MAT 123 Introduction to Calculus | Jason Starr
Core Competency Exam C | Spring 2015
In-lecture Wednesday / Thursday, Week of 5/4-5/8

MAT 123 Practice for Core Competency Exam C With Solutions

Remark. If you are comfortable with all of the following problems, you will be well prepared for Core Competency Exam C. For the Core Competency Exams, passing will be 80% or better.

Exam Policies. You must show up on time for all exams. Please bring your student ID card: ID cards may be checked, and students may be asked to sign a picture sheet when turning in exams. Other policies for exams will be announced at the beginning of the exam.

If you have a university-approved reason for taking an exam at a time different than the scheduled exam (because of a religious observance, a student-athlete event, etc.), please contact your instructor as soon as possible. Similarly, if you have a documented medical emergency which prevents you from showing up for an exam, again contact your instructor as soon as possible.

For excused absences from a Core Competency Exam, since there will be multiple attempts for such an exam, usually the student will simply be asked to pass one of the later attempts. In exceptional circumstances, a make-up exam may be scheduled for the missed exam.

All exams are closed notes and closed book. Once the exam has begun, having notes or books on the desk or in view will be considered cheating and will be referred to the Academic Judiciary.

For all exams, you must bring your Stony Brook ID. The IDs may be checked against picture sheets.

It is not permitted to use cell phones, calculators, laptops, MP3 players, Blackberries or other such electronic devices at any time during exams. If you use a hearing aid or other such device, you should make your instructor aware of this before the exam begins. You must turn off your cell phone, etc., prior to the beginning of the exam. If you need to leave the exam room for any reason before the end of the exam, it is still not permitted to use such devices. Once the exam has begun, use of such devices or having such devices in view will be considered cheating and will be referred to the Academic Judiciary. Similarly, once the exam has begun any communication with a person other than the instructor or proctor will be considered cheating and will be referred to the Academic Judiciary.

Review Topics.
The following are the core skills for the second part of the course.

(1) Understand the graph of $\sin(x)$, $\cos(x)$ and $\tan(x)$ over various intervals. Know where each graph crosses the x-axis, where it is positive, where it is negative, where it achieves a local maximum / minimum (if any), what is the domain and range, and what is the period.

(2) Know elementary identities involving $\sin(x)$ and $\cos(x)$ such as the Pythagorean theorem, and whether each function is even or odd.

(3) Know identities for $\sin(x)$ and $\cos(x)$ involving (horizontal) shifts by simple fractions of a period, e.g., $\cos(x + \pi) = -\cos(x)$ or $\sin(x + \pi/2) = \cos(x)$.

(4) Compute trigonometric functions on angles such as π, $\pi/2$, $\pi/3$, $\pi/4$ and $\pi/6$.

(5) For an unspecified angle such that the value of one trigonometric function is given, e.g., $\tan(\theta) = 3$, $-\pi/2 < \theta < \pi/2$, compute other trigonometric functions using relations betweens trigonometric functions and the Pythagorean theorem, e.g., $\sin(\theta) = 3\sqrt{10}/10$.

(6) Know the definitions of the secondary trigonometric functions, $\cot(x)$, $\sec(x)$, $\csc(x)$, in terms of the primary trigonometric functions, and be able to relate values and identities of the primary trigonometric functions to values and identities for the secondary trigonometric functions.

(7) Know the double angle formulas.

(8) Understand in which quadrants the points on the unit circle have positive / negative x-coordinate and y-coordinate. Relate this to the sign of trigonometric functions for various intervals, e.g., $0 < \theta < \pi/2$.

Practice Problems.

(1) Please look at the graphs of $\sin(x)$, $\cos(x)$ and $\tan(x)$ on each of the following (labelled) intervals of the x-axis: $(-2\pi, -3\pi/2)$, $(-3\pi/2, -\pi)$, $(-\pi, -\pi/2)$, $(-\pi/2, 0)$, $(0, \pi/2)$, $(\pi/2, \pi)$, $(\pi, 3\pi/2)$, $(3\pi/2, 2\pi)$. Be sufficiently familiar so that, if shown a graph on one of these specified intervals, you could identify the function.

