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MAT 544 Problem Set 5 Solutions

Problems.

Problem 1 Let (S, d_S) be a metric space and let X be a bounded, closed subset of a Banach space $(W, \| \bullet \|_W)$. Let $K : S \times X \to X$ be as in Corollary 4 on p. 230, i.e., K is continuous and there exists a positive real number $C < 1$ such that for every $s \in S$, the map $K_s : X \to X$ by $K_s(x) = K(s, x)$ is C-Lipschitz. Denote by $BC(S, X)$ the subset of $BC(S, W)$ parameterizing bounded continuous functions with image in X.

(a) Prove that $BC(S, X)$ is a closed subset of $BC(S, W)$. Combined with Theorem 4.7.5 on p. 218, it follows that $BC(S, X)$ is a complete metric space.

(b) For every $f : S \to X$ in $BC(S, X)$, define $\tilde{K}(f) : S \to X$ by $s \mapsto K(s, f(s))$. Prove that $\tilde{K}(f)$ is an element of $BC(S, X)$.

(c) Prove that the map $\tilde{K} : BC(S, X) \to BC(S, X)$ by $f \mapsto \tilde{K}(f)$ is C-Lipschitz. Apply the contraction mapping fixed point theorem to give a second proof of Corollary 4 (in this context).

Nota Bene. Corollary 4 is more general since X need not be a closed bounded subset of a Banach space. If X is a subset of a Banach space W, then it is valid to replace X by the intersection of the closure of X with a bounded ball in W by the estimates in Corollaries 1 – 3 together with the theorem from lecture that a uniformly continuous (e.g., Lipschitz) function on a metric space X extends to a continuous function on the completion of the domain (i.e., the closure of X in W). In practice the metric spaces X we work with usually are subsets of Banach spaces.

Problem 2 Let $(V, \| \bullet \|_V)$ be a normed vector space, let $(W, \|bullet\|_W)$ be a Banach space. Let $\tilde{V} \subset V$ and $\tilde{W} \subset W$ be open subsets. Let $K : \tilde{V} \times \tilde{W} \to W$ be a continuous function such that for every $\vec{v} \in \tilde{V}$, the induced morphism $K_{\vec{v},\bullet} : \tilde{W} \to W$, $\vec{w} \mapsto K(\vec{v}, \vec{w})$ is differentiable. Let C be a positive real number such that $C < 1$. Assume that for every $\vec{v} \in S$ and for every $\vec{w} \in W$, $\|d(K_{\vec{v},\bullet})_{\vec{w}}\|_{\text{op}} \leq C$ so that $K_{\vec{v},\bullet}$ is C-Lipschitz. Let $\vec{v}_0 \in \tilde{V}$ and $\vec{w}_0 \in \tilde{W}$ be elements such that $K_{\vec{v}_0,\bullet}(\vec{w}_0) = \vec{w}_0$.

(a) Using Corollaries 1 – 3 on pp. 229-230 if necessary, prove that there exist real numbers $\delta_V > 0$ and $\delta_W > 0$ such that

(i) The ball $S = B_{\delta_V}(\vec{v}_0)$ is contained in $\tilde{V}$, and the closed ball $X = B_{\leq \delta_W}(\vec{w}_0)$ is contained in $\tilde{W}$.

(ii) The continuous map K maps $S \times X$ into X.

(b) Denote by $c_{\vec{w}_0} : S \to X$ the constant function $c_{\vec{w}_0}(\vec{v}) = \vec{w}_0$. Apply **Problem 1** to conclude that the sequence $(\tilde{K}^n(c_{\vec{w}_0}))_{n=0,1,2,\dots}$ converges in $BC(S,X)$ to the unique continuous function $f : S \to X$ from Corollary 4.

(c) Finally assume that $G : \tilde{V} \times \tilde{W} \to W$ is a continuous function such that every $G_{\vec{v},\bullet}$ is differentiable and the derivatives $d(G_{\vec{b},\bullet})_{\vec{w}}$ vary continuously in $(\vec{v}, vecw)$. Let $\vec{v}_0 \in \tilde{V}$ and $\vec{w}_0 \in \tilde{W}$ be elements such that $G_{\vec{v}_0,\bullet}(\vec{w}_0) = \vec{0}_W$. Modify (or simply quote) the arguments in the proof of Theorem 4.9.3, pp. 230-231, to show that up to replacing $\tilde{V}$ by a small open ball about $\vec{v}_0$ and up to replacing $\tilde{W}$ by a small open ball about $\vec{w}_0$, the map $K_{\vec{v},\bullet}(\vec{w}) := \vec{w} - T^{-1}(G_{\vec{v},\bullet}(\vec{w}))$ satisfies the hypothesis in (a). As above, conclude that $(\tilde{K}^n(c_{\vec{w}_0}))_{n=0,1,2,\dots}$ converges in $BC(S,X)$ to the unique continuous function $f : S \to X$ such that $G(\vec{v}, f(\vec{v})) = 0$.

