MAT 131 Midterm 1 Review

Exam Policy. Exam 1 will be held on Wednesday, September 28th, from 7:50 PM to 9:15 PM. The rooms are to be announced. The exam is closed book, closed notes, no electronic devices are allowed, and you need only bring a writing implement. You will write directly on the exam booklet. Scratch paper and a stapler will be provided.

Review Topics. Please be familiar with all of the following course outcomes / key skills.

• Definition, basic properties and graphs of elementary functions: powers, exponentials, logarithms, and trigonometric.

• The definition, basic properties and graphs of even and odd functions.

• The definition and meaning of increasing and decreasing for functions and graphs.

• Reflection, translation and scaling of graphs and the corresponding transformation of the functions.

• Definition, basic properties, and graphs of inverse functions. Computation of an inverse function.

• Intuitive definition, basic laws, and techniques for computing limits, one-sided limits, limits using the squeeze theorem, limits equal to infinity, and limits at infinity. Students are NOT expected to work with ϵ-δ notions of the limit on the midterm.

• Identifying all discontinuity points (both the location and type), the domain of a function, and all vertical and horizontal asymptotes. Application of these notions to curve-sketching.

• The statement of the Intermediate Value Theorem and its use in finding zeroes of functions.

• The definition of the derivative as the limit of a difference quotient, and methods for computing derivatives directly from the definition.

• Using the derivative to compute the equations of tangent lines.

• Using the rules of differentiation: the sum rule, the product rule, the quotient rule, the power rule, and derivatives of exponential functions.

Important Note. The best preparation is to understand the material from the textbook and the homework problems, both the assigned and unassigned problems. The exam will have some problems that test understanding of statements of propositions and theorems, but there will be an emphasis on applications of the results to computation.

Problem 1. In each of the following cases, determine whether the limit exists as a finite number, and say its value if it is defined. If the limit does not exist as a finite number, determine whether the limit is positive or negative infinity. If the limit does not exist as a finite number or as positive/negative infinity, explain why.

(a)

$$\lim_{x\to 0} f(x), \text{ where } f(x) = \begin{cases} \sqrt{x}, & x > 0 \\ -\sqrt{-x}, & x \leq 0 \end{cases}$$

Solution to (a)

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = \lim_{x\to 0} f(x) = \boxed{0}.$$

(b)

$$\lim_{x\to 0} \frac{x+ \mid x \mid}{x}.$$

Solution to (b)

$$\lim_{x\to 0^+} \frac{2x}{x} = 2,$$

$$\lim_{x\to 0^-} \frac{0}{x} = 0.$$

Since the one-sided limits exist but are not equal, the limit $\boxed{\text{does not exist}}$.

(c)

$$\lim_{x\to 0} \frac{\sqrt{x^2}}{x}$$

Solution to (c)

$$\lim_{x\to 0^+} \frac{\sqrt{x^2}}{x} = \lim_{x\to 0+} \frac{x}{x} = 1.$$

$$\lim_{x\to 0^-} \frac{\sqrt{x^2}}{x} = \lim_{x\to 0^-} \frac{(-x)}{x} = -1.$$

Since

$$\lim_{x\to 0^+} \frac{\sqrt{x^2}}{x} \neq \lim_{x\to 0^-} \frac{\sqrt{x^2}}{x},$$

thus

$$\lim_{x\to 0} \frac{\sqrt{x^2}}{x} \boxed{\text{ does not exist}}.$$

(d)

$$\lim_{x\to 2} \frac{x^3 - 2x^2 - 4x + 8}{x^2 - 4}$$

Solution to (d)

$$\lim_{x\to 2} \frac{(x-2)(x^2-4)}{(x-2)(x+2)} = \lim_{x\to 2} \frac{x^2-4}{x+2} = \frac{(2)^2-4}{2+2} = \frac{0}{4} = \boxed{0}.$$

(e)

$$\lim_{x\to 1} \frac{\ln(5^x)}{x}$$

Solution to (e)

$$\lim_{x\to 1} \frac{\ln(5^x)}{x} = \lim_{x\to 1} \frac{x\ln(5)}{x} = \lim_{x\to 1} \frac{\ln(5)}{=} \boxed{\ln(5)}.$$

