Name: ______________________________ **Problem 1:** ______________ /10

Problem 1(10 points) In each of the following statements, circle T if it is true and F if it is false. Each part is worth **only 2 points out of 100 total points**. Remember to use your time wisely. There is no need to show your work on this problem.

$\boxed{\text{T}}$ F (a) If differentiable functions $f(x)$ and $g(x)$ are both 0 at $x = a$, then also the derivative of $f(x)g(x)$ equals 0 at $x = a$.

T $\boxed{\text{F}}$ (b) The only everywhere differentiable functions which are equal to their own fourth derivatives are $f(x) = \sin(x)$ and $f(x) = \cos(x)$.

T $\boxed{\text{F}}$ (c) The derivative of every inverse trigonometric function is another inverse trig function.

$\boxed{\text{T}}$ F (d) For differentiable functions $f(x)$ and $g(x)$, if $g'(a)$ equals 0 then the derivative at $x = a$ of $f(g(x))$ equals 0.

T $\boxed{\text{F}}$ (e) For every real number $a > 0$, the derivative of $f(x) = a^x$ is an everywhere positive function.

Name: ______________________ Problem 2: ____________ /20

Problem 2(20 points) In each of the following cases, compute the derivative. You may use the derivative formulas for $\sin(x)$, $\cos(x)$, e^x and $\ln(x)$, and the derivative rules such as the product rule, quotient rule and chain rule. But you must show all other work.

Show your work and write a box or circle around your final $\boxed{\text{answer.}}$

(a)(4 points)

$$\frac{d}{dx}\sqrt{1-\cos^2(x)}, \quad \text{for } 0 < x < \pi/2$$

Solution 1.

For $0<x<\frac{\pi}{2}$, $\sqrt{1-\cos^2(x)}$ equals $\sin(x)$,

$\frac{d}{dx}\sin(x)$ equals $\boxed{\cos(x)}$

Solution 2.

$$\frac{d}{dx}(1-\cos^2(x))^{\frac{1}{2}} = \frac{1}{2}(1-\cos^2(x))^{-\frac{1}{2}}\frac{d}{dx}(1-\cos^2(x))$$

$$= \frac{1}{2}(1-\cos^2(x))^{-\frac{1}{2}}\cdot\left(0-2\cos(x)\frac{d}{dx}\cos(x)\right)$$

$$= \frac{1}{2}(1-\cos^2(x))^{-\frac{1}{2}}\cdot(-2\cos(x))\cdot(-\sin(x))$$

$$= \boxed{\frac{\sin(x)}{\sqrt{1-\cos^2(x)}}\cdot\cos(x)}$$

(b)(6 points)

$$\frac{d}{dx}x^{\ln(x)}$$

Hint. Use logarithms.

$y = x^{\ln(x)}$, $\ln(y) = \ln(x^{\ln(x)}) = \ln(x)\cdot\ln(x) = [\ln(x)]^2$

$\frac{d}{dx}\ln(y) = \frac{d}{dx}[\ln(x)]^2$, $\frac{1}{y}\frac{dy}{dx} = 2[\ln(x)]\cdot\frac{d}{dx}\ln(x) = 2\cdot\ln(x)\cdot\frac{1}{x}$

$$\frac{dy}{dx} = 2\cdot\ln(x)\cdot\frac{1}{x}\cdot y = \boxed{\frac{2\ln(x)}{x}\cdot x^{\ln(x)}}$$

Name: ______________________________ Problem 2, continued

In each of the following cases, compute the derivative. **Show all your work and write a box or circle around your final** $\boxed{\textbf{answer.}}$

(c)(6 points)

$$\frac{d}{dx}\left(\frac{x - x^{-1}}{x + x^{-1}}\right).$$

Solution 1.

$$\frac{x-x^{-1}}{x+x^{-1}} \cdot \frac{x}{x} = \frac{x^2-1}{x^2+1}$$

$$\frac{d}{dx}\left(\frac{x^2-1}{x^2+1}\right) = \frac{\frac{d}{dx}(x^2-1)\cdot(x^2+1) - (x^2-1)\frac{d}{dx}(x^2+1)}{(x^2+1)^2}$$

$$= \frac{2x\cdot(x^2+1) - (x^2-1)2x}{(x^2+1)^2}$$

$$= \boxed{\frac{4x}{(x^2+1)^2}}$$

Solution 2.

