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MAT 544 Problem Set 3

Problems.

Problem 1 The real vector space ℓ^1 is defined to be the vector space of all sequences of real numbers $(x_k)_{k=1,2,\dots}$ which are *absolutely convergent*, i.e., $\sum_{k=1}^{\infty}|x_k|$ is finite. For an absolutely convergent sequence, the ℓ^1-norm is defined by

$$\|(x_k)\|_{\ell^1} := \sum_{k=1}^{\infty}|x_k|.$$

In this exercise you may assume that this defines a real normed vector space. Prove that ℓ^1 is a Banach space, i.e., prove that the metric is complete.

Problem 2 Denote by $\|\bullet\|_1 : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ the usual product norm,

$$\|\vec{x}\|_1 := \sum_{k=1}^{n}|x_k| \text{ for } \vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

Let $\|\bullet\| : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be a norm on $\mathbb{R}^n$.

(a) Prove that there exists a real number $C > 0$ such that for all $\vec{x} \in \mathbb{R}^n$, $\|\vec{x}\| \leq C \cdot \|\vec{x}\|_1$. Conclude that $\|\bullet\| : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a continuous function on $\mathbb{R}^n$ with the topology coming from the norm $\|\bullet\|_1$.

(b) Consider the subset $S = \{\vec{x} \in \mathbb{R}^n | \|\vec{x}\|_1 = 1\}$ of $\mathbb{R}^n$. Prove that the function $\|\bullet\|$ takes on a maximum value $M > 0$ and a minimum value $m > 0$ on S. Thus for all $\vec{x} \in \mathbb{R}^n$ we have $m \cdot \|\vec{x}\|_1 \leq \|\vec{x}\| \leq M \cdot \|\vec{x}\|_1$.

(c) Conclude that the topology on $\mathbb{R}^n$ coming from the norm $\|\bullet\|$ equals the topology coming from the usual norm $\|\bullet\|_1$. Also conclude that the closed unit ball $B_1(\vec{0})$ for the norm $\|\bullet\|$ is compact.

Problem 3 Let $(V, \|\bullet\|)$ be a real normed vector space. Let W be a linear subspace of V which is a closed subset of V and which does not equal all of V.

(a) Use the Hahn-Banach theorem to prove that for every real $\epsilon > 0$ there exists $\vec{v}_\epsilon \in V$ with $\|\vec{v}_\epsilon\| = 1$ and with $\|\vec{v}_\epsilon - \vec{w}\| \geq 1 - \epsilon$ for all $\vec{w} \in W$.

(b) Under the further hypothesis that the closed unit ball in V is compact, prove that there exists $\vec{v}_0 \in V$ with $\|\vec{v}_0\| = 1$ and with $\|\vec{v}_0 - \vec{w}\| \geq 1$ for all $\vec{w} \in W$.

Problem 4 Let $(V, \| \bullet \|)$ be a real normed vector space. Denote by B_V the closed unit ball in V centered at the origin (defined with respect to the norm).

(a) For every linear subspace $W \subset V$ which is finite dimensional, prove that W is closed in V.

Hint. It suffices to prove that the intersection $B_V \cap W = B_W$ is closed in B_V. What does **Problem 2(c)** imply about B_W?

(b) Assume that V is infinite dimensional and let $W_1 \subsetneq W_2 \subsetneq \cdots \subset V$ be an increasing sequence of finite dimensional vector spaces. For every integer $k \geq 2$, by **Problem 3(b)** there exists $\vec{v}_k \in B_{W_k}$ with $\|\vec{v}_k - \vec{w}_{k-1}\| \geq 1$ for all $\vec{w}_{k-1} \in W_{k-1}$. Prove that the sequence $(\vec{v}_k)_{k=2,3,\ldots}$ has no convergent subsequence. Conclude that the ball B_V is not compact if V is infinite dimensional. Combined with **Problem 2(c)**, it follows that for a normed vector space V, B_V is compact if and only if V is finite dimensional.

Problem 5 Let (X, d_X) be a metric space and let $BC(X, \mathbb{R})$ be the set of all bounded, continuous functions on X with the uniform metric. If X is not compact, prove that there exists a sequence $(f_k)_{k=1,2,\ldots}$ in $BC(X, \mathbb{R})$ which is equicontinuous and pointwise bounded, but which has no convergent subsequence. Thus the Arzela-Ascoli theorem holds only if X is compact.

Hint. This is very similar to Problem 2(b) and Problem 5(b) from Problem Set 2.