(f)

$$\lim_{x\to 0^-} (3x + \sqrt{9x^2+6x})$$

Solution to (f)

$$\lim_{x\to 0^-} (3x + \sqrt{9x^2+6x}) = 3\cdot 0 + \sqrt{9\cdot 0^2 + 6\cdot 0} = \boxed{0}.$$

(g)

$$\lim_{x\to 2} \frac{|x-2|}{x-2}$$

Solution to (g)

$$\lim_{x\to 2^+} \frac{|x-2|}{x-2} = \lim_{x\to 2^+} \frac{x-2}{x-2} = 1.$$

$$\lim_{x\to 2^-} \frac{|x-2|}{x-2} = \lim_{x\to 2^-} \frac{-(x-2)}{x-2} = -1.$$

Since

$$\lim_{x\to 2^+} \frac{|x-2|}{x-2} \neq \lim_{x\to 2^-} \frac{|x-2|}{x-2},$$

thus

$$\lim_{x\to 2} \frac{|x-2|}{x-2} \boxed{\text{ does not exist}}.$$

(i)

$$\lim_{x\to 1} \frac{x^2-4x+3}{x^2-1}$$

Solution to (i)

$$\lim_{x\to 1} \frac{x^2-4x+3}{x^2-1} = \lim_{x\to 1} (x-1)(x-3)(x-1)(x+1) = \lim_{x\to 1} \frac{x-3}{x+1} = \frac{-2}{2} = \boxed{-1}.$$

(j)

$$\lim_{x\to 0} \frac{\cos x}{x}$$

Solution to (j)

$$\lim_{x\to 0^+} \frac{\cos(x)}{x} = \lim_{x\to 0^+} \frac{\cos(0)}{x} = \lim_{x\to 0^+} \frac{1}{x} = +\infty$$
$$\lim_{x\to 0^-} \frac{\cos(x)}{x} = \lim_{x\to 0^-} \frac{\cos(0)}{x} = \lim_{x\to 0^-} \frac{1}{x} = -\infty.$$

Each one-sided limit is defined as $+\infty$ or $-\infty$, but the two one-sided limits are not the same. Thus

$$\lim_{x\to 0} \frac{\cos(x)}{x} \boxed{\text{does not exist}},$$

neither as a finite number nor as $+\infty$ nor $-\infty$.

(k)

$$\lim_{x\to 0^+} \frac{\cos(x)}{x}$$

Solution to (k)

$$\lim_{x\to 0^+} \frac{\cos(x)}{x} = \cos(0) \times \lim_{x\to 0^+} \frac{1}{x} = \lim_{x\to 0^+} \frac{1}{x} = \boxed{+\infty}.$$

(l)

$$\lim_{x\to 0} \sin(1/x)$$

Solution to (l)

$$\lim_{x\to 0} \sin(1/x) \boxed{\text{does not exist}}$$

since in every δ-neighborhood of the origin the values oscillate between positive numbers (say $+1$ when x equals $2/(2N+1)\pi$ for N a sufficiently positive integer), and negative numbers (say -1 when x equals $2/(2N-1)\pi$ for N a sufficiently positive integer).

(m)

$$\lim_{x\to 0} \frac{3x^{-3} + 2x^{-1} - 1}{4x^{-3} + 1}$$

Solution to (m)

$$\lim_{x\to 0} \frac{3x^{-3} + 2x^{-1} - 1}{4x^{-3} + 1} = \lim_{x\to 0} \frac{x^3(3x^{-3} + 2x^{-1} - 1))}{x^3(4x^{-3} + 1)} = \lim_{x\to 0} \frac{3 + 2x^2 - x^3}{4 + x^3} = \boxed{\tfrac{3}{4}}.$$

(n)

$$\lim_{x\to 0} \frac{4^{1/x}}{2^{1/x}}$$

Solution to (n) The fraction is $2^{2/x} \cdot 2^{-1/x} = 2^{1/x}$. Also,

$$\lim_{x \to 0^+} 2^{1/x} = +\infty,$$

whereas

$$\lim_{x \to 0^-} 2^{1/x} = 0.$$

Thus, the limit does not exist, neither as a finite number, nor as $+\infty$ nor $-\infty$.

(o)

$$\lim_{x \to 0^+} \ln(x)$$

Solution to (o) This limit equals $-\infty$.

