MAT131 Fall 2007 Midterm 2 Review Sheet

The topics tested on Midterm 2 will be among the following.

(i) Using the rules of differentiation: the sum rule, the product rule, the power rule and the derivatives of exponentials.

(ii) Given the limits $\lim_{h \to 0} \frac{\sin(x)}{x} = 1$ and $\lim_{h \to 0} \frac{1 - \cos(x)}{x} = 0$, finding the formulas for the derivatives of $\sin(x)$ and $\cos(x)$.

(iii) Finding derivatives of other trigonometric functions – $\tan(x)$, $\cot(x)$, $\sec(x)$ and $\csc(x)$ – using the derivatives for $\sin(x)$ and $\cos(x)$ and the rules for differentiation.

(iv) Finding derivatives using the chain rule.

(v) Finding the tangent slope to a parametric curve at a specified point.

(vi) Finding derivatives using implicit differentiation, including derivatives of inverse functions.

(vii) Finding derivatives using logarithmic differentiation.

(viii) Finding the linear approximation to the value of a function, using a known nearby value and derivative or using differentials.

(ix) Understanding differential notation and the geometric interpretation of differentials. Using differentials to approximate values of functions.

(x) Solving related rates problems.

Following are some practice problems. More practice problems are in the textbook as well as on the practice midterm.

Problem 1. What is the differential of $y = x^3$? When $x = 2$, a small change $dx = 0.01$ in x produces what change dy in y?

Solution to Problem 1 Since $y' = 3x^2$,

$$\frac{dy}{dx} = y' = 3x^2, \quad \boxed{dy = 3x^2 dx.}$$

When $x = 2$, this gives $dy = 3(2)^2 dx = 12dx$. Thus a small change $dx = 0.01$ produces a change $dy = 12 \times (0.01)$ or $\boxed{0.12}$ in y. Since $y(2) = 8$, this gives $y(2.01) \approx 8.12$.

Problem 2. The inverse function of

$$f(x) = \frac{1}{2}(e^x + e^{-x})$$

is called the *inverse hyperbolic cosine*, $\cosh^{-1}(x)$. For $y = \cosh^{-1}(x)$, find a formula for the derivative of y that is an expression only involving polynomials in x and square roots.

Hint. After you find an answer that involves y, consider what you get by squaring your formula for y' and squaring your formula for x. Can you relate these two formulas?

Solution to Problem 2 Since $y = f^{-1}(x)$, this gives $f(y) = x$. This is the implicit equation

$$\frac{1}{2}(e^y + e^{-y}) = x.$$

Implicit differentiation gives

$$\frac{1}{2}(e^y - e^{-y})y' = 1, \quad y' = \frac{2}{e^y - e^{-y}}.$$

Squaring both sides gives

$$(y')^2 = \frac{4}{e^{2y} - 2 + e^{2y}}.$$

On the other hand, squaring both sides of the original implicit equation gives

$$\frac{e^{2y} + 2 + e^{-2y}}{4} = x^2.$$

Thus
$$\frac{e^{2y}-2+e^{-2y}}{4}=\frac{e^{2y}+2+e^{-2y}}{4}-1=x^2-1.$$
Substituting this into the equation for $(y')^2$ gives
$$(y')^2=\frac{1}{x^2-1}.$$
On the interval $[1,\infty)$ where $\cosh^{-1}(x)$ is usually defined, y' is given by the positive square root,
$$\boxed{y'=1/\sqrt{x^2-1}.}$$

Problem 3. The double-angle formula for tangent is
$$\tan(2x)=\frac{2\tan(x)}{1-(\tan(x))^2}.$$
Compute the derivative of each side of this equation. Which derivative is easier to compute?

Solution to Problem 3 The derivative of $\tan(u)$ is given by
$$d\tan(u)=\sec^2(u)du.$$
Thus, by the chain rule,
$$d\tan(2x)=\sec^2(2x)d(2x)=\sec^2(2x)2dx.$$
This gives
$$\boxed{\tfrac{d\tan(2x)}{dx}=2\sec^2(2x).}$$
And the quotient rule gives
$$\frac{d}{dx}\left(\frac{2\tan(x)}{1-(\tan(x))^2}\right)=\frac{(1-\tan^2(x))(2\sec^2(x))-(2\tan(x))(-2\tan(x)\sec^2(x))}{(1-\tan^2(x))^2}.$$
After simplification this gives
$$\frac{2\sec^2(x)(1+\tan^2(x))}{(1-\tan^2(x))^2}.$$

