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MAT 544 Problem Set 3 Solutions

Problems.

Problem 1 The real vector space ℓ^1 is defined to be the vector space of all sequences of real numbers $(x_k)_{k=1,2,\dots}$ which are *absolutely convergent*, i.e., $\sum_{k=1}^{\infty} |x_k|$ is finite. For an absolutely convergent sequence, the ℓ^1-norm is defined by

$$\|(x_k)\|_{\ell^1} := \sum_{k=1}^{\infty} |x_k|.$$

In this exercise you may assume that this defines a real normed vector space. Prove that ℓ^1 is a Banach space, i.e., prove that the metric is complete.

Problem 2 Denote by $\| \bullet \|_1 : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ the usual product norm,

$$\|\vec{x}\|_1 := \sum_{k=1}^{n} |x_k| \text{ for } \vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

Let $\| \bullet \| : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be a norm on $\mathbb{R}^n$.

(a) Prove that there exists a real number $C > 0$ such that for all $\vec{x} \in \mathbb{R}^n$, $\|\vec{x}\| \leq C \cdot \|\vec{x}\|_1$. Conclude that $\| \bullet \| : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a continuous function on $\mathbb{R}^n$ with the topology coming from the norm $\| \bullet \|_1$.

(b) Consider the subset $S = \{\vec{x} \in \mathbb{R}^n | \|\vec{x}\|_1 = 1\}$ of $\mathbb{R}^n$. Prove that the function $\| \bullet \|$ takes on a maximum value $M > 0$ and a minimum value $m > 0$ on S. Thus for all $\vec{x} \in \mathbb{R}^n$ we have $m \cdot \|\vec{x}\|_1 \leq \|\vec{x}\| \leq M \cdot \|\vec{x}\|_1$.

(c) Conclude that the topology on $\mathbb{R}^n$ coming from the norm $\| \bullet \|$ equals the topology coming from the usual norm $\| \bullet \|_1$. Also conclude that the closed unit ball $B_1(\vec{0})$ for the norm $\| \bullet \|$ is compact.

Problem 3 Let $(V, \| \bullet \|)$ be a real normed vector space. Let W be a linear subspace of V which is a closed subset of V and which does not equal all of V.

(a) Use the Hahn-Banach theorem to prove that for every real $\epsilon > 0$ there exists $\vec{v}_\epsilon \in V$ with $\|\vec{v}_\epsilon\| = 1$ and with $\|\vec{v}_\epsilon - \vec{w}\| \geq 1 - \epsilon$ for all $\vec{w} \in W$.

(b) Under the further hypothesis that the closed unit ball in V is compact, prove that there exists $\vec{v}_0 \in V$ with $\|\vec{v}_0\| = 1$ and with $\|\vec{v}_0 - \vec{w}\| \geq 1$ for all $\vec{w} \in W$.

Problem 4 Let $(V, \|\bullet\|)$ be a real normed vector space. Denote by B_V the closed unit ball in V centered at the origin (defined with respect to the norm).

(a) For every linear subspace $W \subset V$ which is finite dimensional, prove that W is closed in V.

Hint. It suffices to prove that the intersection $B_V \cap W = B_W$ is closed in B_V. What does **Problem 2(c)** imply about B_W?

(b) Assume that V is infinite dimensional and let $W_1 \subsetneq W_2 \subsetneq \dots \subset V$ be an increasing sequence of finite dimensional vector spaces. For every integer $k \geq 2$, by **Problem 3(b)** there exists $\vec{v}_k \in B_{W_k}$ with $\|\vec{v}_k - \vec{w}_{k-1}\| \geq 1$ for all $\vec{w}_{k-1} \in W_{k-1}$. Prove that the sequence $(\vec{v}_k)_{k=2,3,\dots}$ has no convergent subsequence. Conclude that the ball B_V is not compact if V is infinite dimensional. Combined with **Problem 2(c)**, it follows that for a normed vector space V, B_V is compact if and only if V is finite dimensional.

Problem 5 Let (X, d_X) be a metric space and let $BC(X, \mathbb{R})$ be the set of all bounded, continuous functions on X with the uniform metric. If X is not compact, prove that there exists a sequence $(f_k)_{k=1,2,\dots}$ in $BC(X, \mathbb{R})$ which is equicontinuous and pointwise bounded, but which has no convergent subsequence. Thus the Arzela-Ascoli theorem holds only if X is compact.