(p)

$$\lim_{x \to 0} \ln(|\, x\,|).$$

Solution to (p) By the previous part, this limit equals $+\infty$.

(q)

$$\lim_{x \to 2} \frac{x^2 - 5x + 6}{x^2 - 4}$$

Solution to (q) By factoring, $x^2 - 5x + 6 = (x - 2)(x - 3)$ and $x^2 - 4 = (x - 2)(x + 2)$. Thus,

$$\lim_{x \to 2} \frac{x^2 - 5x + 6}{x^2 - 4} = \lim_{x \to 2} \frac{x - 3}{x + 2} = \frac{2 - 3}{2 + 2} = \boxed{-1/4}.$$

(r)

$$\lim_{x \to 3} \frac{x^2 - 6x + 9}{2x - 6}$$

Solution to (r) By factoring, $x^2 - 6x + 9 = (x - 3)^2$ and $2x - 6 = 2(x - 3)$. Thus,

$$\lim_{x \to 3} \frac{x^2 - 6x + 9}{2x - 6} = \lim_{x \to 3} \frac{x - 3}{2} = \frac{3 - 3}{2} = \boxed{0}.$$

(s)

$$\lim_{x \to 1} \frac{\ln(x)}{\ln(x^2)}$$

Solution to (s)

$$\lim_{x \to 1} \frac{\ln(x)}{\ln(x^2)} = \lim_{x \to 1} \frac{\ln(x)}{2\ln(x)} = \lim_{x \to 1} \frac{1}{2} = \boxed{1/2}.$$

(t)

$$\lim_{x\to 0}\left(\frac{x+1}{x}+1+\frac{x-1}{x}\right)$$

Solution to (t) Clearing denominators gives,

$$\lim_{x\to 0}\left(\frac{x+1}{x}+1+\frac{x-1}{x}\right)=\lim_{x\to 0}\left(\frac{x+1}{x}+\frac{x}{x}+\frac{x-1}{x}\right)=\lim_{x\to 0}\frac{(x+1)+x+(x-1)}{x}.$$

Thus,

$$\lim_{x\to 0}\left(\frac{x+1}{x}+1+\frac{x-1}{x}\right)=\lim_{x\to 0}\frac{3x}{x}=\boxed{3}.$$

(u)

$$\lim_{x\to 0}\left(\frac{1}{\frac{1}{x}-\frac{x^2+1}{x^3}}\right)$$

Solution to (u) Multiplying numerator and denominator by the common factor of x^3 gives,

$$\frac{1}{\frac{1}{x}-\frac{x^2+1}{x^3}}=\frac{x^3}{x^2-(x^2+1)}=-x^3$$

for all $x\neq 0$. Therefore,

$$\lim_{x\to 0}\left(\frac{1}{\frac{1}{x}-\frac{x^2+1}{x^3}}\right)=\lim_{x\to 0}-x^3=\boxed{0}.$$

(v)

$$\lim_{x\to 0^+}\left(\sqrt{1+x^{-2}}-x^{-1}\right)$$

$$\lim_{x\to 0^-}\left(\sqrt{1+x^{-2}}-x^{-1}\right)$$

Solution to (v) Using difference of squares,

$$(u-v)(u+v)=u^2-v^2$$

with the substitutions $u=\sqrt{1+x^{-2}}$, $v=x^{-1}$ yields,

$$\left(\sqrt{1+x^{-2}}-x^{-1}\right)\left(\sqrt{1+x^{-2}}+x^{-1}\right)=(1+x^{-2})-x^{-2}=1.$$

Dividing gives,

$$\left(\sqrt{1+x^{-2}}-x^{-1}\right)=\frac{1}{\sqrt{1+x^{-2}}+x^{-1}}.$$

Thus

$$\lim_{x\to 0^+}\left(\sqrt{1+x^{-2}}-x^{-1}\right) = \lim_{x\to 0^+}\frac{x}{\sqrt{x^2+1}+1} = \boxed{0}.$$

On the other hand, clearly,

$$\lim_{x\to 0^{-1}}\left(\sqrt{x^2+1}-x\right) = \infty+\infty = \boxed{\infty}.$$

Since the two one-sided limits are 0 and ∞, the (two-sided) limit does not exist, neither as a finite real number nor as $+\infty$ nor $-\infty$.