Using the identity that

$$1 + \tan^2(x) = \sec^2(x),$$

this becomes

$$\frac{d}{dx}\left(\frac{2\tan(x)}{1-(\tan(x))^2}\right) = \boxed{2\sec^4(x)/(1-\tan^2(x))^2.}$$

Problem 4. Let a be a positive constant and consider the parametric curve

$$\begin{cases} x(t) &= 2at \\ y(t) &= \frac{2a}{1+t^2} \end{cases}$$

Compute the slope of the tangent line at the point where $t = 1/\sqrt{3}$.

Solution to Problem 4 The derivatives are

$$\begin{cases} dx/dt &= 2a \\ dy/dt &= \frac{-4at}{(1+t^2)^2} \end{cases}$$

Thus,

$$dx = 2adt, \quad dy = \frac{-4at}{(1+t^2)^2}dt.$$

Since dx is nonzero, this gives

$$\frac{dy}{dx} = \frac{-4at}{(1+t^2)^2}\frac{1}{2a} = \frac{-2t}{(1+t^2)^2}$$

at the point $(x(t), y(t))$. In particular, when $t = 1/\sqrt{3}$,

$$\frac{dy}{dx} = \frac{-2}{\sqrt{3}(1+(1/\sqrt{3})^2)^2} = \boxed{-3\sqrt{3}/8.}$$

Problem 5. Compute each of the following derivatives.

(a)

$$y = \ln(\ln(x)), \ x > 1$$

Solution to (a)

$$y' = \frac{1}{\ln(x)}\frac{1}{x} = \boxed{1/(x\ln(x)).}$$

(b)

$$y = e^{e^x}$$

Solution to (b)

$$y' = e^{e^x} e^x = \boxed{e^{x+e^x}.}$$

(c)

$$y = \frac{2x}{1+x^2}$$

Solution to (c)

$$y' = \frac{(1+x^2)(2) - (2x)(2x)}{(1+x^2)^2} = \boxed{2(1-x^2)/(1+x^2)^2.}$$

(d)

$$y = \frac{x^3\sqrt{\sin(x)}}{\sqrt{\cos(x)}}$$

Solution to (d) Denote by u the logarithm

$$u = \ln(y) = 3\ln(x) + \frac{1}{2}\ln(\sin(x)) - \frac{1}{2}\ln(\cos(x)).$$

Then

$$\frac{du}{dx} = \frac{3}{x} + \frac{1}{2}\frac{1}{\sin(x)}\cos(x) - \frac{1}{2}\frac{1}{\cos(x)}(-\sin(x)).$$

Simplifying, this becomes

$$\frac{du}{dx} = \frac{6\sin(x)\cos(x) + 1}{2x\sin(x)\cos(x)}.$$

And since also

$$du = \frac{1}{y}dy, \quad dy = y du$$

this gives

$$\frac{dy}{dx} = y\frac{du}{dx} = \boxed{x^2(6\sin(x)\cos(x)+1)/(2\sqrt{\sin(x)\cos^3(x)}).}$$

(e)

$$y = \ln(-x + \sqrt{x^2 - 1})$$

(Simplify your answer as much as possible.)

Solution to (e)

$$y' = \frac{1}{-x + \sqrt{x^2-1}} \frac{d}{dx}(-x+\sqrt{x^2-1}) = \frac{1}{-x + \sqrt{x^2-1}} \left(-1 + \frac{x}{\sqrt{x^2-1}} \right) = \frac{1}{-x + \sqrt{x^2-1}} \frac{x - \sqrt{x}}{\sqrt{x^2}}$$

Simplifying gives

$$y' = \boxed{1/\sqrt{x^2-1}.}$$

Problem 6. Using your known formulas for the derivatives of $\sin(x)$ and $\cos(x)$, find the limit

$$\lim_{h \to 0} \frac{\cos(h) - 1}{h}$$

by relating it to the derivative of $\cos(x)$ for some value of x.