Hint. This is very similar to Problem 2(b) and Problem 5(b) from Problem Set 2.

Solutions to Problems.

Solution to (1) A real normed vector space $(V, \|\bullet\|)$ is complete if and only if every series $\sum_{n=1}^{\infty} \mathbf{x}_n$ in V converges, i.e., the sequence $(\sum_{n=1}^{k} \mathbf{x}_n)_{k=1,2,\dots}$ converges, whenever the series is *absolutely convergent*, i.e., whenever the series $\sum_{n=1}^{\infty} \|\mathbf{x}_n\|$ of nonnegative real numbers is convergent. For the normed vector space ℓ^1, each element $\mathbf{x}$ is itself an absolutely convergent sequence of real numbers $\mathbf{x} = (x_k)_{k=1,2,\dots}$, and the ℓ^1-norm is defined to be the convergent series of nonnegative real numbers

$$\|\mathbf{x}\|_{\ell^1} := \sum_{k=1}^{\infty} |x_k|.$$

Let

$$(\mathbf{x}_n)_{n=1,2,\dots} = (\ (x_{n,k})_{k=1,2,\dots}\ )_{n=1,2,\dots}$$

be a sequence of elements in ℓ^1 whose associated series is absolutely convergent, i.e., the following series of nonnegative real numbers is convergent,

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |x_{n,k}| < \infty$$

Every subseries of an absolutely convergent series is also absolutely convergent. So for every $K = 1, 2, \dots,$ the series $\sum_{n=1}^{\infty} x_{n,K}$ is absolutely convergent, i.e.,

$$\sum_{n=1}^{\infty} |x_{n,K}| \leq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |x_{n,k}| < \infty.$$

The normed vector space $(\mathbb{R}, |\bullet|)$ is complete. Thus, since the series $\sum_{n=1}^{\infty} x_{n,K}$ is absolutely convergent, it is convergent. Define $x_{\infty,K}$ to be the associated limit,

$$\mathbf{x}_{\infty,K} := \sum_{n=1}^{\infty} x_{n,K} = \lim_{N\to\infty} \sum_{n=1}^{N} x_{n,K}.$$

And define $\mathbf{x}_\infty$ to be the sequence of these series, $\mathbf{x}_\infty = (x_{\infty,K})_{K=1,2,\dots}$.

The first claim is that $\mathbf{x}_\infty$ is in ℓ^1. Since $t \mapsto |t|$ is continuous, also $(|\sum_{n=1}^{N} x_{n,K}|)$ converges to $|x_{\infty,K}|$. Since for every $N = 1, 2, \dots$, the triangle inequality gives

$$|\sum_{n=1}^{N} x_{n,K}| \leq \sum_{n=1}^{N} |x_{n,K}| \leq \sum_{n=1}^{\infty} |x_{n,K}|,$$

also in the limit we have

$$|x_{\infty,K}| \leq \sum_{n=1}^{\infty} |x_{n,K}|.$$

Thus we also have inequalities among the partial sums,

$$\sum_{K=1}^{L} |x_{\infty,K}| \leq \sum_{K=1}^{L} \sum_{n=1}^{\infty} |x_{n,K}|.$$

And this second sum is bounded by the infinite sum,

$$\sum_{K=1}^{\infty} \sum_{n=1}^{\infty} |x_{n,K}|,$$

which by hypothesis is finite. Thus for every L, we have an upper bound independent of L,

$$\sum_{K=1}^{L} |x_{\infty,K}| \leq \sum_{K=1}^{\infty} \sum_{n=1}^{\infty} |x_{n,K}|.$$

Since this is an upper bound for each of the partial sums, it follows that the infinite series is bounded. And again since $\mathbb{R}$ is complete (or using the least upper bound property), it follows that the infinite series converges,

$$\sum_{K=1}^{\infty} |x_{\infty,K}| \leq \sum_{K=1}^{\infty} \sum_{n=1}^{\infty} |x_{n,K}| < \infty.$$

Therefore the series is absolutely convergent, hence $\mathbf{x}_\infty$ is an element of ℓ^1. This proves the first claim.