(w)

$$\lim_{x\to 0^+}\left(\sqrt{1+x^{-2}}+x^{-1}\right)$$

$$\lim_{x\to 0^-}\left(\sqrt{1+x^{-2}}+x^{-1}\right)$$

Solution to (w) By the same method as above, the first one-sided limit gives $\boxed{+\infty}$, and the second one-sided limit gives $\boxed{0}$. Thus, the (two-sided) limit does not exist, neither as a finite real number nor as $+\infty$ nor $-\infty$.

(x)

$$\lim_{x\to 0}\frac{x^9-1}{x-1}$$

Solution to (x) There are many ways to evaluate this. One method is to use the formula for a geometric sum,

$$\lim_{x\to 0}(1+x+x^2+x^3+x^4+x^5+x^6+x^7+x^8) = \boxed{1}.$$

Notice that the limit as x approaches 1 equals 9 (this is probably what was originally intended with this problem – the limit as x approaches 0 is less relevant than the limit as x approaches 1).

(y)

$$\lim_{x\to 0}\frac{1}{\sin(x)}$$

$$\lim_{x\to 0}\frac{1}{|\sin(x)|}$$

Solution to (y) Of course,

$$\lim_{x\to 0^+}\frac{1}{\sin(x)} = +\infty$$

and

$$\lim_{x\to 0^-}\frac{1}{\sin(x)} = -\infty.$$

Therefore $\lim_{x\to 0}(1/\sin(x))$ is $\boxed{\text{undefined both as a finite number and as } +\infty \text{ or } -\infty}$.

On the other hand, both

$$\lim_{x\to 0^+} \frac{1}{|\sin(x)|} = +\infty$$

and

$$\lim_{x\to 0^-} \frac{1}{|\sin(x)|} = +\infty.$$

Therefore $\lim_{x\to 0}(1/|\sin(x)|)$ equals $\boxed{\infty}$.

(z)

$$\lim_{x\to 0} \ln(x^2)$$

$$\lim_{x\to 0^+} [\ln(x)]^2 .$$

Solution to (z) Since $\lim_{x\to 0}(x^2)$ equals 0^+,

$$\lim_{x\to 0} \ln(x^2) = \lim_{y\to 0^+} \ln(y) = \boxed{-\infty}.$$

On the other hand, since

$$\lim_{z\to -\infty} z^2 = \infty,$$

also

$$\lim_{x\to 0^+} [\ln(x)]^2 = \boxed{\infty}.$$

(α)

$$\lim_{x\to\infty} \frac{e^{x+1} - e^{x-1}}{e^{x+1} + e^{x-1}}$$

$$\lim_{x\to -\infty} \frac{e^{x+1} - e^{x-1}}{e^{x+1} + e^{x-1}}$$

Solution to (α) The fraction factors as

$$\frac{e^x(e^1 - e^{-1})}{e^x(e^1 + e^{-1})} = \frac{e^1 - e^{-1}}{e^1 + e^{-1}}.$$

This is a constant. Thus, both limits equal this constant, $\boxed{(e - e^{-1})/(e + e^{-1})}$.

(β)

$$\lim_{x\to\infty} e^{x+3}e^{2-2x}e^{x-5}$$

$$\lim_{x\to -\infty} e^{x+3}e^{2-2x}e^{x-5}$$

Solution to (β) The product simplifies to $e^{(x+3)+(2-2x)+(x-5)}$, i.e., $e^0 = 1$. This is a constant. Thus, both limits equal this constant, $\boxed{1}$.

Problem 2 For the following function, state the domain, whether the function is even, odd or neither, and the location and type of any and all discontinuities.

$$f(x) = \frac{1-\sqrt{1-4x^2}}{2x}.$$

Solution to Problem 2 The domain is the union of $[-1/2, 0)$ and $(0, 1/2]$. The function is odd. Note that

$$\lim_{x\to 0} \frac{1-\sqrt{1-4x^2}}{2x} = \lim_{x\to 0} \frac{1^2-(1-4x^2)}{2x(1+\sqrt{1-4x^2})} = \lim_{x\to 0} \frac{4x^2}{2x(1+\sqrt{1-4x^2})} =$$

$$\lim_{x\to 0} \frac{2x}{1+\sqrt{1-4x^2}} = \frac{2\cdot 0}{1+\sqrt{1}} = \frac{0}{2} = 0.$$

Thus $f(x)$ has a removable discontinuity at $x=0$.