Problem 3 With the same notation as above, let $V = W = \mathbb{R}$ and let $G : V \times W \to W$ be the function $G(x,y) = (1+x) - (1+y)^2$, so that $G(x_0,y_0) = 0$ for the point $(x_0,y_0) = (0,0)$. Compute T and T^{-1}. Compute $K(x,y)$ and compute $\tilde{K}(f(x))$. Starting with the constant function $c_0(x) = 0$, compute the first three iterates $\tilde{K}(c_0)$, $\tilde{K}(\tilde{K}(c_0))$ and $\tilde{K}(\tilde{K}(\tilde{K}(c_0)))$. How do these compare to the Taylor approximations to $\sqrt{1+x} - 1$ about $x_0 = 0$?

Problem 4 Let n be a positive integer. Let V and W both be the vector space $L(\mathbb{R}^n, \mathbb{R}^n)$ of linear operators on $\mathbb{R}^n$. Denote by $\mathrm{Id}_{\mathbb{R}^n}$ the identity matrix. Let $G : V \times W \to W$ be the function $G(X,Y) = (\mathrm{Id}_{\mathbb{R}^n} + X) \circ (\mathrm{Id}_{\mathbb{R}^n} + Y) - \mathrm{Id}_{\mathbb{R}^n}$, so that $G(X_0,Y_0) = 0$ for the point $(X_0,Y_0) = (0,0)$. Compute T and T^{-1}. Compute $K(X,Y)$ and compute $\tilde{K}(f(X))$. Starting with the constant function $c_0(X) = 0$, compute the first three iterates $\tilde{K}(c_0)$, $\tilde{K}(\tilde{K}(c_0))$ and $\tilde{K}(\tilde{K}(\tilde{K}(c_0)))$. How do these compare to the "Taylor approximations" to $(\mathrm{Id}_{\mathbb{R}^n} + X)^{-1}$ about $X_0 = 0$?

Problem 5 Find an example of a continuously differentiable function $G : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that $G(0,0) = 0$, yet with $(dG_{0,\bullet})_0$ noninvertible and such that there is no continuous function $f : (-\epsilon_V, \epsilon_V) \to (-\epsilon_W, \epsilon_W)$ with $G(x, f(x)) = 0$.

Solutions to Problems.

Solution to (1)

Solution to (a) The simplest argument uses sequences. Let $(f_n)_{n=1,2,\dots}$ be a sequence of elements of $BC(S,X)$ which converges to f in $BC(S,W)$ with respect to the uniform norm. Because the sequence converges uniformly, for every $s \in S$, the sequence $(f_n(s))_{n=1,2,\dots}$ converges to $f(s)$ with respect to the norm on W. Since this is a sequence of elements in X, and since X is closed, the limit $f(s)$ is also in X. Thus for every $s \in S$, $f(s)$ is in X, i.e., f is in $BC(S,X)$.

Solution to (b) There are three things that need to be checked: that $\tilde{K}(f)$ is continuous, that $\tilde{K}(f)$ maps S into X, and that f is bounded. Since K is continuous and since f is continuous, also $K(s, f(s))$ is continuous. Since the image of K is in X, for every $s \in S$, $K(s, f(s))$ is in X. Finally, since X is bounded, every function into X is bounded. Hence $\tilde{K}(f)$ is in $BC(S,X)$.

Solution to (c) Let f_1, f_2 be elements of $BC(S,X)$. By definition of the uniform norm,

$$\|\tilde{K}(f_1) - \tilde{K}(f_2)\|_{\mathrm{un}} = \mathrm{lub}\{\|K_s(f_1(s)) - K_s(f_2(s))\|_W : s \in S\}.$$

Since K_s is C-Lipschitz by hypothesis, we have

$$\|K_s(f_1(s)) - K_s(f_2(s))|_W \leq C\|f_1(s) - f_2(s)\|_W \leq C\|f_1 - f_2\|_{\text{un}}.$$

Since $C\|f_1 - f_2\|_{\text{un}}$ is an upper bound for every $s \in S$, we conclude that it is greater than or equal to the least upper bound, i.e.,

$$\|\tilde{K}(f_1) - \tilde{K}(f_2)\|_{\text{un}} \leq C\|f_1 - f_2\|_{\text{un}}.$$

Therefore $\tilde{K}$ is C-Lipschitz. Since $C < 1$, it follows that $\tilde{K}$ is a contraction. And by **(a)**, $BC(S,X)$ is a complete metric space. Therefore by the contraction mapping fixed point theorem, there exists a unique fixed point f in $BC(S,X)$, i.e., there exists a unique continuous function $f : S \to X$ such that $\tilde{K}(f) = f$. This precisely says that for every $s \in S$, $K_s(f(s)) = f(s)$, i.e., $f(s)$ is a fixed point of K_s. Of course we know that this fixed point is unique, so $f(s)$ is *the* fixed point of K_s. Since f is in $BC(S,X)$, it is continuous. So we conclude that the function $f : S \to X$ sending every s to the unique fixed point $f(s)$ of K_s is a continuous function.