$$\frac{\frac{d}{dx}(x-x^{-1})\cdot(x+x^{-1}) - (x-x^{-1})\frac{d}{dx}(x+x^{-1})}{(x+x^{-1})^2}$$

$$= \frac{(1+x^{-2})\cdot(x+x^{-1}) - (x-x^{-1})(1-x^{-2})}{(x+x^{-1})^2}$$

$$= \frac{(x+x^{-1}+x^{-1}+x^{-3}) - (x-x^{-1}-x^{-1}+x^{-3})}{(x+x^{-1})^2}$$

$$= \boxed{\frac{4x^{-1}}{(x+x^{-1})^2}}$$

(d)(4 points)

$$\frac{d}{dx}\cos(\arctan(x)).$$

Solution 1.

$$\cos(\arctan(x)) = (1+x^2)^{-\frac{1}{2}}$$

$$\frac{d}{dx}\cos(\arctan(x)) = \frac{d}{dx}\left[(1+x^2)^{-\frac{1}{2}}\right]$$

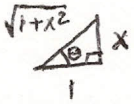
$$= -\frac{1}{2}(1+x^2)^{-\frac{3}{2}}\frac{d}{dx}(1+x^2)$$

$$= -\frac{1}{2}(1+x^2)^{-\frac{3}{2}}(0+2x)$$

$$= \boxed{\frac{-x}{(1+x^2)^{3/2}}}$$

$\theta = \arctan(x)$

$\tan(\theta) = x = \frac{x}{1}$



$\sin(\arctan(x)) =$

$\sin(\theta) = \frac{x}{\sqrt{1+x^2}}$

$\cos(\arctan(x)) =$

$\cos(\theta) = \frac{1}{\sqrt{1+x^2}}$

Solution 2

$$\frac{d}{dx}\cos(\arctan(x))$$

$$= -\sin(\arctan(x))\cdot\frac{d}{dx}\arctan(x)$$

$$= -\sin(\arctan(x))\cdot\frac{1}{1+x^2}$$

$$= -\left(\frac{x}{\sqrt{1+x^2}}\right)\cdot\frac{1}{1+x^2}$$

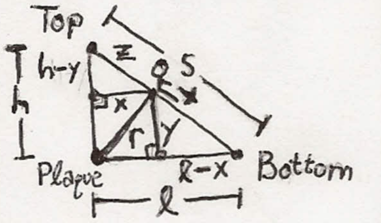
$$= \boxed{\frac{-x}{(1+x^2)^{3/2}}}$$

Name: ______________________ **Problem 3:** ____________ /20

Problem 3(20 points) A playground slide has the shape of a diagonal line whose top is a point 6 feet directly above a plaque on the ground and whose bottom is a point on the ground 8 feet horizontally distant from the plaque. A child on the slide moves away from the top at a speed of 1/2 feet per second at the moment when she is halfway down the slide. At that moment, how fast is the distance from the child to the plaque increasing?

Show all your work, including a labeled diagram and a list of equations relating the various quantities of the problem. Write your answer as a fraction, a/b feet per second. Write a box or circle around your final $\boxed{\text{answer.}}$

Diagram



Known.

$h = 6\,ft$, $\ell = 8\,ft$

At $t = t_0$ when

$x(t_0) = \frac{1}{2}\ell$, $y(t_0) = \frac{1}{2}h$, $z(t_0) = \frac{1}{2}s$,

also $\frac{dz}{dt}(t_0) = \frac{1}{2}\,ft/sec$

To find. $\frac{dr}{dt}(t_0) = ?$

Constraint Equations

Similar Triangles

① $\frac{x}{z} = \frac{\ell}{s}$, $x = \frac{\ell}{s}\cdot z$

② $\frac{h-y}{z} = \frac{h}{s}$, $y = -\frac{h}{s}\cdot z + h$

Pythagorean Theorem

③ $r^2 = x^2 + y^2$

Summary

① $x = \frac{\ell}{s} z$

② $y = -\frac{h}{s} z + h$

③ $r^2 = x^2 + y^2$

Implicit Differentiation.