Problem 3 For each of the following functions, state the domain of the function, and the location and type of any and all discontinuities.

(a)

$$y = \frac{x}{\sqrt{x^2}-x}$$

Solution to (a) The function is defined except when $\sqrt{x^2}-x=0$, i.e., when x is nonnegative. So the domain of the function is precisely the interval $(-\infty, 0)$. On this domain, the fraction equals

$$\frac{x}{-x-x} = -1/2.$$

Thus, the function is a constant function on $(-\infty, 0)$, hence it is continuous at every point of $(-\infty, 0)$.

(b)

$$y = \frac{x+2}{x^3+x^2-2x}$$

Solution to (b) Factoring gives $x^3+x^2-2x = x(x^2+x-2) = x(x-1)(x+2)$. Thus the denominator is 0 for $x=-2$, $x=0$ and $x=1$. So $f(x)$ is undefined when $x=-2$, $x=0$ and $x=1$. The discontinuities $x=0$ and $x=1$ are each infinite discontinuities. But since

$$\lim_{x\to -2} \frac{x+2}{x(x-1)(x+2)} = \lim_{x\to -2} \frac{1}{x(x-1)} = \frac{1}{(-2)(-3)} = \frac{1}{6},$$

$x=-2$ is a removable discontinuity.

(c)

$$y = \frac{x}{1+\cos(x)}$$

Solution to (c) The denominator equals 0, and thus the expression is undefined, precisely if $\cos(x)$ equals -1. This holds if and only if x equals $(2n+1)\pi$ for some whole number n. At each of these points, there is an infinite discontinuity.

Problem 4 Find the equations of all tangent lines to the graph of $y = x^2$ which contain the point $(3, 5)$. Please note this point is *not* on the graph. You may compute the derivative by any (correct) method you know.

Note. If this review problem is discussed in lecture, we will draw a picture. For a nice Java applet illustrating this problem, scan down to the "Archimedes triangle" section of this webpage on the parabola.

Solution to 4. The derivative of $y = x^2$ at $x = a$ is

$$y'(a) = 2a.$$

Thus the equation of the tangent line to $y = x^2$ at (a, a^2) is

$$y - a^2 = 2a(x - a), \quad y = 2ax - a^2.$$

Substituting in $(x, y) = (3, 5)$, the point $(3, 5)$ lies on the tangent line at (a, a^2) if and only if

$$5 = 2a(3) - a^2.$$

Rewriting gives

$$a^2 - 6a + 5 = 0.$$

This factors as $(a - 5)(a - 1) = 0$. Thus $a = 1$ or $a = 5$. The equations of the corresponding tangent lines are

$$\boxed{y = 2x - 1} \text{ and } \boxed{y = 10x - 25}.$$

Problem 5 In each of the following cases, use the definition of the derivative as a limit of a difference quotient to compute the derivative of $y = f(x)$ at the point $x = a$. Then find the equation of the tangent line to the graph of $y = f(x)$ at the point $(a, f(a))$.

(a) $y = \sqrt{x+1}$ at $x = 3$

Solution to (a) The difference quotient is

$$\frac{1}{h}(y(3+h) - y(3)) = \frac{1}{h}(\sqrt{(3+h)+1} - \sqrt{3+1}) = \frac{1}{h}(\sqrt{4+h} - \sqrt{4}) =$$

$$\frac{1}{h}\frac{(4+h) - (4)}{\sqrt{4+h} + \sqrt{4}} = \frac{1}{\sqrt{4+h} + \sqrt{4}}$$

for $h \neq 0$. Therefore

$$y'(3) = \lim_{h \to 0} \frac{1}{h}(y(3+h) - y(3)) = \lim_{h \to 0} \frac{1}{\sqrt{4+h} + \sqrt{4}} = \frac{1}{\sqrt{4} + \sqrt{4}} = \frac{1}{2\sqrt{4}} = \boxed{1/4}.$$