Solution to Problem 6 By definition, the derivative of $\cos(x)$ at $x = 0$ is

$$\lim_{h \to 0} \frac{\cos(h) - \cos(0)}{h}.$$

Since $\cos(0) = 1$, this gives

$$\frac{d\cos(x)}{dx}\Big|_{x=0} = \lim_{h \to 0} \frac{\cos(h) - 1}{h}.$$

On the other hand, the formula for the derivative is

$$\frac{d\cos(x)}{dx} = -\sin(x).$$

Since $\sin(0) = 0$, this gives

$$\lim_{h \to 0} \frac{\cos(h) - 1}{h} = \boxed{0.}$$

Problem 7. The value of $1/(1 + \sqrt{10})$ is close to $1/4 = 0.25$. Using a linear approximation or differentials, estimate whether the true value is closer to 0.2 or closer to 0.3.

Solution to Problem 7 Observe 10 is close to 9 and $\sqrt{9} = 3$. Thus let

$$y = \frac{1}{1 + \sqrt{x}}.$$

The differential of y is

$$dy = \frac{-1}{2\sqrt{x}(1 + \sqrt{x})^2} dx.$$

Thus at $x = 9$ the differential is

$$dy = \frac{-1}{2(3)(1 + 3)^2} dx = \frac{-1}{96} dx.$$

So the change $dx = 10 - 9 = 1$ gives a change $dy = (-1/96)(1)$ for x close to 9. This gives a linear approximation

$$\frac{1}{1 + \sqrt{10}} \approx \frac{1}{4} - \frac{1}{96} = \boxed{23/96}$$

which is closer to $\boxed{0.2}$ than to 0.3.

Problem 8. For the curve with implicit equation

$$y + \frac{1}{y} = 4x + 2x^2$$

find the slope of the tangent line to the curve at the point $(x, y) = (1/2, 2)$.

Solution to Problem 8 Differentiating both sides gives

$$\frac{y^2 - 1}{y^2} y' = 4 + 4x.$$

When y is not 0 or ± 1, we can divide and get

$$y' = \frac{4(x + 1)y^2}{y^2 - 1}.$$

Plugging in $(x, y) = (1/2, 2)$ gives

$$y' = \boxed{8.}$$

Problem 9. A calculus instructor of height 170 cm is moving at speed 5.2 km/hr away from a street lamp of height 3m. What is the speed of the shadow of the instructor's head?

Solution to Problem 9 Denote the displacement between the base of the lamp and the instructor's feet by x. And denote the displacement between the base of the lamp and the tip of the shadow by y. Thus the displacement between the instructor's feet and the tip of the shadow is $y - x$. By similar triangles,

$$\frac{y}{300 \text{ cm}} = \frac{y - x}{130 \text{ cm}}.$$

Solving for y gives

$$y = \frac{30}{7}x.$$

Implicitly differentiating with respect to t gives

$$\frac{dy}{dt} = \frac{30}{7}\frac{dx}{dt}.$$

Thus the speed of the shadow is

$$\frac{dy}{dt} = \boxed{156/7 \text{ km/hr} \approx 22.3 \text{ km/hr.}}$$

Problem 10. A long, straight piece of wood of length 7 meters rests with its foot on the ground and its midpoint on the top of a 3 meter tall fence. The foot of the plank begins to slide straight away from the fence with the plank still touching the top of the fence. At the moment when the foot of the plank is 4 meters from the base of the fence, the distance between the top of the plank and the ground is decreasing at 0.3 meters per second. At what speed is the foot of the plank moving away from the base of the fence?

Solution to Problem 10 Denote the displacement between the foot of the plank and the fence by x. Denote the distance between the top of the plank and a point on the ground directly below by y. This point, the top of the plank and the foot of the plank are the vertices of a right triangle. The vertical leg of this right triangle has length y. There is a smaller but similar right triangle with vertices the base of the fence, the top of the fence and the foot of the plank. The vertical leg of this smaller triangle has length 3 meters, and the horizontal leg of this smaller triangle has length x. Thus, by similar triangles, the horizontal leg of the large triangle is

$$z = \frac{xy}{3 \text{ m}}.$$

By the Pythagorean theorem,

$$7^2 \text{ m}^2 = y^2 + z^2 = y^2 + \frac{x^2y^2}{9 \text{ m}^2} = \frac{(x^2 + 9 \text{ m}^2)y^2}{9 \text{ m}^2}.$$

