, 2e \leq i$, and we only need to check integer valued e . By [Ben13, Lemma 3.3], for any smooth complete intersection Z , we have that $\dim H^p(Z, \Omega_Z^q(t)) = 0$, for all $p + q < \dim Z, t < q - p$. The statement follows. $\square$

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