So the equation of the tangent line is

$$y - \sqrt{4} = (1/4)(x-3), \quad \boxed{y = (1/4)x + (5/4)}.$$

(b) $y = x + \frac{1}{x}$ at $x = -1$

Solution to (b) The difference quotient is

$$\frac{1}{h}(y(-1+h) - y(-1)) = \frac{1}{h}((-1+h) + \frac{1}{-1+h} - (-2)) = \frac{1}{h}(1 + h + \frac{1}{-1+h}).$$

Clearing denominators gives,

$$\frac{1}{h}(1+h+\frac{1}{-1+h}) = \frac{1}{h}\frac{h^2-1}{-1+h} + \frac{1}{-1+h} = \frac{1}{h}\frac{(h^2-1)+1}{-1+h} = \frac{1}{h}\frac{h^2}{-1+h} = \frac{h}{-1+h}.$$

Therefore,

$$y'(-1) = \lim_{h\to 0} \frac{h}{-1+h} = \boxed{0}.$$

So the equation of the tangent line is

$$\boxed{y = -2.}$$

(c) $y = x^3 + x^2$ at $x = -2$

Solution to (c) The difference quotient is

$$\frac{1}{h}(y(2+h) - y(2)) = \frac{1}{h}((2+h)^3 + (2+h)^2 - 12) = \frac{1}{h}((8 + 12h + 6h^2 + h^3) + (4 + 4h + h^2) - 12) =$$

$$\frac{1}{h}(16h + 7h^2 + h^3) = 16 + 7h + h^2.$$

Therefore,

$$y'(2) = \lim_{h\to 0}(16 + 7h + h^2) = \boxed{16}.$$

So the equation of the tangent line is

$$y - 12 = 16(x-2), \quad \boxed{y = 16x - 20}.$$

(d) $y = \frac{x+1}{x-1}$ at $x = 0$

Solution to (d) The difference quotient is

$$\frac{1}{h}(y(h) - y(0)) = \frac{1}{h}\left(\frac{h+1}{h-1} - (-1)\right) = \frac{1}{h}\left(\frac{h+1}{h-1} + \frac{h-1}{h-1}\right) = \frac{1}{h}\frac{(h+1)+(h-1)}{h-1} =$$

$$\frac{1}{h}\frac{2h}{h-1} = \frac{2}{h-1}.$$

Therefore,

$$y'(0) = \lim_{h\to 0} \frac{2}{h-1} = \boxed{-2}.$$

So the equation of the tangent line is

$$y + 1 = -2x, \quad \boxed{y = -2x - 1}.$$

Problem 6 Use the definition of the derivative as a limit of a difference quotient to compute the derivative of $y = x^2 + \ln(1)\sin(x)$ at the point $x = 7$.

Solution to Problem 6 Since $\ln(1)$ equals 0, this is the same as $y = x^2$. The difference quotient is

$$\frac{y(7+h) - y(7)}{h} = \frac{(7+h)^2 - 7^2}{h} = \frac{(49 + 14h + h^2) - 49}{h} = \frac{14h + h^2}{h} = 14 + h$$

for $h \neq 0$. Thus

$$y'(7) = \lim_{h\to 0} \frac{y(7+h) - y(7)}{h} = \lim_{h\to 0}(14 + h) = \boxed{14}.$$

Problem 7 Use the definition of the derivative as a limit of a difference quotient to compute the derivative at $x = 0$ for the following function

$$y = \begin{cases} x^2, & x > 0 \\ 0, & x = 0 \\ -x^2, & x < 0 \end{cases}$$

Note. The derivative *is* defined at this point.

Solution to Problem 7 For $h > 0$, the difference quotient is

$$\frac{y(h) - y(0)}{h} = \frac{h^2 - 0}{h} = h.$$

And for $h < 0$, the difference quotient is

$$\frac{y(h) - y(0)}{h} = \frac{-h^2 - 0}{h} = -h.$$

Thus

$$\lim_{h\to 0^+} \frac{y(h) - y(0)}{h} = \lim_{h\to 0^+} h = 0,$$

and

$$\lim_{h\to 0^-} \frac{y(h) - y(0)}{h} = \lim_{h\to 0^+} (-h) = 0.$$

Since

$$\lim_{h\to 0^+} \frac{y(h) - y(0)}{h} = \lim_{h\to 0^-} \frac{y(h) - y(0)}{h} = 0,$$

also

$$y'(0) = \lim_{h\to 0} \frac{y(h) - y(0)}{h} = \boxed{0}.$$

Problem 8 Determine whether or not the following function is continuous at $x = 0$.

$$y = \begin{cases} x^2 \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Also determine whether or not the derivative of $y = f(x)$ is defined at $x = 0$. If it is defined, compute it. If it is not defined, explain why not.

