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Fall 2011

MAT 544 Problem Set 4 Solutions

Problems.

Problem 1 Look over the problems and solutions from Midterm 1, now posted on the exams page of the course webpage. Understand the correct solutions to each of the problems. Each student will get full credit for this problem; there is nothing to write up or turn in (but I do expect each student to actually understand the solutions).

Problem 2 For an $\mathbb{R}$-vector space U, recall that two norms $\|\bullet\|_U$ and $\|\bullet\|'_U$ are *equivalent* if there exist real numbers $0 < m, M$ such that for every $\vec{u} \in U$,

$$m\|\vec{u}\|_U \leq \|\vec{u}\|'_U \leq M\|\vec{u}\|_U.$$

For normed vector spaces $(V, \|\bullet\|_V)$ and $(W, \|\bullet\|_W)$, an open subset $O \subset V$, and a continuous function $F : O \to W$, recall that the function F is (Frechet) differentiable at $\vec{v}_0 \in O$ if there exists a bounded linear operator

$$dF_{\vec{v}_0} : V \to W$$

such that for every real number $\epsilon > 0$, there exists a real number $\delta > 0$ with

$$\|F(\vec{v}_0 + \vec{v}) - F(\vec{v}_0) - dF_{\vec{v}_0}(\vec{v})\|_W \leq \epsilon\|\vec{v}\|_V$$

whenever $\|\vec{v}\|_V < \delta$. In this case the bounded linear operator $dF_{\vec{v}_0}$ is called the (Frechet) derivative of F at $\vec{v}_0$.
Prove that if $\|\bullet\|'_V$ is a norm on V which is equivalent to $\|\bullet\|_V$, then F is differentiable at $\vec{v}_9$ with respect to $\|\bullet\|_V$ if and only if it is differentiable at $\vec{v}_0$ with respect to $\|\bullet\|'_V$, and in this case the derivatives are equal. Similarly, if $\|\bullet\|'_W$ is a norm on W which is equivalent to $\|\bullet\|_W$, prove that F is differentiable at $\vec{v}_0$ with respect to $\|\bullet\|_W$ if and only if it is so with respect to $\|\bullet\|'_W$, and in this case the derivatives are equal. In particular, if V and W are finite dimensional, then all norms on V, respectively on W, are equivalent, hence differentiability is an intrinsic property.

Problem 3 Let V be a finite dimensional real vector space. Denote by $L(V, V)$ the vector space of linear transformations $T : V \to V$ (all of which are automatically bounded with respect to all norms on V). Denote by $\mathrm{Det}(T)$ and $\mathrm{Tr}(T)$ the determinant and trace of T (with respect to one, and hence any, basis for V). Thus

$$\mathrm{Det} : L(V, V) \to \mathbb{R}$$

is a continuous function and

$$\mathrm{Tr} : L(V, V) \to \mathbb{R}$$

is a (bounded) linear transformation. Prove that Det is differentiable at Id_V and $d(Det)_{\mathrm{Id}_V}$ equals Tr.

Problem 4 Let (X, d_X) be a metric space. For this problem, an *almost contraction* is a continous map $T : (X, d_X) \to (X, d_X)$ such that for all $x, x' \in X$ with $x \neq x'$, we have

$$d_X(T(x), T(x')) < d_X(x, x'),$$

where this is strict inequality.

(a) Prove that an almost contraction has at most one fixed point x, and this is the same as a point of X at which the following function attains a minimum

$$f : X \to \mathbb{R}_{\geq 0}, \quad f(x) = d_X(x, T(x)).$$

(b) Prove that every almost contraction of a compact metric space has a fixed point.

Problem 5 Let $d_{\mathbb{R}}$ be the usual distance function on $\mathbb{R}$, namely $d_{\mathbb{R}}(s, t) = |t - s|$. Find an example of an almost contraction $f : \mathbb{R} \to \mathbb{R}$ which has no fixed point. Conclude that in Problem 4 it does not suffice to replace "compact" by "complete" (although, of course, "complete" is sufficient in the Banach contraction mapping fixed point theorem).