The second claim is that the sequence of partial sums $(\sum_{n=1}^N \mathbf{x}_n)_{N=1,2,\dots}$ converges to $\mathbf{x}_\infty$ in the norm $\| \bullet \|_{\ell^1}$. Let $\epsilon > 0$ be a real number. Since the series of nonnegative real number

$$\sum_{k=1}^{\infty} \left(\sum_{n=1}^{\infty} |x_{n,k}| \right)$$

converges, the tails converge to 0. Therefore there exists an integer L_1 such that

$$\sum_{k=L_1+1}^{\infty} \left(\sum_{n=1}^{\infty} |x_{n,k}| \right) < \frac{\epsilon}{4}.$$

In particular, for every $N = 1, 2, \dots,$ we also have the inequality

$$\sum_{k=L_1+1}^{\infty} \left(\sum_{n=1}^{N} |x_{N,k}| \right) \leq \sum_{k=L_1+1}^{\infty} \left(\sum_{n=1}^{\infty} |x_{n,k}| \right) < \frac{\epsilon}{4}.$$

Similarly, since the series

$$\sum_{k=1}^{\infty} |x_{\infty,k}|$$

converges, its tails converge to 0. Therefore there exists an integer L_2 such that

$$\sum_{k=L_2+1}^{\infty} |x_{\infty,k}| < \frac{\epsilon}{4}.$$

Define $L = \max(L_1, L_2)$. And for every integer $N \geq 1$ and every integer $k \geq L + 1$, we have

$$|x_{\infty,k} - \sum_{n=1}^{N} x_{N,k}| \leq |x_{\infty,k}| + \sum_{n=1}^{N} |x_{N,k}|$$

by the triangle inequality. Summing over $k \geq L + 1$ gives,

$$\sum_{k=L+1}^{\infty} |x_{\infty,k} - \sum_{n=1}^{N} x_{n,k}| \leq \sum_{k=L+1}^{\infty} |x_{\infty,k}| + \sum_{k=L+1}^{\infty} \left(\sum_{n=1}^{N} |x_{N,k}| \right) < \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2}.$$

Finally, for every $k = 1, \dots, L$, since the sequence $(\sum_{n=1}^N x_{n,k})_{N=1,2,\dots}$ converges to $x_{\infty,k}$ by hypothesis, there exists an integer M_k such that for every $N \geq M_k$ we have

$$|x_{\infty,k} - \sum_{n=1}^{N} x_{n,k}| < \frac{\epsilon}{2L}.$$

Define $M = \max(M_1, \dots, M_L)$. Then for every integer $N \geq M$, for every $k = 1, \dots, L$, we have $|x_{\infty,k} - \sum_{n=1}^N x_{n,k}| < \frac{\epsilon}{2L}$. Summing over k gives,

$$\sum_{k=1}^{L} |x_{\infty,k} - \sum_{n=1}^{N} x_{n,k}| < \sum_{k=1}^{L} \frac{\epsilon}{2L} = \frac{\epsilon}{2}.$$

And recall we already had

$$\sum_{k=L+1}^{\infty} |x_{\infty,k} - \sum_{n=1}^{N} x_{n,k}| < \frac{\epsilon}{2}.$$

So putting the two inequalities together gives for every integer $N \geq M$,

$$\|\mathbf{x}_\infty - \sum_{n=1}^{N} \mathbf{x}_n\|_{\ell^1} = \sum_{k=1}^{\infty} |x_{\infty,k} - \sum_{n=1}^{N} x_{N,k}| = \left(\sum_{k=1}^{L} |x_{\infty,k} - \sum_{n=1}^{N} x_{N,k}|\right) + \left(\sum_{k=L+1}^{\infty} |x_{\infty,k} - \sum_{n=1}^{N} x_{N,k}|\right) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Therefore the sequence $(\sum_{n=1}^N \mathbf{x}_n)_{N=1,2,\dots}$ converges to $\mathbf{x}_\infty$ in the norm $\| \bullet \|_{\ell^1}$.