Solution to Problem 8 The function is squeezed between $+x^2$ and $-x^2$ since $\sin(1/x)$ is trapped between $+1$ and -1. Since

$$\lim_{x\to 0} x^2 = \lim_{x\to 0}(-x^2) = 0,$$

by the Squeeze Theorem,

$$\lim_{x\to 0} x^2 \sin(1/x) = 0.$$

Since $y(0) = 0$ also, y is continuous at $x = 0$.

Moreover, $(y(h) - y(0))/h = h\sin(1/h)$ for $h \neq 0$. Since this is squeezed between $|h|$ and $-|h|$, and since

$$\lim_{h\to 0} |h| = \lim_{h\to 0}(-|h|) = 0,$$

also the derivative

$$y'(0) = \lim_{h\to 0} h\sin(1/h) = \boxed{0}$$

by the Squeeze Theorem.

Problem 9 In each of the following cases, use the definition of the derivative as a limit of a difference quotient to compute the *derivative function*.

(a)

$$f(x) = \frac{1}{x+3}, \text{ for } x \neq 3, \ f'(x) = ?$$

Solution to (a) By definition,

$$f'(x) = \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{(x+h)+3} - \frac{1}{x+3}\right) = \lim_{h\to 0} \frac{1}{h}\frac{(x+3)-(x+h+3)}{(x+h+3)(x+3)} =$$

$$\lim_{h\to 0} \frac{1}{h}\frac{-h}{(x+h+3)(x+3)} = \lim_{h\to 0} \frac{-1}{(x+h+3)(x+3)} = \boxed{-1/(x+3)^2}.$$

(b)

$$g(x) = 2x^2 - 4, \quad g'(x) = ?$$

Solution to (b) By definition,

$$g'(x) = \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = \lim_{h\to 0}\frac{(2(x+h)^2-4)-(2x^2-4)}{h} =$$

$$\lim_{h\to 0}\frac{(2(x^2+2xh+h^2)-4)-2x^2+4}{h} = \lim_{h\to 0}\frac{4xh+2h^2}{h} = \lim_{h\to 0}(4x+2h) = \boxed{4x}.$$

(c)

$$f(x) = \sqrt{2x-7}, \quad f'(x) = ?$$

Solution to (c) By definition,

$$f'(x) = \lim_{h\to 0}\frac{1}{h}(\sqrt{2x+2h-7}-\sqrt{2x-7}) = \lim_{h\to 0}\frac{1}{h}\frac{(2x+2h-7)-(2x-7)}{\sqrt{2x+2h-7}+\sqrt{2x-7}} =$$

$$\lim_{h\to 0}\frac{2}{\sqrt{2x+2h-7}+\sqrt{2x-7}} = \frac{2}{2\sqrt{2x-7}} = \boxed{1/\sqrt{2x-7}}.$$

(d)

$$i(x) = \frac{1}{x+1}-\frac{1}{x-1}, \quad i'(x) = ?$$

Solution to (d) By definition,

$$i'(x) = \lim_{h\to 0}\frac{1}{h}(\frac{1}{x+h+1}-\frac{1}{x+h-1}-\frac{1}{x+1}+\frac{1}{x-1}) =$$

$$\lim_{h\to 0}\frac{1}{h}(\frac{x+1}{(x+1)(x+h+1)}-\frac{x-1}{(x+h-1)(x-1)}-\frac{x+h+1}{(x+1)(x+h+1)}+\frac{x+h-1}{(x-1)(x+h-1)}) =$$

$$\lim_{h\to 0}\frac{1}{h}(\frac{(x+1)-(x+h+1)}{(x+1)(x+h+1)}-\frac{(x-1)-(x+h-1)}{(x-1)(x+h-1)}) =$$

$$\lim_{h\to 0}\frac{1}{h}(\frac{-h}{(x+1)(x+h+1)}-\frac{-h}{(x-1)(x+h-1)} =$$

$$\lim_{h\to 0}(\frac{1}{(x-1)(x+h-1)}-\frac{1}{(x+1)(x+h+1)}) = \boxed{\tfrac{1}{(x-1)^2}-\tfrac{1}{(x+1)^2}.}$$

Problem 10 Sketch the graph of a function $f(x)$ satisfying all of the following properties.