Solutions to Problems.

Solution to (2) First we prove that differentiability is preserved by equivalent norms on the domain vector space V. Because "equivalence of norms" is indeed an equivalence relation, and in particular because it is symmetric, it suffices to prove that if $\| \bullet \|'_V$ is equivalent to $\| \bullet \|_V$, then differentiability of F at $\vec{v}_0$ with respect to $\| \bullet \|_V$ implies differentiability of F at $\vec{v}_0$ with respect to $\| \bullet \|'_V$, and that the derivative of F at $\vec{v}_0$ with respect to $\| \bullet \|_V$ equals the derivative $dF_{\vec{v}_0}$ of F at $\vec{v}_0$ with respect to $\| \bullet \|'_V$. Thus assume that F is differentiable at $\vec{v}_0$ with respect to $\| \bullet \|_V$, and denote the derivative by $dF_{\vec{v}_0}$. We must prove that for every positive real ϵ' there exists a postive real δ' such that for every $\vec{v} \in V$ with $\|\vec{v}\|'_V < \delta'$, we have

$$\|F(\vec{v}_0 + \vec{v}) - F(\vec{v}_0) - dF_{\vec{v}_0}(\vec{v})\|_W \leq \epsilon' \|\vec{v}\|'_V.$$

Since ϵ' and m are positive real numbers, so is $\epsilon := m\epsilon'$. Since F is differentiable at $\vec{v}_0$ with respect to $\| \bullet \|_V$, there exists a real number $\delta > 0$ such that for every $\vec{v} \in V$ with $\|\vec{v}\|_V < \delta$ we have

$$\|F(\vec{v}_0 + \vec{v}) - F(\vec{v}_0) - dF_{\vec{v}_0}(\vec{v})\|_W \leq \epsilon \|\vec{v}\|_V.$$

Since δ and m are positive real numbers, $\delta' := m\delta$ is a positive real number. By hypothesis, $\|\vec{v}\|'_V < \delta'$ implies that

$$\|\vec{v}\|_V \leq (1/m)\|\vec{v}\|'_V < (\delta'/m) = \delta,$$

which in turn implies that

$$\|F(\vec{v}_0 + \vec{v}) - F(\vec{v}_0) - dF_{\vec{v}_0}(\vec{v})\|_W \leq \epsilon\|\vec{v}\|_V.$$

Also by hypothesis,

$$\epsilon\|\vec{v}\|_V = \epsilon'(m\|\vec{v}\|_V) \leq \epsilon'\|\vec{v}\|'_V.$$

Therefore the hypothesis $\|\vec{v}\|'_V < \delta'$ implies that

$$\|F(\vec{v}_0 + \vec{v}) - F(\vec{v}_0) - dF_{\vec{v}_0}(\vec{v})\|_W \leq \epsilon'\|\vec{v}\|'_V,$$

just as needed.

The argument for equivalent norms on W is similar and easier. The details are left to the reader.

Solution to (3) There are two simple facts which are useful for this problem: a bounded linear transformation is always differentiable with derivative equal to the transformation, and a function to the space of (bounded) linear operators on a finite dimensional vector space is continuous if and only if each of its "column vector" coordinate functions is continuous. So I will formulate these as lemmas.

Lemma 0.1. *Let $(U, \|\bullet\|_U)$ and $(W, \|\bullet\|_W)$ be normed vector spaces. Every bounded linear operator $T : U \to W$ is differentiable at every $\vec{u}_0$ in U and the derivative $dT_{\vec{u}_0}$ equals T.*

Proof. For every $\vec{u}$ in U, since T is linear we have

$$T(\vec{u}_0 + \vec{u}) - T(\vec{u}_0) - T(\vec{u}) = [T(\vec{u}_0) + T(\vec{u})] - T(\vec{u}_0) - T(\vec{u}) = 0_W.$$