Solution to (2)

Solution to (a) Because $\tilde{W}$ is open and $\vec{w}_0$ is in $\tilde{W}$, there exists a real number $r > 0$ such that the open ball $B_r(\vec{w}_0)$ is in $\tilde{W}$. Therefore for any positive real $\delta_W < r$, say $\delta_W := r/2$, the closed ball $B_{\leq \delta_W}(\vec{w}_0)$ is in $\tilde{W}$. Since K is continuous, in particular the function $K_{\bullet,\vec{w}_0} : \tilde{V} \to W$ by $\vec{v} \mapsto K(\vec{v}, \vec{w}_0)$ is also continuous. And $K_{\bullet,\vec{w}_0}(\vec{v}_0)$ equals $K(\vec{v}_0, \vec{w}_0)$, which equals $\vec{w}_0$ by hypothesis. Therefore there exists a real number $\delta_V > 0$ such that for every $\vec{v} \in B_{\delta_V}(\vec{v}_0)$, $K(\vec{v}, \vec{w}_0) = K_{\bullet,\vec{w}_0}(\vec{v})$ is in the ball $B_{(1-C)\delta_W}(\vec{w}_0)$, i.e.,

$$d_W(K(\vec{v}, \vec{w}_0), \vec{w}_0) < (1 - C)\delta_W.$$

Also $K_{\vec{v},\bullet}$ is C-Lipschitz. So for every $\vec{w} \in B_{\leq \delta_W}(\vec{w}_0)$, also we have

$$d_W(K(\vec{v}, \vec{w}), K(\vec{v}, \vec{w}_0)) \leq C d_W(\vec{w}, \vec{w}_0) \leq C\delta_W.$$

Thus by the triangle inequality, for every $\vec{v} \in B_{\delta_V}(\vec{v}_0)$ and for every $\vec{w} \in B_{\leq \delta_W}(\vec{w}_0)$,

$$d_W(K(\vec{v}, \vec{w}), \vec{w}_0) \leq d_W(K(\vec{v}, \vec{w}), K(\vec{v}, \vec{w}_0)) + d_W(K(\vec{v}, \vec{w}_0), \vec{w}_0) < C\delta_W + (1 - C)\delta_W = \delta_W.$$

Thus $K(\vec{v}, \vec{w})$ is in $B_{\leq \delta_W}(\vec{w}_0)$. Therefore, for $S = B_{\delta_V}(\vec{v}_0)$ and for $X = B_{\leq \delta_W}(\vec{w}_0)$, K maps $S \times X$ into X.

Solution to (a) By **Problem 1**, the map $\tilde{K} : BC(S,X) \to BC(S,X)$ is a contraction. Therefore, by the contraction mapping fixed point theorem, for any function $f_0 \in BC(S,X)$, the sequence $(\tilde{K}^n(f_0))_{n=0,1,2,\ldots}$ converges in $BC(S,X)$ to the unique fixed point. And again by **Problem 1**, that fixed point is the continuous function $f : S \to X$ from Corollary 4.

Solution to (b) With the choice of S and X from **(a)**, and using that K is C-Lipschitz by the mean value theorem, K satisfies the hypotheses of **Problem 1**. Thus, by that problem, $(\tilde{K}^n(c_{\vec{w}_0}))_{n=0,1,2,\ldots}$ converges in $BC(S,X)$ to the unique continuous function $f : S \to X$ from Corollary 4.

Solution to (c) The argument for this is as in the proof of Theorem 4.9.3, pp. 230-231. Since $d(K_{\vec{v}_0,\bullet})_{\vec{w}_0}$ is 0 by construction, and thus has operator norm 0, and since the derivative map is continuous, so that also the operator norms of the derivatives vary continuously, for any specified real number $C > 0$ (in particular, for C with $0 < C < 1$), for all $\vec{v}$ in a sufficiently small ball about $\vec{v}_0$ and for all $\vec{w}$ in a sufficiently small ball about $\vec{w}_0$, the operator norm of $d(K_{\vec{v},\bullet})_{\vec{w}}$ is $\leq C$. Combined with the mean value theorem, this implies that $K_{\vec{v},\bullet}$ is C-Lipschitz on this small ball. Now apply **(a)** and **Problem 1**.