1. $\lim_{x\to 1^+} f(x) = 1$
2. $\lim_{x\to 1^-} f(x) = 0$
3. $f(1) = 1$

4. $\lim_{x \to -\infty} f(x) = 2$
5. $f(-2) = 4$
6. $\lim_{x \to -1^-} f(x) = -\infty$
7. $\lim_{x \to -1^+} f(x) = \infty$
8. $\lim_{x \to \infty} f(x) = -1$

Problem 11 In each of the following cases, say whether the statement is true or false for an everywhere continuous function $f(x)$ satisfying the stated hypothesis. If the statement is false, sketch a graph demonstrating it is false.

1. If $y = f(x)$ is increasing, then $y = -f(x)$ is increasing. FALSE
2. If $y = f(x)$ is increasing, then $y = -f(x)$ is decreasing. TRUE
3. If $y = f(x)$ is increasing, then $y = f(-x)$ is increasing. FALSE
4. If $y = f(x)$ is increasing, then $y = f(-x)$ is decreasing. TRUE
5. If $y = f(x)$ is even, it cannot be everywhere decreasing. TRUE
6. If $y = f(x)$ is odd, it cannot be everywhere decreasing. FALSE
7. An inverse function $y = f^{-1}(x)$ defined on an interval $[a, b]$ cannot be both increasing on (a, c) and decreasing on (c, b). TRUE
8. If there exists a function $y = g(x)$ defined on the set of all real numbers whose restriction to the range of $f(x)$ is an inverse of $f(x)$, then the domain of the inverse of $f(x)$ is the set of all real numbers and $g(x)$ satisfies the Horizontal Line Test on the domain of all real numbers. FALSE

Problem 12 In each of the following cases, compute the derivative using derivative rules (or limits of difference quotients, if you show all work). Explain what derivative rules you use at each step.

(a)

$$\frac{df(x)}{dx} = g(x), \quad \frac{d(f(x))^2}{dx} = ?$$

Solution to (a) By the Product Rule / Leibniz Rule, the derivative equals $2f(x)g(x)$.

(b)

$$\frac{df(x)}{dx} = g(x), \quad \frac{d(f(x))^{-1}}{dx} = ?$$

Solution to (b) By the Quotient Rule, the derivative equals $\boxed{-g(x)/(f(x))^2}$.

(c)

$$\frac{d(x^n)}{dx} = ?, \quad n = 0, 1, 2, \ldots$$

Solution to (c) By the Power Rule, this equals $\boxed{nx^{n-1}}$. This can also be proved using the Product Rule / Leibniz Rule and proof by induction on n. Finally, it can be computed directly as a limit of a difference quotient using the Binomial Theorem or various other methods.

(d)

$$\frac{d(x^{-n})}{dx} = ?, \quad n = 0, 1, 2, \ldots$$

Solution to (d) Using the Quotient Rule and the Power Rule, the derivative equals $\boxed{-nx^{-n-1}}$.

(e)

$$f(x) = \frac{(x-2)x(x+2)}{(x-1)(x+1)}, \quad f'(x) = ?$$

Solution to (e) Iteratively applying the Product Rule and the Quotient Rule gives $\boxed{(x^4 + x^2 + 4)/(x^2 - 1)^2}$.

(f)

$$f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \quad f'(x) = ?$$

Solution to (f) Iteratively applying the Quotient Rule and the Exponential Rule gives $\boxed{4/(e^x + e^{-x})^2}$.

(g)

$$f'(x) = \cos(x), \quad \frac{d(f(x))^{-1}}{dx} = ?$$

Solution to (g) By the Quotient Rule, the derivative equals $\boxed{-\cos(x)/(f(x))^2}$.

(h)

$$\frac{d}{dx}(1 + x + x^2 + \cdots + x^n) = ?, \quad n = 2, 3, \ldots,$$

Solution to (h) By the formula for a finite geometric sum and the Quotient Rule, this derivative equals $\boxed{(nx^{n+1} - (n+1)x^n + 1)/(x-1)^2}$.