MAT131 Fall 2022 Paper HW 10

Due the week of November 14 – November 18. For all problem sets, students are allowed to work together. However, the final answer you turn in must be based on your own understanding and must be in your words. Per university policy, all instances of suspected academic dishonesty will be referred to the academic judiciary.

Problem 1. Riemann sums **do** exist for a Riemann integrable function $f(x)$ on an interval $[a, b]$ with respect to partitions into subintervals that do **not** necessarily have equal length. Moreover, for every $\epsilon > 0$ there exists $\delta > 0$ such that the Riemann integral of f over $[a, b]$ is estimated to within ϵ by the Riemann sum of any partition whose (possibly varying) lengths of subintervals are all $< \delta$. Work through the following example.

Let p be a positive integer, let $f(x) = x^{p-1}$, let $a = 1$, let $b = e^B > 1$ for some $B > 0$, let n be a positive integer, and let $1 = x_0 < \cdots < x_k < \cdots < x_n = b$ be the partition with $x_k = b^{k/n} = e^{kB/n} = q_n^k$ for $q_n = b^{1/n} = e^{B/n}$. For each $k = 1, \ldots, n$, the length $\Delta x_k = x_k - x_{k-1}$ depends on k, but the fractions $(1/x_k)\Delta x_k = 1 - (x_{k-1}/x_k)$ and $(1/x_{k-1})\Delta x_k = (x_k/x_{k-1}) - 1$ do not. Here are the left-endpoint and right-endpoint Riemann sums,

$$R_n = \sum_{k=1}^{n} (x_k)^p (1/x_k)\Delta x_k = \sum_{k=1}^{n} (q_n^p)^{(k-1)} \cdot \frac{q_n^p \Delta x_k}{x_k},$$

$$L_n = \sum_{k=1}^{n} (x_{k-1})^p (1/x_{k-1})\Delta x_k = \sum_{k=1}^{n} (q_n^p)^{(k-1)} \cdot \frac{\Delta x_k}{x_{k-1}}.$$

Evaluate both sums explicitly using the geometric sum formula,

$$1 + q_n^p + q_n^{2p} + \cdots + q_n^{(n-1)p} = \frac{q_n^{np} - 1}{q_n^p - 1} = \frac{b^p - 1}{q_n^p - 1}.$$

Finally, evaluate these limits using the limit computed earlier in the semester as part of the proof of the Power Law,

$$\lim_{q\to 1}\frac{q-1}{q^p-1} = 1/\lim_{q\to 1}(1+q+\cdots+q^{p-1}) = 1/p.$$

Problem 2. For p equal to 3 and for p equal to 4, compute the left-endpoint Riemann sum L_n and the right-endpoint Riemann sum R_n of $f(x) = x^{p-1}$ on the interval $[1,b]$ for the **uniform partition** of $[1,b]$ into n subintervals of equal length $\Delta x = (b-1)/n$. Explain the summation formulas that you are using, and compute the limit as n tends to ∞ of both R_n and L_n.