Solution to Problem 2

Solution to (a) Denote the standard basis vectors in $\mathbb{R}^n$ by $(\vec{e}_1, \dots, \vec{e}_n)$ so that every vector is $\vec{x} = \sum_{k=1}^n x_k \vec{e}_k$ for unique real numbers $x_1, \dots, x_n$, which are simply the coordinates. Thus $\|\vec{x}\|_1$ equals $\sum_{k=1}^n |x_k|$. Let C be any positive real number with

$$\max(\|\vec{e}_1\|, \dots, \|\vec{e}_n\|) \leq C.$$

The claim is that this constant works, i.e.,

$$\|\vec{x}\| \leq C \sum_{k=1}^{n} |x_k|.$$

Since $\| \bullet \|$ is a norm, it satisfies the triangle inequality. Combined with induction, it follows that for every integer $m > 0$ and for every collection $\vec{v}_1, \dots, \vec{v}_m \in \mathbb{R}^n$, we have

$$\left\| \sum_{k=1}^{m} \vec{v}_k \right\| \leq \sum_{k=1}^{m} \|\vec{v}_k\|.$$

For every vector $\vec{x} = \sum_{k=1}^n x_k \vec{e}_k$, take m above to equal n and take $\vec{v}_k = x_k \vec{e}_k$ to obtain

$$\|\vec{x}| = \left\| \sum_{k=1}^{n} x_k \vec{e}_k \right\| \leq \sum_{k=1}^{n} \|x_k \vec{e}_k\|.$$

Since $\| \bullet \|$ is a norm, it is homogeneous; in particular $\|x_k \vec{e}_k\| = |x_k| \|\vec{e}_k\|$ for every $k = 1, \dots, n$. Substituting this in the inequality above gives

$$\|\vec{x}\| \leq \sum_{k=1}^{n} |x_k| \|\vec{e}_k\| \leq \sum_{k=1}^{n} |x_k| C = C \|\vec{x}\|_1,$$

proving the claim.

Solution to (b) By Problem 2 on Problem Set 1, the metric on $\mathbb{R}^n$ coming from $\|\bullet\|_1$ is continuous for the topology coming from that metric. And $\|\vec{x}\|_1$ is simply the distance from $\vec{0}$. Therefore the function $\|\bullet\|_1 : \mathbb{R}^n \to \mathbb{R}$ is continuous for the topology coming from $\|\bullet\|_1$. And the singleton set $\{1\}$ is a closed subset of $\mathbb{R}$. Hence its inverse image under this continuous function is a closed subset of $\mathbb{R}^n$, i.e., S is a closed subset of $\mathbb{R}^n$. Moreover, S is bounded, in fact contained in the multi-interval $[-1,1]^n$. So by Heine-Borel and Bolzano-Weierstrass, S is compact.

By **(a)**, the function $\|\bullet\| : \mathbb{R}^n \to \mathbb{R}$ is continuous. And the continuous image of a compact space is compact. Therefore the set

$$\|S\| := \{\|\vec{x}\| | \vec{x} \in S\}$$

is a compact subset of $\mathbb{R}$. By Heine-Borel, $\|S\|$ is bounded, and thus has both a least upper bound M and a greatest lower bound m. And since $\|S\|$ is closed, both M and m are in $\|S\|$. Therefore there exist vectors $\vec{x}_m, \vec{x}_M \in S$ such that $\|\vec{x}_m| = m$ and $|\vec{x}_M| = M$. Note in particular that since $\|\vec{x}_m\|_1 = \|\vec{x}_M\|_1 = 1 \neq 0$, by positive definiteness of $\|\bullet\|_1$ we have $\vec{x}_m \neq \vec{0}$ and $\vec{x}_M \neq \vec{0}$. And then by positive definiteness of $\|\bullet\|$, also m and M are positive real numbers.