So with $dT_{\vec{u}_0}$ defined to be T, for every positive real ϵ, for any positive real δ, for every $\vec{u}$ in U with $\|\vec{u}\|_U < \delta$, we have

$$\|T(\vec{u}_0 + \vec{u}) - T(\vec{u}_0) - dT_{\vec{u}_0}(\vec{u})\|_W = \|0_W\|_W = 0 \leq \epsilon\|\vec{u}\|_U.$$

□

For normed vector spaces $(U, \| \bullet \|_U)$ and $(W, \| \bullet \|_W)$ there is a natural map

$$c_{U,W} : L(U, W) \times U \to W, \quad (T, \vec{u}) \mapsto T(\vec{u}).$$

The following goes (almost) without saying.

Lemma 0.2. *The map c_U is continuous.*

Proof. Let $(T, \vec{u})$ be an element in $L(U, W) \times U$. Let ϵ be a positive real. Define $\delta_L := \min(\sqrt{\epsilon/3}, \epsilon/3(1+ \|\vec{u}\|_U))$ and define $\delta_U := \min(\sqrt{\epsilon/3}, \epsilon/3(1 + \|T\|_{\text{op}}))$. These are positive real numbers. Let $T' \in L(U, W)$ and $\vec{u}' \in U$ be elements such that $\|T' - T\|_{\text{op}} < \delta_L$ and $\|\vec{u}' - \vec{u}\|_U < \delta_U$. Then we have $\|T' - T\|_{\text{op}}\|\vec{u}' - \vec{u}\|_U < \sqrt{\epsilon/3}\sqrt{\epsilon/3} = \epsilon/3$. Similarly we have $\|T' - T\|_{\text{op}}\|\vec{u}\|_U < (\epsilon/3(1 +$

$\|\vec{u}\|_U))\|\vec{u}\|_U < \epsilon/3$. And finally we have $\|T\|_{\text{op}}\|\vec{u}' - \vec{u}\|_U < \|T\|_{\text{op}}(\epsilon/3(1+\|T\|_{\text{op}})) < \epsilon/3$. Therefore we have (using the triangle inequality),

$$\|T'(\vec{u}') - T(\vec{u})\|_W = \|(T'-T)(\vec{u}'-\vec{u}) + (T'-T)(\vec{u}) + T(\vec{u}'-\vec{u})\|_W \leq$$
$$\|(T'-T)(\vec{u}'-\vec{u})\|_W + \|(T'-T)(\vec{u})\|_U + \|T(\vec{u}'-\vec{u})\|_W \leq$$
$$\|T'-T\|_{\text{op}}\|\vec{u}'-\vec{u}\|_U + \|T'-T\|_{\text{op}}\|\vec{u}\|_U + \|T\|_{\text{op}}\|\vec{u}'-\vec{u}\|_U \leq \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.$$

□

Now let (X, d_X) be a metric space and let there be given a function

$$\tau : X \to L(U, W), \quad x \mapsto (\tau_x : U \to W)$$

Form the new function

$$(\tau \times \text{Id}_U) : X \times U \to L(U, W) \times U, \quad (x, \vec{u}) \mapsto (\tau_x, \vec{u}).$$

Finally, form the composition $\tilde{\tau} := c_{U,V} \circ (\tau \times \text{Id}_U)$, i.e.,

$$\tilde{\tau} : X \times U \to U, \quad \tilde{\tau}(x, \vec{u}) = \tau_x(\vec{u}).$$

If τ is continuous, so is $\tau \times \text{Id}_U$. And since $c_{U,W}$ is continuous, then also $\tilde{\tau}$ is continuous. The useful result about linear maps is a converse of this observation.