Solution to (3) For fixed $x \in \mathbb{R}$, the function $G_{x,\bullet}(y) = (1+x)-(1+y)^2$ is a polynomial function in y. The usual single variable derivative is the partial derivative,

$$\frac{\partial G}{\partial y} = -2(1+y).$$

The derivative linear transformation $d(G_{x,\bullet})_y : W \to W$ is the linear transformation $\Delta y \mapsto -2(1+y)\Delta y$. In particular, $T := d(G_{0,\bullet})_0$ is the linear transformation $\Delta y \mapsto -2\Delta y$. So the inverse linear transformation is $T^{-1}(\Delta y) = (-1/2)\Delta y$. Therefore we have

$$K(x,y) = y - T^{-1}(G(x,y)) = y - \frac{-1}{2}((1+x)-(1+y)^2) = y + \frac{1}{2}(x-2y-y^2) = \frac{1}{2}(x-y^2).$$

Therefore for a function $f_n(x)$, the function $f_{n+1} := \tilde{K}(f_n)$ is

$$f_{n+1}(x) = \frac{1}{2}(x-(f_n(x))^2).$$

Starting with $f_0(x) = c_0(x) = 0$, this gives first

$$f_1(x) = \frac{1}{2}(x-0^2) = \frac{1}{2}x,$$

next

$$f_2(x) = \frac{1}{2}\left(x - \left(\frac{1}{2}x\right)^2\right) = \frac{1}{2}x - \frac{1}{8}x^2,$$

and finally

$$f_3(x) = \frac{1}{2}\left(x - \left(\frac{1}{2}x - \frac{1}{8}x^2\right)^2\right) = \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{1}{128}x^4.$$

On the other hand, the Taylor series of $f(x) = \sqrt{1+x}-1$ is

$$0 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \dots$$

So the degree n Taylor approximation of $f_n(x)$ equals the degree n Taylor approximation of $f(x)$ for each $n = 0,1,2,3$.

Solution to (4) There are a number of elementary results from multivariable calculus which are useful for computing derivatives. One is that every constant function is differentiable with zero derivative. Another is that the sum of two differentiable functions is differentiable and the derivative of the sum equals the sum of the derivatives. A final result, used previously, is that every bounded linear transformation is differentiable with derivative equal to that same linear transformation. In particular,

$$G_{X,\bullet}(Y) = (\mathrm{Id}_{\mathbb{R}^n} + X) \circ Y + X$$

is the sum of the linear transformation $(\mathrm{Id}_{\mathbb{R}^n} + X) \circ Y$ and the constant function X. Thus the derivative is the sum of the derivatives, the first of which is $(\mathrm{Id}_{\mathbb{R}^n} + X) \circ Y$ and the second of which is 0. Thus we have

$$d(G_{X,\bullet})_Y(\Delta Y) = (\mathrm{Id}_{\mathbb{R}^n} + X) \circ \Delta Y.$$

In particular, the derivative at $(X_0, Y_0) = (0, 0)$ is

$$T(\Delta Y) = d(G_{X_0,\bullet})_{Y_0}(\Delta Y) = (\mathrm{Id}_{\mathbb{R}^n} + 0) \circ \Delta Y = \Delta Y.$$

In other words, T is the identity transformation. So the inverse linear transformation T^{-1} is also the identity transformation. Therefore we have

$$K(X,Y) = Y - T^{-1}(G(X,Y)) = Y - G(X,Y) = Y - (X + Y + X \circ Y) = -X - X \circ Y.$$

Therefore for a function $f_n(X)$, the function $f_{n+1} := \tilde{K}(f_n)$ is

$$f_{n+1}(X) = -X - X \circ f_n(X).$$

Starting with $f_0(X) = c_0(X) = 0$, this gives first

$$f_1(X) = -X - X \circ 0 = -X - 0 = -X,$$

next

$$f_2(X) = -X - X \circ (-X) = -X + X^2,$$

and finally

$$f_3(X) = -X - X \circ (-X + X^2) = -X + X^2 - X^3.$$

On the other hand, the Taylor series of $f(X) = (\mathrm{Id}_{\mathbb{R}^n} + X)^{-1} - \mathrm{Id}_{\mathbb{R}^n}$ is

$$0 - X + X^2 - X^3 + X^4 + \cdots + (-1)^n X^n + \dots$$

So the degree n Taylor approximation of $f(x)$ equals $f_n(X)$ for each $n = 0, 1, 2, 3$.

Solution to (5) There are many examples. One such is $G(x, y) = x - y^2$. Of course the partial derivative with respect to y at $(0, 0)$ is

$$\left.\frac{\partial G}{\partial y}\right|_{(0,0)} = (-2y|_{(0,0)} = 0.$$

Moreover, for every $x < 0$, there is no choice of y such that $y^2 = x$. So there is no function, continuous or otherwise, defined on $(-\epsilon_W, 0)$ with $G(x, f(x)) = 0$.