Finally it remains to prove that for every $\vec{x} \in \mathbb{R}$,

$$m \cdot \|\vec{x}\|_1 \leq \|\vec{x}\| \leq M \cdot \|\vec{x}\|_1.$$

The claim is that there exists $\hat{x} \in S$ such that $\vec{x} = \|\vec{x}\|_1 \cdot \hat{x}$. If $\vec{x}$ equals $\vec{0}$, then $\|\vec{0}\|_1 = 0$ so that any element $\hat{x} \in S$ satisfies the equation. And if $\vec{x} \neq \vec{0}$, then by positive definiteness $\|\vec{x}\|_1 \neq 0$ so that there is a unique vector $\hat{x} = (1/\|\vec{x}\|_1) \cdot \vec{x}$ satisfying the equation. Finally, by homogeneity of $\|\bullet\|_1$ we have

$$\|\hat{x}\|_1 = \left\| \frac{1}{\|\vec{x}\|_1} \cdot \vec{x} \right\|_1 = \frac{1}{\|\vec{x}\|_1} \|\vec{x}\|_1 = 1.$$

Therefore $\hat{x}$ is in S. This proves the claim. By homogeneity of $\|\bullet\|$, we have

$$\|\vec{x}\| = \|\vec{x}\|_1 \cdot \|\hat{x}\|.$$

And by definition of m and M, we have

$$m \leq \|\hat{x}\|_1 \leq M$$

for every $\hat{x} \in S$. Combined with the equation above this gives

$$m \cdot \|vecx\|_1 \leq \|\vec{x}\| \leq M \cdot \|\vec{x}\|_1.$$

Solution to (c) Because of **(b)**, the metrics $d_{\|\bullet\|_1}$ and $d_{\|\bullet\|}$ defined by $\|\bullet\|_1$ and $\|\bullet\|$ satisfy the inequalities

$$m \cdot d_{\|\bullet\|_1}(\vec{v}, \vec{w}) \leq d_{\|\bullet\|}(\vec{v}, \vec{w} \leq M \cdot d_{\|\bullet\|_1}(\vec{v}, \vec{w})$$

for every $\vec{v}, \vec{w} \in \mathbb{R}^n$. A bit more generally, for any set X and any pair of metrics d_1, d on this set, the metrics are *equivalent* if there exist real numbers $m, M > 0$ such that for every $x, y \in X$,

$$m \cdot d_1(x,y) \leq d(x,y) \leq M \cdot d_1(x,y).$$

Observe that this relation is symmetric since we have

$$\frac{1}{M} \cdot d(x,y) \leq d_1(x,y) \leq \frac{1}{m} \cdot d(x,y).$$

The claim is that equivalent metrics define the same topology on X. Since the open sets in the metric topology are precisely the unions of balls, it suffices to prove that the balls $B_{r,d}(x)$ for the metric d are unions of balls $B_{r,d_1}(x)$ for the metric d_1 and vice versa. In fact, using that the relation of equivalence of metrics is symmetric, it suffices to check the first of these. Let y be any element of $B_{r,d}(x)$. By the triangle inequality, for $s = r - d(x,y) > 0$, $B_{s,d}(y)$ is contained in $B_{r,d}(x)$. And by the inequality above, $B_{s/M,d_1}(y) \subset B_{s,d}(y)$, i.e., $d_1(z,y) < s/M$ implies that $d(z,y) \leq M \cdot d_1(z,y) < s$. Thus every element y in $B_{r,d}(x)$ is contained in a ball $B_{s/M,d_1}(y)$ which is contained in $B_{r,d}(x)$. Therefore $B_{r,d}(x)$ is a union of balls $B_{s/M,d_1}(y)$, which proves that the metric topology for d_1 equals the metric topology for d. In particular, the metric topology for $\| \bullet \|_1$ on $\mathbb{R}^n$ equals the metric topology for $\| \bullet \|$ on $\mathbb{R}^n$.

Since the "closed unit ball" $B_{\leq 1}(\vec{0})$ is closed for the metric topology coming from $\|\bullet\|$, and since this topology is the usual topology, $B_{\leq 1}(\vec{0})$ is closed for the usual topology. And since $\|\vec{v}\| \leq M \cdot \|\vec{v}\|_1$, also the ball $B_{\leq 1}(\vec{0})$ for the norm $\| \bullet \|$ is contained in the ball $B'_{\leq M}(\vec{0})$ for the norm $\| \bullet \|_1$. Thus $B_{\leq 1}(\vec{0})$ is bounded. So by Heine-Borel and Bolzano-Weierstrass, $B_{\leq 1}(\vec{0})$ is compact.