Lemma 0.3. *Assume that U is finite dimensional so that the closed unit ball in U is compact. Let $\tau : X \to L(U, W)$ be a function such that the associated map $\tilde{\tau} : X \times U \to W$ is continuous. Then τ is continuous.*

Proof. Assume that $\tilde{\tau}$ is continuous. Let x_0 be an element of X. Since addition and subtraction are continuous in the normed vector space $L(U, W)$, for the function $\sigma : X \to L(U, W)$ defined by $\sigma_x := \tau_x - \tau_{x_0}$, the function $\tilde{\sigma}$ is continuous. So for every positive real ϵ, for the open ball $B_{\epsilon/2}(0_W)$ in W, the subset $A_\epsilon := \tilde{\sigma}^{-1}B_{\epsilon/2}(0_W)$ is an open subset of $X \times U$. Since σ_{x_0} equals 0, for every $\vec{u}$ in the closed unit ball $B_{\leq 1}(0_U)$ in U, $(x_0, \vec{u})$ is in A_ϵ. Since A_ϵ is open, there exists an open neighborhood $X_{\epsilon,\vec{u}}$ of x_0 in X and there exists a positive real $\delta_{\vec{u}}$ such that $X_{\epsilon,\vec{u}} \times B_{\delta_{\vec{u}}}(\vec{u})$ is in A_ϵ. The following collection of open sets cover $B_{\leq 1}(0_U)$ in U

$$\{B_{\delta_{\vec{u}}}(\vec{u}) | \vec{u} \in B_{\leq 1}(0_U)\}.$$

By hypothesis $B_{\leq 1}(0_U)$ is compact. Hence there exists a finite subset $F \subset B_{\leq 1}(0_U)$ such that the following collection of open sets also covers $B_{\leq 1}(0_U)$,

$$\{B_{\delta_{\vec{u}}}(\vec{u}) | \vec{u} \in F\}.$$

So for every $\vec{u}' \in B_{\leq 1}(0_U)$, there exists $\vec{u} \in F$ such that $\vec{u}' \in B_{\delta_{\vec{u}}}(\vec{u})$. Define X_ϵ to be the finite intersection

$$X_\epsilon = \cap_{\vec{u} \in F} X_{\epsilon,\vec{u}}.$$

As a finite intersection of open sets, X_ϵ is open. Since each of these open sets contains x_0, also X_ϵ contains x_0. And since $X_{\epsilon,\vec{u}} \times B_{\delta_{\vec{u}}}(\vec{u})$ is in A_ϵ, in particular $X_\epsilon \times \{\vec{u}'\}$ is in A_ϵ, i.e., for every $x \in X_\epsilon$, for every $\vec{u}' \in B_{\leq 1}(0_U)$,

$$\|\tau_x(\vec{u}') - \tau_{x_0}(\vec{u}')\|_W < \epsilon/2.$$

But this precisely says that for every x in the open neighborhood X_ϵ of x_0, $\|\tau_x - \tau_{x_0}\|_{\text{op}} \leq \epsilon/2 < \epsilon$. This is precisely the definition of $\tau : X \to L(U,W)$ being continuous at x_0. Since this holds for every x_0 in X, τ is continuous. $\square$

One can use the Hahn-Banach theorem to prove that the hypothesis of compactness of the closed unit ball is essential in this proof. The desire to "weaken" this compactness hypothesis leads to the "weak topology" on $L(U,W)$, which is the finest topology on $L(U,W)$ such that for every topological space X and for every function $\tau : X \to L(U,W)$, τ is continuous if and only if the associated function $\tilde{\tau} : X \times U \to W$ is continuous. At any rate, Lemma0.3 exactly applies when U is finite dimensional.

Returning to the solution of the problem, there is undoubtedly a "coordinate-free" solution to this problem. But it seems simplest to use coordinates. Let $\underline{b} = (\vec{b}_1, \dots, \vec{b}_n)$ be an ordered basis for V. This basis determines a linear isomorphism of vector spaces

$$e_{\underline{b}} : L(V,V) \to V^n = (V \times \cdots \times V), \quad e_{\underline{b}}(T) = (T(\vec{b}_1), \dots, T(\vec{b}_n)).$$

Because all norms on a finite dimensional vector space are equivalent, this linear isomorphism is an equivalence of normed vector spaces (in the sense of "equivalent norms" as above). Hence differentiability of Det is equivalent to differentiability of the following function $Det \circ e_{\underline{b}}^{-1}$, i.e.,

$$\text{Det}_{\underline{b}} : V^n \to \mathbb{R}, \ \text{Det}(T) = \text{Det}_{\underline{b}}(T(\vec{b}_1), \dots, T(\vec{b}_n)).$$

The usual axioms for the determinant function presented in undergraduate linear algebra courses are as follows.