Solving for x^2 gives

$$x^2 = \frac{441 \text{ m}^4}{y^2} - 9 \text{ m}^2.$$

Implicitly differentiating with respect to t gives a second equation

$$2x\frac{dx}{dt} = -\frac{882 \text{ m}^4}{y^3}\frac{dy}{dt}.$$

Moreover, plugging in $x(t_0) = 4$ m into the first equation for x^2 gives

$$(4 \text{ m})^2 = \frac{441 \text{ m}^4}{y(t_0)^2} - 9 \text{ m}^2.$$

Solving for $y(t_0)$ gives

$$y(t_0) = \frac{21}{5} \text{ m}.$$

Plugging in $x(t_0) = 4$ m, $y(t_0) = (21/5)$ m and $dy/dt(t_0) = -0.3$ m/sec into the second equation gives

$$8\frac{dx}{dt} \text{ m} = -\frac{882 \text{ m}^4}{(21/5)^3 \text{ m}^3}(-0.3\text{m/sec}).$$

Solving gives

$$\frac{dx}{dt} = \boxed{(25/56) \text{ meters per second.}}$$

Problem 11. Find the derivative y' of the function y in each of the following cases.

(a)

$$y = \sin(\sqrt{x^2 + 1})$$

Solution to (a)

$$\frac{dy}{dx} = \cos(\sqrt{x^2+1})\left(\frac{1}{2}(x^2+1)^{-1/2}\right)(2x) = \boxed{x\cos(\sqrt{x^2+1})/\sqrt{x^2+1}.}$$

(b)

$$y = (\ln(x^2))^2$$

Solution to (b) First of all,

$$y = (2\ln(x))^2 = 4(\ln(x))^2.$$

Thus,

$$\frac{dy}{dx} = 4(2\ln(x))\left(\frac{1}{x}\right) = \boxed{8\ln(x)/x.}$$

(c)

$$y = (\sqrt{x})^{\cos(x)}$$

Solution to (c) Setting $u = \ln(y)$ gives

$$u = \ln(y) = \ln(\sqrt{x}^{\cos(x)}) = \cos(x)\ln(\sqrt{x}) = \frac{1}{2}\cos(x)\ln(x).$$

The derivative of this is

$$\frac{du}{dx} = \frac{1}{2}(-\sin(x))\ln(x) + \frac{1}{2}\cos(x)\left(\frac{1}{x}\right) = \frac{\cos(x) - x\sin(x)\ln(x)}{2x}.$$

Since also $u' = y'/y$, this gives $y' = yu'$ or

$$\frac{dy}{dx} = y\frac{du}{dx} = (\sqrt{x})^{\cos(x)}\frac{\cos(x) - x\sin(x)\ln(x)}{2x} = \boxed{(\cos(x) - x\sin(x)\ln(x))(\sqrt{x})^{\cos(x)}/2x.}$$

(d)

$$y = \frac{x^2e^x}{x^2+2x+1}$$

Solution to (d) Setting $u = \ln(y)$ gives

$$u = \ln(y) = \ln(x^2e^x/(x^2+2x+1)) = 2\ln(x)+x-\ln(x^2+2x+1) = 2\ln(x)+x-2\ln(x+1)$$

The derivative of this is

$$\frac{du}{dx} = \frac{2}{x} + 1 - 2\frac{1}{x+1} = \frac{x^2+x+2}{x(x+1)}.$$

Since also $u' = y'/y$, this gives $y' = yu'$ or

$$\frac{dy}{dx} = y\frac{du}{dx} = \frac{x^2e^x}{(x+1)^2}\frac{x^2+x+2}{x(x+1)} = \boxed{x(x^2+x+2)e^x/(x+1)^3.}$$

(e)

$$x^2 + 4y^2 = 5$$

Solution to (e) Implicitly differentiating both sides with respect to x gives

$$2x + 8y\frac{dy}{dx} = 0.$$

Solving for the derivative gives

$$\frac{dy}{dx} = \boxed{-x/4y.}$$

Solving for y as well gives an answer depending only on x,

$$\frac{dy}{dx} = \boxed{-x/(2\sqrt{5-x^2}).}$$

(f)

$$y = \frac{(x^2+1)^3}{x^2-1}$$

Solution to (f) Rewrite this as

$$y = \frac{(x^2+1)^3}{(x+1)(x-1)}.$$

Setting $u = \ln(y)$ gives

$$u = \ln(y) = \ln\left(\frac{(x^2+1)^3}{(x+1)(x-1)}\right) = 3\ln(x^2+1) - \ln(x+1) - \ln(x-1).$$

The derivative of this is

$$\frac{du}{dx} = 3\frac{1}{x^2+1}(2x) - \frac{1}{x+1} - \frac{1}{x-1} = \frac{4x(x^2-2)}{(x^2+1)(x+1)(x-1)}$$

Since also $u' = y'/y$, this gives $y' = yu'$ or

$$\frac{dy}{dx} = y\frac{du}{dx} = \frac{(x^2+1)^3}{(x+1)(x-1)}\frac{4x(x^2-2)}{(x^2+1)(x+1)(x-1)} = \boxed{4x(x^2-2)(x^2+1)^2/((x+1)(x-1)).}$$

Problem 12 In each of the following two cases, find the linearization of $f(x)$ near the point $x = a$.