① $\frac{d}{dt}(x) = \frac{d}{dt}\left(\frac{\ell}{s}\cdot z\right)$, $\frac{dx}{dt} = \frac{\ell}{s}\frac{dz}{dt}$

② $\frac{d}{dt}(y) = \frac{d}{dt}\left(-\frac{h}{s}z + h\right)$, $\frac{dy}{dt} = -\frac{h}{s}\frac{dz}{dt}$

③ $\frac{d}{dt}(r^2) = \frac{d}{dt}(x^2 + y^2)$,

$2r\frac{dr}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2x\left(\frac{\ell}{s}\frac{dz}{dt}\right) + 2y\left(\frac{-h}{s}\frac{dz}{dt}\right)$

$= \frac{2(x\ell - yh)}{s}\frac{dz}{dt}$;

$\frac{dr}{dt} = \frac{(x\ell - yh)}{rs}\frac{dz}{dt}$

At time $t = t_0$.

$y = \frac{1}{2}h$, so $h - y = \frac{1}{2}h$.

So $r^2 = (\frac{1}{2}\ell)^2 + (\frac{1}{2}h)^2$ and

$z^2 = (\frac{1}{2}\ell)^2 + (\frac{1}{2}h)^2$.

So $r(t_0) = z(t_0) = \frac{1}{2}s$.

So $\frac{x\ell - yh}{rs} = \frac{(\frac{1}{2}\ell)\ell - (\frac{1}{2}h)h}{(\frac{1}{2}s)s}$

$= \frac{\ell^2 - h^2}{s^2}$.

Also $s = \sqrt{\ell^2 + h^2} = \sqrt{64ft^2 + 36ft^2}$.

So $s = 10ft$. Therefore

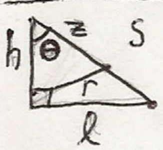
$\frac{dr}{dt}(t_0) = \frac{\ell^2 - h^2}{s^2}\frac{dz}{dt}$

$= \frac{64ft^2 - 36ft^2}{100\,ft^2} \cdot \frac{1}{2}\frac{ft}{sec}$

$= \frac{28ft^2}{100ft^2} \cdot \frac{1}{2}\frac{ft}{sec}$

$= \boxed{\frac{7}{50}\frac{ft}{sec}}$

Second Solution



$\sin(\theta) = \ell/s$

$\cos(\theta) = h/s$

Constraint Equation

Law of Cosines

$r^2 = z^2 - 2zh\cos(\theta) + h^2$

$2r\frac{dr}{dt} = 2z\frac{dz}{dt} - 2h\cos(\theta)\frac{dz}{dt}$

$= 2(z - h\cos(\theta))\frac{dz}{dt}$

$= \frac{2(sz - h^2)}{s}\frac{dz}{dt}$

$\frac{dr}{dt} = \frac{sz - h^2}{rs}\frac{dz}{dt}$

$\frac{dr}{dt}(t_0) = \frac{s(\frac{1}{2}s) - h^2}{(\frac{1}{2}s)s}\frac{dz}{dt}$

$= \frac{s^2 - 2h^2}{s^2}\frac{dz}{dt}$

$= \frac{\ell^2 - h^2}{s^2}\frac{dz}{dt} = \frac{7}{50}\frac{ft}{sec}$.

Name: ______________________ **Problem 4:** ____________ /15

Problem 4(15 points) The inverse function of $\tan(x)$ is $\tan^{-1}(x) = \arctan(x)$. Derive the formula for the derivative of $\arctan(x)$.

You are allowed to use the formulas you know for the derivatives of $\sin(x)$, $\cos(x)$ or $\tan(x)$. You may also use the trigonometric formula relating $\sin^2(x)$ and $\cos^2(x)$.