(i) The function $\text{Det}_{\underline{b}}$ is n-multilinear, i.e., for every $k = 1, \dots, n$, for every collection $\underline{a}_k = (\vec{a}_1, \dots, \vec{a}_{k-1}, \vec{a}_{k+1}, \dots, \vec{a}_n)$ in V^{n-1}, the function $\text{Det}_{\underline{b},k,\underline{a}_k}(\vec{v}_k) := \text{Det}_{\underline{b}}(\vec{a}_1, \dots, \vec{a}_{k-1}, \vec{v}_k, \vec{a}_{k+1}, \dots, \vec{a}_n)$ is $\mathbb{R}$-linear.

(ii) The function $\text{Det}_{\underline{b}}$ is alternating, i.e., $\text{Det}_{\underline{b}}(\vec{v}_1, \dots, \vec{v}_n)$ equals 0 if there exist integers k, l with $1 \leq k < l \leq n$ such that $\vec{v}_k$ equals $\vec{v}_l$.

(iii) The function $\text{Det}_{\underline{b}}$ is "calibrated" in the sense that $\text{Det}_{\underline{b}}(\vec{b}_1, \dots, \vec{b}_n)$ equals 1.

For differentiability, Axiom (i) is particularly important.

By Theorem 3.8.2 on p. 154 of the textbook, $\text{Det}_{\underline{b}}$ is continuously differentiable if and only if for every $k = 1, \dots, n$ and for every $\underline{a}_k = (\vec{a}_1, \dots, \vec{a}_{k-1}, \vec{a}_{k+1}, \dots, \vec{a}_n)$ in V^{n-1}, the function $\text{Det}_{\underline{b},k,\underline{a}_k}(\vec{v}_k) := \text{Det}_{\underline{b}}(\vec{a}_1, \dots, \vec{a}_{k-1}, \vec{v}_k, \vec{a}_{k+1}, \dots, \vec{a}_n)$ is differentiable and the associated function

$$d\text{Det}_{\underline{b}}^k : V^n \to L(V,V), \quad (\vec{a}_1, \dots, \vec{a}_{k-1}, \vec{v}_k, \vec{a}_{k+1}, \dots, \vec{a}_n) \mapsto d(\text{Det}_{\underline{b},k,\underline{a}_k})_{\vec{v}_k}$$

is continuous. By Axiom (i) above, the function $\mathrm{Det}_{\underline{b},k,\underline{a}_k}$ is linear. Since V is a finite dimensional vector space, this linear operator is automatically bounded. Thus by Lemma 0.1, the function is differentiable and the derivative is simply $d(\mathrm{Det}_{\underline{b},k,\underline{a}_k})_{\vec{v}_k} = \mathrm{Det}_{\underline{b},k,\underline{a}_k}$, which in particular is independent of the argument $\vec{v}_k$. Hence the function $d\mathrm{Det}^k_{\underline{b}}$ factors through the (continuous) linear projection $V^n \to V^{n-1}$ sending $(\vec{a}_1, \dots, \vec{a}_{k-1}, \vec{v}_k, \vec{a}_{k+1}, \dots, \vec{a}_n)$ to $\underline{a}_k$. Thus $d\mathrm{Det}^k_{\underline{b}}$ is continuous if the associated function

$$\mathrm{Det}^k_{\underline{b}} : V^{n-1} \to L(V, V), \quad \underline{a}_k \mapsto \mathrm{Det}^k_{\underline{b},k,\underline{a}_k}$$

is a continuous function. Again since V is finite dimensional, by Lemmas 0.2 and 0.3, the function $\mathrm{Det}^k_{\underline{b}}$ is continuous if and only if the following function is continuous

$$\widetilde{\mathrm{Det}^k_{\underline{b}}} : V^{n-1} \times V \to V, \quad (\underline{a}_k, \vec{v}_k) \mapsto \mathrm{Det}_{\underline{b},k,\underline{a}_k}(v_k) = \mathrm{Det}_{\underline{b}}(\vec{a}_1, \dots, \vec{a}_{k-1}, \vec{v}_k, \vec{a}_{k+1}, \dots, \vec{a}_n).$$