(2) Compute each of the following trigonometric functions.

(i) $\sin(2\pi/3)$

$$(a)\ 0,\quad (b)\ 1/2,\quad (c)\ \sqrt{2}/2,\quad (d)\ \sqrt{3}/2,\quad (e)\ 1.$$

(ii) $\sin(3\pi/2)$

$$(a)\ 0,\quad (b)\ 1,\quad (c)\ -1,\quad (d)\ \text{undefined}.$$

(iii) $\cos(\pi/4)$

$$(a)\ 0,\quad (b)\ 1/2,\quad (c)\ \sqrt{2}/2,\quad (d)\ \sqrt{3}/2,\quad (e)\ 1.$$

(iv) $\tan(\pi/3)$

$$(a)\ 1,\quad (b)\ \sqrt{3},\quad (c)\ \sqrt{3}/3,\quad (d)\ 0,\quad (e)\ 2.$$

(v) $\cos(-\pi/6)$

$$(a)\ 0,\quad (b)\ -1/2,\quad (c)\ \sqrt{2}/2,\quad (d)\ \sqrt{3}/2,\quad (e)\ -1.$$

(vi) $\tan(\pi/2)$

$$(a)\ 0,\quad (b)\ 1,\quad (c)\ -1,\quad (d)\ \text{undefined}.$$

(vii) $\sin(5\pi/4)$

$$(a)\ 0,\quad (b)\ 1/2,\quad (c)\ -\sqrt{2}/2,\quad (d)\ \sqrt{3}/2,\quad (e)\ -1.$$

(viii) $\tan(3\pi/4)$

$$(a)\ 0,\quad (b)\ 1,\quad (c)\ -1,\quad (d)\ -\sqrt{3},\quad (e)\ 1/\sqrt{3}.$$

Solution to (2) These are the correct answers: **(i)** (d), **(ii)** (c), **(iii)** (c), **(iv)** (b), **(v)** (d), **(vi)** (d), **(vii)** (c), **(viii)** (c).

(3) In each of the following cases, simplify the given expression.

(i) $\sin(x - 2\pi)$

$$(a)\ \sin(x),\quad (b)\ \cos(x),\quad (c)\ -\sin(x),\quad (d)\ -\cos(x).$$

(ii) $\cos(x - \pi)$

$$(a)\ \sin(x),\quad (b)\ \cos(x),\quad (c)\ -\sin(x),\quad (d)\ -\cos(x).$$

(iii) $\tan(x + \pi)$

$$(a)\ \tan(x),\quad (b)\ -\tan(x),\quad (c)\ 1/\tan(x),\quad (d)\ -1/\tan(x).$$

(iv) $\cos(x + 2\pi)$

$$(a)\ \sin(x),\quad (b)\ \cos(x),\quad (c)\ -\sin(x),\quad (d)\ -\cos(x).$$

(v) $\tan(-x)$

$$(a)\ \tan(x),\quad (b)\ -\tan(x),\quad (c)\ 1/\tan(x),\quad (d)\ -1/\tan(x).$$

(vi) $\sin(\pi/2 - x)$

$$(a)\ \sin(x),\quad (b)\ \cos(x),\quad (c)\ -\sin(x),\quad (d)\ -\cos(x).$$

(vii) $\cos(x + 3\pi/2)$

$$(a)\ \sin(x),\quad (b)\ \cos(x),\quad (c)\ -\sin(x),\quad (d)\ -\cos(x).$$

(viii) $\tan(x - \pi/2)$

$$(a)\ \tan(x),\quad (b)\ -\tan(x),\quad (c)\ 1/\tan(x),\quad (d)\ -1/\tan(x).$$

(ix) $\cos(-x)$

$$(a)\ \sin(x),\quad (b)\ \cos(x),\quad (c)\ -\sin(x),\quad (d)\ -\cos(x).$$

Solution to (3) These are the correct answers: **(i)** (a), **(ii)** (d), **(iii)** (a), **(iv)** (b), **(v)** (b), **(vi)** (b), **(vii)** (a), **(viii)** (d), **(ix)** (b).

(4) In each of the following cases, compute the unknown trigonometric function given the specified trigonometric function.