(a)

$$f(x) = x^{2/5}, \quad a = 32.$$

Solution to (a) The derivative is

$$f'(x) = \frac{2}{5}x^{-3/5}.$$

Plugging in $x = 32$ so that $x^{1/5} = 2$, this gives

$$f'(32) = \frac{2}{5}2^{-3} = \frac{1}{20}.$$

So the linearization is

$$f(x) \approx f(a) + f'(a)(x-a) = \boxed{2 + (1/20)(x-32).}$$

(b)

$$f(x) = \frac{1}{\sqrt{1+x^2}}, \quad a = \sqrt{3}.$$

Solution to (b) The derivative is

$$f'(x) = \frac{-1}{2}(1+x^2)^{-3/2}(2x) = \frac{-x}{(1+x^2)^{3/2}}.$$

Plugging in $a = \sqrt{3}$ so that $f(\sqrt{3}) = 1/\sqrt{1+3} = 1/2$, the derivative is

$$f'(\sqrt{3}) = \frac{-\sqrt{3}}{(1/2)^3} = -8\sqrt{3}.$$

So the linearization is

$$f(x) \approx f(a) + f'(a)(x-a) = \boxed{(1/2) - 8\sqrt{3}(x - \sqrt{3}).}$$

Problem 13 Using differentials or an appropriate linear approximation, approximate the following number.

$$\frac{1}{\sqrt{25.1}}$$

Solution to Problem 13 Let $f(x) = x^{-1/2}$ and let $a = 25$ so that $f(a) = 1/5$. The differential is

$$dy = df(x) = \frac{-1}{2} x^{-3/2} dx.$$

When $x = 25$ this gives

$$dy = \frac{-1}{2} 5^{-3} dx = -\frac{1}{250} dx.$$

Thus the linear approximation is

$$f(25.1) \approx 1/5 - \frac{1}{250}(0.1) = \boxed{0.1996} = \boxed{499/2500.}$$

Problem 14 In a flat ceiling two hooks are fastened 21 cm apart. A length of 27 cm of inextensible wire is suspended between the hooks. A heavy weight is hung from the wire near the first hook and slides along the wire toward a point equidistant from both hooks, pulling the wire taut at each moment. At the moment when the weight is 10 cm from the first hook, it is moving away from the first hook at a speed of 1 cm/sec.

(a) With what speed is the weight moving towards the second hook at this moment?

Solution to (a) Denote by A the distance of the weight from the first hook. And denote by B the distance of the weight from the second hook. The total length of wire is 27 cm. Thus

$$A + B = 27 \text{ cm}.$$

Differentiating with respect to t gives

$$\frac{dA}{dt} + \frac{dB}{dt} = 0.$$

Thus

$$\frac{dB}{dt}(t_0) = -\frac{dA}{dt}(t_0) = \boxed{-1 \text{ cm/sec.}}$$

(b) What is distance between the weight and the ceiling at this moment?

Solution to (b) Denote by C the total distance 21 cm between the two hooks. Denote by y the distance between the weight and the ceiling. Denote by α the angle between the wire and the ceiling at the first hook. And denote by β the angle between the wire and the ceiling at the second hook. The law of cosines precisely states that

$$B^2 = A^2 + C^2 - 2AC\cos(\alpha).$$

Solving for $A\cos(\alpha)$ gives

$$A\cos(\alpha) = \frac{C^2 + A^2 - B^2}{2C}.$$

On the other hand,

$$y = A\sin(\alpha).$$

Thus again by the Pythagorean theorem,

$$y^2 = A^2\sin^2(\alpha) = A^2 - A^2\cos^2(\alpha) = A^2 - \frac{(C^2 + A^2 - B^2)^2}{4C^2}.$$

Simplifying gives

$$y^2 = \frac{2A^2B^2 + 2C^2(A^2 + B^2) - (A^4 + B^4) - C^4}{4C^2}.$$

At the moment t_0, $A(t_0)$ equals 10 cm. Thus $B(t_0) = 17$ cm. Plugging in to the equation gives

$$y^2 = 64 \text{ cm}^2.$$

Therefore

$$y = \boxed{8 \text{ cm.}}$$

(c) With what speed is the weight moving away from the ceiling (i.e., what is the rate of change of the distance between the weight and the ceiling)?