But up to the permutation

$$V^{n-1} \times V \to V^n, \quad (\underline{a}_k, \vec{v}_k) \mapsto (\vec{a}_1, \dots, \vec{a}_{k-1}, \vec{v}_k, \vec{a}_{k+1}, \dots, \vec{a}_n),$$

the function $\widetilde{\mathrm{Det}^k_{\underline{b}}}$ is precisely $\mathrm{Det}_{\underline{b}}$, which is continuous (since it is a polynomial function, for instance). Since for every $k = 1, \dots, n$, the function $\mathrm{Det}_{\underline{b},k,\underline{a}_k}$ is differentiable and the derivative function $d\mathrm{Det}^k_{\underline{b}}$ is continuous, by Theorem 3.8.2, p. 154, the function $\mathrm{Det}_{\underline{b}}$ is differentiable (and the derivative is continuous). Moreover for each $\underline{a} = (\vec{a}_1, \dots, \vec{a}_n) \in V^n$, we have that $d(\mathrm{Det}_{\underline{b}})_{\underline{a}}(\vec{v}_1, \dots, \vec{v}_k, \dots, \vec{v}_n)$ equals

$$\sum_{k=1}^{n} d(\mathrm{Det}_{\underline{b},k,\underline{a}_k})(\vec{v}_k) = \sum_{k=1}^{n} \mathrm{Det}_{\underline{b}}(\vec{a}_1, \dots, \vec{a}_{k-1}, \vec{v}_k, \vec{a}_{k+1}, \dots, \vec{a}_n).$$

Now consider the special case that $\underline{a}$ equals $\underline{b}$, i.e., $(\vec{b}_1, \dots, \vec{b}_n)$. For a vector $\vec{v} \in V$, denote by $x_1(\vec{v}), \dots, x_n(\vec{v})$ the unique real numbers such that

$$\vec{v} = \sum_{j=1}^{n} x_j(\vec{v})\vec{b}_j,$$

i.e., $(x_1, \dots, x_n)$ are the "coordinates with respect to $\vec{b}$", which then form the ordered basis of $L(V, \mathbb{R})$ which is "dual" to the ordered basis $(\vec{b}_1, \dots, \vec{b}_n)$ of V. Then for every $k = 1, \dots, n$, Axiom (i) implies that

$$\mathrm{Det}_{\underline{b}}(\vec{b}_1, \dots, \vec{b}_{k-1}, \vec{v}_k, \vec{b}_{k+1}, \dots, \vec{b}_n) = \sum_{j=1}^{n} x_j(\vec{v}_k)\mathrm{Det}_{\underline{b}}(\vec{b}_1, \dots, \vec{b}_{k-1}, \vec{b}_j, \vec{b}_{k+1}, \dots, \vec{b}_n).$$

By Axiom (ii), the term in the j^{th} summand $\mathrm{Det}_{\underline{b}}(\vec{b}_1, \dots, \vec{b}_{k-1}, \vec{b}_j, \vec{b}_{k+1}, \dots, \vec{b}_n)$ equals 0 unless j equals k. So only the k^{th} summand is nonzero. And for $j = k$, by Axiom (iii), the term $\mathrm{Det}_{\underline{b}}(\vec{b}_1, \dots, \vec{b}_{k-1}, \vec{b}_k, \vec{b}_{k+1}, \dots, \vec{b}_n)$ equals 1. Therefore the sum equals

$$\mathrm{Det}_{\underline{b}}(\vec{b}_1, \dots, \vec{b}_{k-1}, \vec{v}_k, \vec{b}_{k+1}, \dots, \vec{b}_n) = x_k(\vec{v}_k).$$