Solution to Problem 3

Solution to (a) Since W does not equal all of V, there exists $\vec{u} \in V$ which is not in W. Since W is a closed, linear subspace, by the Hahn-Banach theorem there exists a bounded linear functional

$$\phi : V \to \mathbb{R}$$

such that each of the following hold

(i) for every $\vec{w} \in W$, $\phi(\vec{w}) = 0$,

(ii) the operator norm $\|\phi\|_{\text{op}}$ equals 1, and

(iii) $\phi(\vec{u}) = \inf(\|\vec{u} - \vec{w}\|)_{\vec{w} \in W}$.

Since the operator norm $|\phi\|_{\text{op}}$ is ≤ 1, for every $\vec{x} \in V$ we have $|\phi(\vec{x})| \leq \|x\|$. And since the operator norm equals precisely 1, for every real $\epsilon > 0$, the operator norm is not $\leq 1 - \epsilon$, i.e., there exists $\vec{x}_\epsilon$ such that $|\phi(\vec{x}_\epsilon)| > (1-\epsilon)\|\vec{x}_\epsilon\|$. Now $\phi(\vec{0}) = 0$ and $\|\vec{0}\| = 0$ so that $|\phi(\vec{0})| = (1-\epsilon)\|\vec{0}\|$, and thus also $\vec{x}_\epsilon \neq \vec{0}$. By positive definiteness, $\|\vec{x}_\epsilon\| > 0$. Defining $\vec{v}_\epsilon := \vec{x}_\epsilon / \|\vec{x}_\epsilon\|$, it follows (by homogeneity of $\| \bullet \|$) that $\|\vec{v}_\epsilon\| = 1$.

Also we have
$$|\phi(\vec{v}_\epsilon)| > (1-\epsilon)\cdot\|\vec{v}_\epsilon\| = (1-\epsilon)\cdot 1 = 1-\epsilon.$$
Now for every $\vec{w}\in W$, consider $\vec{x} = \vec{v}_\epsilon - \vec{w}$. Since $\|\phi\|_{\text{op}} \leq 1$, we have $|\phi(\vec{x})| \leq \|\vec{x}\|$, i.e.,
$$|\phi(\vec{v}_\epsilon - \vec{w})| \leq \|\vec{v}_\epsilon - \vec{w}\|.$$
Since ϕ is linear, $\phi(\vec{v}_\epsilon - \vec{w})$ equals $\phi(\vec{v}_\epsilon) - \phi(\vec{w})$. And by property (i) above, $\phi(\vec{w})$ equals 0. Thus $|\phi(\vec{v}_\epsilon - \vec{w})| = |\phi(\vec{v}_\epsilon)| > 1-\epsilon$. Therefore
$$1-\epsilon < |\phi(\vec{v}_\epsilon - \vec{w})| \leq \|\vec{v}_\epsilon - \vec{w}\|$$
for every $\vec{w}\in W$.

Solution to (b) By **(a)**, for every integer $k > 1$, taking $\epsilon = 1/k$, there exists a vector $\vec{v}_k \in V$ with $\|\vec{v}_k\| = 1$ and such that for every $\vec{w}\in W$, $|\vec{v}_k - \vec{w}| > 1-(1/k)$. Since $\|\vec{v}_k\|$ equals 1, $\vec{v}_k$ is in the closed unit ball $B_{\leq 1}(\vec{0})$. By hypothesis the closed unit ball is compact. Thus the sequence $(\vec{v}_k)_{k=1,2,\dots}$ has a convergent subsequence, $(\vec{v}_{k_j})_{j=1,2,\dots}$. Denote the limit by $\vec{v}_0$. Since the norm $\|\bullet\|$ is a continuous function (for the topology defined by the norm), the sequence $(\|\vec{v}_{k_j}\|)_{j=1,2,\dots}$ converges to $\|\vec{v}_0\|$. But of course $\|\vec{v}_{k_j}\|$ equals 1, so that this is the constant sequence $(1,1,1,\dots)$ which converges to 1. Thus $|\vec{v}_0\|$ equals 1. Similarly, for every $\vec{w}\in W$, the sequence $(\|\vec{v}_{k_j} - \vec{w}\|)_{j=1,2,\dots}$ converges to $\|\vec{v}_0 - \vec{w}\|$. And by hypothesis $|\vec{v}_{k_j} - \vec{w}\| > 1-(1/k_j)$ for every $j = 1,2,\dots$. Since the sequence $(\|\vec{v}_{k_j} - \vec{w}\|)_{j=1,2,\dots}$ is term-by-term greater than or equal to the sequence $(1-(1/k_j))_{j=1,2,\dots}$, also the limit $\|\vec{v}_0 - \vec{w}\|$ is greater than or equal to the limit 1. Therefore for every $\vec{w}\in W$, $\|\vec{v}_0 - \vec{w}\| \geq 1$.

