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MAT 544 Problem Set 5

Problems.

Problem 1 Let (S, d_S) be a metric space and let X be a bounded, closed subset of a Banach space $(W, \| \bullet \|_W)$. Let $K : S \times X \to X$ be as in Corollary 4 on p. 230, i.e., K is continuous and there exists a positive real number $C < 1$ such that for every $s \in S$, the map $K_s : X \to X$ by $K_s(x) = K(s, x)$ is C-Lipschitz. Denote by $BC(S, X)$ the subset of $BC(S, W)$ parameterizing bounded continuous functions with image in X.

(a) Prove that $BC(S, X)$ is a closed subset of $BC(S, W)$. Combined with Theorem 4.7.5 on p. 218, it follows that $BC(S, X)$ is a complete metric space.

(b) For every $f : S \to X$ in $BC(S, X)$, define $\tilde{K}(f) : S \to X$ by $s \mapsto K(s, f(s))$. Prove that $\tilde{K}(f)$ is an element of $BC(S, X)$.

(c) Prove that the map $\tilde{K} : BC(S, X) \to BC(S, X)$ by $f \mapsto \tilde{K}(f)$ is C-Lipschitz. Apply the contraction mapping fixed point theorem to give a second proof of Corollary 4 (in this context).

Nota Bene. Corollary 4 is more general since X need not be a closed bounded subset of a Banach space. If X is a subset of a Banach space W, then it is valid to replace X by the intersection of the closure of X with a bounded ball in W by the estimates in Corollaries 1 – 3 together with the theorem from lecture that a uniformly continuous (e.g., Lipschitz) function on a metric space X extends to a continuous function on the completion of the domain (i.e., the closure of X in W). In practice the metric spaces X we work with usually are subsets of Banach spaces.

Problem 2 Let $(V, \| \bullet \|_V)$ be a normed vector space, let $(W, \|bullet\|_W)$ be a Banach space. Let $\tilde{V} \subset V$ and $\tilde{W} \subset W$ be open subsets. Let $K : \tilde{V} \times \tilde{W} \to W$ be a continuous function such that for every $\vec{v} \in \tilde{V}$, the induced morphism $K_{\vec{v},\bullet} : \tilde{W} \to W$, $\vec{w} \mapsto K(\vec{v}, \vec{w})$ is differentiable. Let C be a positive real number such that $C < 1$. Assume that for every $\vec{v} \in S$ and for every $\vec{w} \in W$, $\|d(K_{\vec{v},\bullet})_{\vec{w}}\|_{\text{op}} \leq C$ so that $K_{\vec{v},\bullet}$ is C-Lipschitz. Let $\vec{v}_0 \in \tilde{V}$ and $\vec{w}_0 \in \tilde{W}$ be elements such that $K_{\vec{v}_0,\bullet}(\vec{w}_0) = \vec{w}_0$.

(a) Using Corollaries 1 – 3 on pp. 229-230 if necessary, prove that there exist real numbers $\delta_V > 0$ and $\delta_W > 0$ such that

(i) The ball $S = B_{\delta_V}(\vec{v}_0)$ is contained in $\tilde{V}$, and the closed ball $X = B_{\leq \delta_W}(\vec{w}_0)$ is contained in $\tilde{W}$.

(ii) The continuous map K maps $S \times X$ into X.

(b) Denote by $c_{\vec{w}_0} : S \to X$ the constant function $c_{\vec{w}_0}(\vec{v}) = \vec{w}_0$. Apply **Problem 1** to conclude that the sequence $(\tilde{K}^n(c_{\vec{w}_0}))_{n=0,1,2,\dots}$ converges in $BC(S,X)$ to the unique continuous function $f : S \to X$ from Corollary 4.

(c) Finally assume that $G : \tilde{V} \times \tilde{W} \to W$ is a continuous function such that every $G_{\vec{v},\bullet}$ is differentiable and the derivatives $d(G_{\vec{b},\bullet})_{\vec{w}}$ vary continuously in $(\vec{v}, vecw)$. Let $\vec{v}_0 \in \tilde{V}$ and $\vec{w}_0 \in \tilde{W}$ be elements such that $G_{\vec{v}_0,\bullet}(\vec{w}_0) = 0_W$. Modify (or simply quote) the arguments in the proof of Theorem 4.9.3, pp. 230-231, to show that up to replacing $\tilde{V}$ by a small open ball about $\vec{v}_0$ and up to replacing $\tilde{W}$ by a small open ball about $\vec{w}_0$, the map $K_{\vec{v},\bullet}(\vec{w}) := \vec{w} - T^{-1}(G_{\vec{v},\bullet}(\vec{w}))$ satisfies the hypothesis in (a). As above, conclude that $(\tilde{K}^n(c_{\vec{w}_0}))_{n=0,1,2,\dots}$ converges in $BC(S,X)$ to the unique continuous function $f : S \to X$ such that $G(\vec{v}, f(\vec{v})) = 0$.

Problem 3 With the same notation as above, let $V = W = \mathbb{R}$ and let $G : V \times W \to W$ be the function $G(x,y) = (1+x)-(1+y)^2$, so that $G(x_0,y_0) = 0$ for the point $(x_0,y_0) = (0,0)$. Compute T and T^{-1}. Compute $K(x,y)$ and compute $\tilde{K}(f(x))$. Starting with the constant function $c_0(x) = 0$, compute the first three iterates $\tilde{K}(c_0)$, $\tilde{K}(\tilde{K}(c_0))$ and $\tilde{K}(\tilde{K}(\tilde{K}(c_0)))$. How do these compare to the Taylor approximations to $\sqrt{1+x} - 1$ about $x_0 = 0$?

Problem 4 Let n be a positive integer. Let V and W both be the vector space $L(\mathbb{R}^n, \mathbb{R}^n)$ of linear operators on $\mathbb{R}^n$. Denote by $\mathrm{Id}_{\mathbb{R}^n}$ the identity matrix. Let $G : V \times W \to W$ be the function $G(X,Y) = (\mathrm{Id}_{\mathbb{R}^n} + X) \circ (\mathrm{Id}_{\mathbb{R}^n} + Y) - \mathrm{Id}_{\mathbb{R}^n}$, so that $G(X_0,Y_0) = 0$ for the point $(X_0,Y_0) = (0,0)$. Compute T and T^{-1}. Compute $K(X,Y)$ and compute $\tilde{K}(f(X))$. Starting with the constant function $c_0(X) = 0$, compute the first three iterates $\tilde{K}(c_0)$, $\tilde{K}(\tilde{K}(c_0))$ and $\tilde{K}(\tilde{K}(\tilde{K}(c_0)))$. How do these compare to the "Taylor approximations" to $(\mathrm{Id}_{\mathbb{R}^n} + X)^{-1}$ about $X_0 = 0$?

Problem 5 Find an example of a continuously differentiable function $G : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that $G(0,0) = 0$, yet with $(dG_{0,\bullet})_0$ noninvertible and such that there is no continuous function $f : (-\epsilon_V, \epsilon_V) \to (-\epsilon_W, \epsilon_W)$ with $G(x, f(x)) = 0$.