Substituting this into the formula for $d(\text{Det}_{\underline{b}})_{\underline{b}}$ gives

$$d(\text{Det}_{\underline{b}})_{\underline{b}}(\vec{v}_1, \dots, \vec{v}_n) = x_1(\vec{v}_1) + \cdots + x_n(\vec{v}_n).$$

Now $e_{\underline{b}}(\text{Id}_V)$ exactly equals $(\vec{b}_1, \dots, \vec{b}_n) = \underline{b}$. And for T in $L(V,V)$ with $e_{\underline{b}}(T) = (T(\vec{b}_1), \dots, T(\vec{b}_n)) = (\vec{v}_1, \dots, \vec{v}_n)$, the expression

$$x_1(\vec{v}_1) + \cdots + x_n(\vec{v}_n) = x_1(T(\vec{b}_1)) + \cdots + x_n(T(\vec{b}_n))$$

is precisely $\text{Tr}(T)$. Making these substitutions gives

$$d\text{Det}_{\text{Id}_V}(T) = d(\text{Det}_{\underline{b}})_{e_{\underline{b}}(\text{Id}_V)}(e_{\underline{b}}(T)) = \text{Tr}(T).$$

As a final note, this one computation will allow us to compute $d\text{Det}_R$ for every R in $L(V,V)$. Observe for every $S \in L(V,V)$ that the function

$$L_S : L(V,V) \to L(V,V), \quad T \mapsto S \circ T$$

is a linear operator of a finite dimensional vector space, hence bounded. In fact, even in the infinite dimensional case this is bounded with $\|L_S\|_{\text{op}} \leq \|S\|_{\text{op}}$ – one of the axioms for a Banach *algebra*. Thus by Lemma 0.1, L_S is continuously differentiable with constant derivative function equal to L_S. Thus by the chain rule, the composition $\text{Det} \circ L_S$ is also continuously differentiable with

$$d(\text{Det} \circ L_S)_R = d(\text{Det})_{L_S(R)} \circ d(L_S)_R = d(\text{Det})_{SR} \circ L_S.$$

In particular, when R is invertible and when $S = R^{-1}$, this gives

$$d(\text{Det} \circ L_S)_R(T) = d(\text{Det})_{\text{Id}_V}(ST) = \text{Tr}(ST).$$

On the other hand, since Det is multiplicative,

$$(\text{Det} \circ L_S)(T) = \text{Det}(ST) = \text{Det}(S)\text{Det}(T) = (\text{Det}(S)\text{Det})(T).$$

And by linearity, $d(\text{Det}(S)\text{Det}) = \text{Det}(S)d\text{Det}$. Putting the pieces together,

$$\text{Det}(S)d\text{Det}_R(T) = \text{Tr}(ST).$$

Since R is invertible, $\text{Det}(S)$ is invertible with inverse $\text{Det}(R)$. So this finally gives

$$d\text{Det}_R(T) = \text{Tr}([\text{Det}(R)R^{-1}]T).$$

By Cramer's Rule, $\text{Det}(R)R^{-1}$ is the *cofactor transformation* R^{cof} whose entries are defined and continuous functions of R (even polynomial functions of the entries of R) for all R, not only when R invertible. So we conclude that for all invertible R in $L(V,V)$,

$$d\text{Det}_R = \text{Tr} \circ L_{R^{\text{cof}}}.$$

Since both sides are continuous functions $L(V,V) \to L(L(V,V),\mathbb{R})$, the subset of $L(V,V)$ on which these functions are equal is closed. Since it contains the dense open set of invertible operators R, this closed set is all of $L(V,V)$. Therefore for every R in $L(V,V)$,

$$d\mathrm{Det}_R(T) = \mathrm{Tr}(R^{\mathrm{cof}}T).$$

Solution to (4)

Solution to (a) The proof that there is at most one fixed point is exactly as in the case of a contraction. Let x_1 and x_2 be fixed points, i.e., $T(x_i) = x_i$ for $i = 1, 2$. Then $d_X(T(x_1), T(x_2))$ equals $d_X(x_1, x_2)$. But if $x_1 \neq x_2$, then by definition of an almost contraction, $d_X(T(x_1), T(x_2))$ is strictly less than $d_X(x_1, x_2)$. Therefore x_1 must equal x_2.