Show all of the steps in your derivation. If a step is missing or is not clearly written, you will lose points.

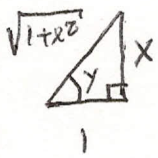
$$y = \tan^{-1}(x)$$

$$\tan(y) = x$$

$$\frac{d}{dx}\tan(y) = \frac{d}{dx}(x)$$

$$\sec^2(y)\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\sec^2(y)} = \cos^2(y)$$



$$,\quad \cos(y) = \frac{1}{\sqrt{1+x^2}}$$

$$\frac{dy}{dx} = \left(\frac{1}{\sqrt{1+x^2}}\right)^2,$$

$$\boxed{\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}}$$

Name: ______________________ **Problem 5:** __________ /20

Problem 5(20 points) Consider the smooth curve in the plane with implicit equation

$$y^2 = x^3 - 3x - 1.$$

(a)(10 points) Find the equation of the tangent line to the curve at the point $(2, 1)$.
Show all your work. Write your answer in slope-intercept form $y = mx + b$.

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(x^3 - 3x - 1)$$

$$2y\frac{dy}{dx} = 3x^2 - 3 = 3(x^2 - 1)$$

$$\frac{dy}{dx} = \frac{3(x^2-1)}{2y}$$

$$\left.\frac{dy}{dx}\right|_{(2,1)} = \frac{3((2)^2-1)}{2\cdot 1} = \frac{3(4-1)}{2\cdot 1} = \frac{3\cdot 3}{2} = \frac{9}{2}$$

Tangent line

$$y - y_0 = m(x - x_0)$$

$$y - 1 = \frac{9}{2}(x-2) = \frac{9}{2}x - 9$$

$$y = \frac{9}{2}x - 8$$

$$m_{tgt} = \frac{9}{2}$$

$$\boxed{y = \frac{9}{2}x - 8}$$

(b)(10 points) Find **both** the x- and y-coordinates of all points on the curve where the tangent line has slope 0. **Show all your work. Write each point as** $(x, y) = (a, b)$ **where** a **and** b **are numbers.**

$$2y\frac{dy}{dx} = 3(x^2 - 1)$$

Slope $\frac{dy}{dx} = 0$ implies

$$2y \cdot 0 = 3(x^2 - 1)$$

so

$$x^2 - 1 = 0$$

$$x^2 = 1$$

$x = -1$ or $x = +1$.

Case I. $x = -1$

$$y^2 = x^3 - 3x - 1 = (-1)^3 - 3(-1) - 1 = -1 + 3 - 1 = +1$$

Since $y^2 = +1$,

$y = -1$ or $y = +1$

$(x,y) = (-1,-1)$ or $(-1,+1)$

Case II. $x = +1$

$$y^2 = (+1)^3 - 3\cdot(+1) - 1 = 1 - 3 - 1 = -3.$$

$y^2 = -3$ has no solutions

Therefore the points on the curve $y^2 = x^3 - 3x - 1$ where tangent line is horizontal are

$\boxed{(x,y) = (-1,-1) \text{ and } (-1,+1)}$

Name: ______________________________ **Problem 6:** ______________ /15

Problem 6(15 points) Find a whole number whose fourth power is close to 15. Use this to find the approximate value of $(15)^{3/4}$ using a linear approximation or differentials.

Show all your work. Write your approximate answer as a proper fraction $a\frac{b}{c}$.

$f(x) = x^{3/4}$, $f'(x) = \frac{3}{4}x^{-1/4}$. Since $2^4 = 16$, $f(16) = 2^3 = 8$.

And $f'(16) = \frac{3}{4}2^{-1} = \frac{3}{8}$. So the linear approximation at $x=16$

is $f(x) \approx f(a) + f'(a)(x-a)$

$= 8 + \frac{3}{8}(x-16)$.

Therefore $(15)^{3/4} = f(15)$ approximately equals $8 + \frac{3}{8}(15-16) = 8 - \frac{3}{8}$

$= 7\frac{5}{8}$

$(15)^{3/4}$ **approximately equals** $\boxed{7\ 5/8}$