(i) $\sin(x) = 1/3$, $0 < x < \pi/2$, $\cos(x) =?$

$$(a)\ 2/3,\quad (b)\ 2\sqrt{2}/3,\quad (c)\ 3,\quad (d)\ \sqrt{1-(1/3)}.$$

(ii) $\cos(x) = -2/3$, $\pi < x < 3\pi/2$, $\sin(x) =?$

$$(a)\ 1/3,\quad (b)\ -1/3,\quad (c)\ -\sqrt{5}/3,\quad (d)\ -\sqrt{1-(-2/3)}.$$

(iii) $\sin(x) = 1/4$, $0 < x < \pi/2$, $\tan(x) =?$

$$(a)\ \sqrt{15},\quad (b)\ 1/3,\quad (c)\ \sqrt{15}/4,\quad (d)\ 1/\sqrt{15}.$$

(iv) $\tan(x) = 4$, $-\pi < x < -\pi/2$, $\cos(x) =?$

$$(a)\ 1/\sqrt{17},\quad (b)\ -4/\sqrt{17},\quad (c)\ 4,\quad (d)\ -1/\sqrt{17}.$$

(v) $\sin(x) = 0$, $\pi/2 < x < 3\pi/2$, $\cos(x) =?$

$$(a)\ 1,\quad (b)\ -1,\quad (c)\ 0,\quad (d)\ \sqrt{1-0}.$$

(vi) $\sin(x) + \cos(x) = \sqrt{2}$, $0 < x < \pi/2$, $\cos(x) = ?$

$$(a)\ 1,\quad (b)\ 1/\sqrt{2},\quad (c)\ 1/2,\quad (d)\ \sqrt{3}/2.$$

Solution to (4) (i) For $0 < x < \pi/2$, $\cos(x)$ is positive. By the Pythagorean theorem, $(\sin(x))^2 + (\cos(x))^2$ equals 1. Thus $(\cos(x))^2$ equals $1 - (1/3)^2 = 8/9$. Therefore $\cos(x)$ equals $2\sqrt{2}/3$; the answer is (b).

(ii) For $\pi < x < 3\pi/2$, $\sin(x)$ is negative. By the Pythagorean theorem, $(\sin(x))^2 + (\cos(x))^2$ equals 1. Thus $(\sin(x))^2$ equals $1 - (-2/3)^2 = 5/9$. Therefore $\sin(x)$ equals $-\sqrt{5}/3$; the answer is (c).

(iii) For $0 < x < \pi/2$, $\sin(x)$, $\cos(x)$ and $\tan(x)$ are positive. By the Pythagorean theorem, $(\sin(x))^2 + (\cos(x))^2$ equals 1. Thus $(\cos(x))^2$ equals $1 - (1/4)^2 = 15/16$. Therefore $\cos(x)$ equals $\sqrt{15}/4$. Thus $\tan(x) = \sin(x)/\cos(x)$ equals $1/\sqrt{15}$; the answer is (d).

(iv) Since $-\pi < x < -\pi/2$, both $\sin(x)$ and $\cos(x)$ are negative. Since $\sin(x)/\cos(x)$ equals $\tan(x) = 4$, $\sin(x)$ equals $4\cos(x)$. From the Pythagorean theorem,

$$1 = (\sin(x))^2 + (\cos(x))^2 = (4\cos(x))^2 + (\cos(x))^2 = 16(\cos(x))^2 + (\cos(x))^2 = 17(\cos(x))^2.$$

Thus $(\cos(x))^2$ equals $1/17$. Therefore $\cos(x)$ equals $-1/\sqrt{17}$; the answer is (d).

(v) Since $\pi/2 < x < 3\pi/2$, $\cos(x)$ is negative. Since $\sin(x)$ equals 0, by the Pythagorean theorem also $(\cos(x))^2 = 1$. Therefore $\cos(x)$ equals -1; the answer is (b).