Since d_X takes on only nonnegative values, for a fixed point x of X, the value $d_X(x, T(x)) = d_X(x, x) = 0$ is a minimum value of $f(x) = d_X(x, T(x))$. Conversely, let x be an element of X such that $f(x)$ is a minimum value of f. If $T(x)$ does not equal x, then since T is an almost contraction we have $d(T(T(x)), T(x))$ is strictly less than $d(T(x), x)$. But $d(T(T(x)), T(x))$ equals $f(T(x))$. So $f(T(x)) < f(x)$, contradicting that $f(x)$ is a minimum value of f. Therefore $T(x)$ equals x, i.e., x is a fixed point.

Solution to (b) Since $d_X : X \times X \to \mathbb{R}$ and $T : X \to X$ are continuous, also the function $f(x) = d_X(x, T(x))$ is a continuous function $f : X \to \mathbb{R}$. If X is compact, then also $f(X)$ is a compact subset of $\mathbb{R}$. By the Heine-Borel theorem, $f(X)$ is closed and bounded. In particular, by the least upper bound property (or rather the greatest lower bound property), $f(X)$ contains its infimum. Therefore there exists $x \in X$ such that $f(x)$ is the infimum of $f(X)$, i.e., $f(x)$ is a minimum value of f. By **(a)**, x is a fixed point of T.

Solution to (5) There are many such examples. For instance, begin with a function

$$g_{\mathbb{Z}} : \mathbb{Z} \to (-1/2, 1/2)$$

which is strictly decreasing, e.g., $g_{\mathbb{Z}}(n) = -(1/2)(1 - (1/2)^n)$ for $n \geq 0$ and $g_{\mathbb{Z}}(-n) = +(1/2)(1 - (1/2)^n)$ for $-n \leq 0$. Then for every integer n, we have $-1 < g_{\mathbb{Z}}(n+1) - g_{\mathbb{Z}}(n) < 1$, $0 < 1 + g_{\mathbb{Z}}(n+1) - g_{\mathbb{Z}}(n) < 1$, and $g_{\mathbb{Z}}(n) > -1$. Then the function

$$f_{\mathbb{Z}} : \mathbb{Z} \to \mathbb{R}, \quad f_{\mathbb{Z}}(n) = n + 2 + g_{\mathbb{Z}}(n)$$

is stricly increasing and is an almost contraction in the sense that

$$f_{\mathbb{Z}}(n+1) - f_{\mathbb{Z}}(n) = 1 + g_{\mathbb{Z}}(n+1) - g_{\mathbb{Z}}(n) < 1 = (n+1) - n.$$

And also $f_{\mathbb{Z}}(n) = n + 2 + h_{\mathbb{Z}}(n) > n + 2 + (-1) = n + 1$. Now extend $f_{\mathbb{Z}}$ to the unique continuous function $f : \mathbb{R} \to \mathbb{R}$ which is linear on each interval $[n, n+1]$, i.e.,

$$f(n+t) = f_{\mathbb{Z}}(n) + t(f_{\mathbb{Z}}(n+1) - f_{\mathbb{Z}}(n))$$

for $0 \leq t \leq 1$. Since $f_{\mathbb{Z}}$ is strictly increasing, so is f. Since $f_{\mathbb{Z}}$ is an almost contraction, so is f. Finally, for every integer n and for every t with $0 \leq t \leq 1$, $f(n+t) \geq f(n) = f_{\mathbb{Z}}(n) > n+1 > n+t$. Thus $n + t$ is not a fixed point of f.