Solution to Problem 4

Solution to (a) Let $(\vec{w}_n)_{n=1,2,\dots}$ be a sequence in W which has a limit $\vec{u}$ in V. The goal is to prove that $\vec{u}$ is in W. The first claim is that there exists a real number $r > 0$ such that the sequence $((1/r)\cdot\vec{w}_n)_{n=1,2,\dots}$ is a sequence in $W\cap B_V$. Since the sequence converges, there exists an integer N such that for every $n > N$, $\|\vec{w}_n - \vec{u}\| \leq 1$. By the triangle inequality,
$$\|\vec{w}_n\| \leq \|\vec{w}_n - \vec{u}\| + |\vec{u}\| = 1 + \|\vec{u}\|.$$
Define $r = \max(\|\vec{w}_1\|,\dots,\|vecw_N\|, 1+\|\vec{u}\|)$. Then for every $n = 1,2,\dots$, $\|\vec{w}_n\| \leq r$ so that $\|(1/r)\vec{w}_n\| \leq 1$. This proves the first claim.

Now, by continuity of the function $\vec{x}\mapsto (1/r)\vec{x}$, the sequence $((1/r)\vec{w}_n)$ converges to $(1/r)\vec{u}$. Since W is linear, it is stable under scaling by $1/r$. Thus $((1/r)\vec{w}_n)$ is a sequence in the closed unit ball $B_V\cap W = B_W$ in W. But as proved in **Problem 2(c)**, this ball is compact. So there exists a subsequence $((1/r)\vec{w}_{n_k})_{k=1,2,\dots}$ which converges to a limit $\vec{w}$ in B_W. But since $((1/r)\vec{w}_n)$ converges to $(1/r)\vec{u}$, also the subsequence $((1/r)\vec{w}_{n_k})$ converges to $(1/r)\vec{u}$. Thus $(1/r)\vec{u}$ equals $\vec{w}$, which in W. Therefore $\vec{u}$ equals $r\cdot\vec{w}$. Since W is linear, it is stable under scaling by r. Thus $r\cdot\vec{w}$ is in W, i.e., $\vec{u}$ is in W. Since the limit in V of every convergent sequence of elements in W is again in W, W is a closed subset of V.

Solution to (b) By way of contradiction, assume that $(\vec{v}_k)_{k=2,3,\dots}$ has a subsequence $(\vec{v}_{k_j})_{j=1,2,\dots}$ which converges to a limit $\vec{v}_\infty$. Then there exists an integer J such that for every integer $j \geq J$, $\|\vec{v}_\infty - \vec{v}_{k_j}\| < 1/2$. Since $j+1 > j \geq J$, the same holds for $\vec{v}_{k_{j+1}}$. Then by the triangle inequality,

$$\|\vec{v}_{k_{j+1}} - \vec{v}_{k_j}\| \leq \|\vec{v}_{k_{j+1}} - \vec{v}_\infty\| + \|\vec{v}_\infty - \vec{v}_{k_j}\| < 1/2 + 1/2 = 1.$$

Yet $\vec{v}_{k_j} \in W_{k_j} \subset W_{k_{j+1}-1}$, contradicting the hypothesis that $\|\vec{v}_{k_{j+1}} - \vec{w}_{k_{j+1}-1}\| \geq 1$ for every $\vec{w}_{k_{j+1}-1} \in W_{k_{j+1}-1}$. Therefore $(\vec{v}_k)_{k=2,3,\dots}$ has no convergent subsequence. Since this is a sequence in B_V, it follows that B_V is not compact.

Solution to Problem 5 The sequence from the solution to Problem 5 on Problem Set 2 gives an example of an equicontinuous, pointwise bounded sequence which has no subsequence which is convergent in the uniform norm (since the uniform norms of the elements in the sequence form a strictly increasing, unbounded sequence of real numbers).