Solution to (c) Implicitly differentiating the equation

$$y^2 = A^2 - \frac{(C^2 + A^2 - B^2)^2}{4C^2}$$

with respect to t gives

$$2y\frac{dy}{dt} = 2A\frac{dA}{dt} + \frac{1}{4C^2}2(C^2 + A^2 - B^2)(2A\frac{dA}{dt} - 2B\frac{dB}{dt}).$$

Solving for dy/dt gives

$$\frac{dy}{dt} = \boxed{32/7 \text{ cm per sec.}}$$

Problem 15 A car approaches a large hotel at night, driving along a semi-circular driveway which becomes tangent to the front wall of the hotel at the entrance. The headlights of the car illuminate a spot on the front wall. If the radius of the driveway is 10 meters, and if the car is moving at a speed of 10 km/hour, with what speed is the spot moving when the angle between the entrance, the center of the circle and the car is $\pi/3$ radians, i.e., 60 degrees?

Solution to Problem 15 Denote by r the radius, i.e., $r = 10$ meters. Denote by ϕ the angle between the radius joining the center of the circle to the entrance and the radius joining the center of the circle to the car. Denote half of this angle by θ, i.e., $\theta = \phi/2$. Then the distance of driveway between the car and the entrance is

$$s = r\phi = 2r\theta.$$

Also the distance from the entrance to the spot on the wall is

$$x = r\tan(\theta).$$

Implicitly differentiating each of these with respect to time t gives

$$\frac{ds}{dt} = 2r\frac{d\theta}{dt}.$$

and

$$\frac{dx}{dt} = r\sec^2(\theta)\frac{d\theta}{dt}.$$

The speed of the car is ds/dt, which is 10 km/hour. Thus the rate of change of θ is

$$\frac{d\theta}{dt} = \frac{1}{2r}\frac{ds}{dt} = 500 \text{ radians/hour.}$$

Thus

$$\frac{dx}{dt} = 10\sec^2(\theta)(500) \text{ meters/hour.}$$

When ϕ equals $\pi/3$, then θ equals $\pi/6$. Thus

$$\cos(\theta) = \cos(\pi/6) = \frac{\sqrt{3}}{2}.$$

Plugging in

$$\sec^2(\theta) = \frac{1}{\cos^2(\theta)} = \frac{1}{3/4} = \frac{4}{3}.$$

Plugging in gives

$$\frac{dx}{dt} = \boxed{20/3 \text{ kilometers per hour.}}$$

Problem 16 A cube of ice rests on a hot plate. It melts in such a way that its shape at every moment is a cube and the rate of decrease of the volume is a constant multiple of the area of the face resting on the hot plate. If after 5 minutes the volume of the cube is one eighth of its initial volume, how much longer is required before the cube melts entirely?

Solution to Problem 16 Denote the edge length of the cube by x, the area of a face by A and the volume by V. Then

$$A = x^2, \quad V = x^3.$$

Implicitly differentiating gives

$$\frac{dV}{dt} = 3x^2\frac{dx}{dt}.$$

By hypothesis, this is a constant multiple c times the area A of the face. Thus

$$3x^2\frac{dx}{dt} = \frac{dV}{dt} = cA = cx^2.$$

Dividing both sides by $3x^2$ gives

$$\frac{dx}{dt} = \frac{c}{3}.$$

Since the tangent slope is constant, x must be linear. In other words,

$$x = mt + b$$

for some slope m and some constant b. Plugging in

$$V(t) = (mt + b)^3.$$

By hypothesis,

$$(m5 + b)^3 = V(5) = \frac{1}{8}V(0) = \frac{b^3}{8}.$$

Taking cube roots, $m5+b$ equals $b/2$ or $5m = -b/2$. Thus $m = -b/10$. Thus $x(t)$ equals 0 when $-bt/10+b$ equals 0, i.e., $t = 10$ minutes. Since 5 minutes have already elapsed, the cube melts entirely after a further $\boxed{\text{5 minutes.}}$