(vi) Since $0 < x < \pi/2$, both $\sin(x)$ and $\cos(x)$ are positive. Since $\sin(x) + \cos(x)$ equals $\sqrt{2}$, also $\sin(x)$ equals $\sqrt{2} - \cos(x)$. From the Pythagorean theorem,

$$1 = (\sin(x))^2 + (\cos(x))^2 = (\sqrt{2} - \cos(x))^2 + (\cos(x))^2 = 2 - 2\sqrt{2}\cos(x) + 2(\cos(x))^2.$$

Thus $c = \cos(x)$ satisfies the quadratic equation,

$$2c^2 - 2\sqrt{2}c + 1 = 0, \text{ i.e., } (\sqrt{2}\cos(x) - 1)^2 = 0.$$

Therefore $\cos(x)$ equals $1/\sqrt{2}$; the answer is (b).

(5) In each of the following cases, compute the specified secondary trigonometric function.

(i) $\sec(\pi/3) =$

$$(a)\ 2,\quad (b)\ 2/\sqrt{3},\quad (c)\ 1/\sqrt{3},\quad (d)\ \sqrt{3}.$$

(ii) $\cot(\pi/6) =$

$$(a)\ 2,\quad (b)\ 2/\sqrt{3},\quad (c)\ 1/\sqrt{3},\quad (d)\ \sqrt{3}.$$

(iii) $\csc(\pi/4) =$

$$(a)\ 1/\sqrt{2},\quad (b)\ \sqrt{2},\quad (c)\ 1,\quad (d)\ \sqrt{3}.$$

(iv) $\cos(x) = 3/5$, $0 < x < \pi/2$, $\csc(x) =$?

$$(a)\ 4/5,\quad (b)\ 5/3,\quad (c)\ 5/4,\quad (d)\ 3/4.$$

(v) $\tan(x) = 4/3$, $0 < x < \pi/2$, $\sec(x) =$?

$$(a)\ 3/4,\quad (b)\ 5/3,\quad (c)\ 5/4,\quad (d)\ \sqrt{7}/3.$$

Solution to (5) These are the answers: **(i)** (a), **(ii)** (d), **(iii)** (b), **(iv)** (c), **(v)** (b).

(6) Simplify the given expression.

(i) $\sin(x)\sec(x)$.

$$(a)\ \tan(x),\quad (b)\ \cos(x),\quad (c)\ 1,\quad (d)\ \cot(x).$$

(ii) $\tan(x)/\sin(x)$.

$$(a)\ \cos(x),\quad (b)\ \csc(x),\quad (c)\ 1,\quad (d)\ \sec(x).$$

(iii) $\sin^2(x)\cot(x)\sec(x)$.

$$(a)\ \cos(x),\quad (b)\ \tan(x),\quad (c)\ \sin(x),\quad (d)\ \cot(x).$$

(iv) $\tan(x)\cot(x)$.

$$(a)\ \sin(x),\quad (b)\ \cos(x),\quad (c)\ 1,\quad (d)\ \csc(x).$$

Solution to (6) **(i)** Since $\sec(x)$ equals $1/\cos(x)$, also $\sin(x)\sec(x)$ equals $\sin(x)/\cos(x)$. This is the same as $\tan(x)$; the answer is (a).

(ii) Since $\tan(x)$ equals $\sin(x)/\cos(x)$, also $\tan(x)/\sin(x)$ equals $1/\cos(x)$. This is the same as $\sec(x)$; the answer is (d).

(iii) Since $\cot(x)$ equals $\cos(x)/\sin(x)$, and since $\sec(x)$ equals $1/\cos(x)$, this simplifies to

$$\sin^2(x)\cot(x)\sec(x) = \sin^2(x)\cdot\frac{\cos(x)}{\sin(x)}\cdot\frac{1}{\cos(x)} = \frac{\sin^2(x)\cos(x)}{\sin(x)\cos(x)}.$$

Thus, the simplifed form is $\sin(x)$; 0nswer is (c).

(iv) Since $\cot(x)$ equals $1/\tan(x)$, the answer is (c).

(7) In each of the following cases, simplify the expression.

(i) $\sin(2x)$

(a) $\cos^2(x) - \sin^2(x)$, (b) $2\cos^2(x) - 1$, (c) $2\sin(x)\cos(x)$, (d) $2\tan(x)/(1-\tan^2(x))$.

(ii) $\cos(2x)$

(a) $\cos^2(x) + \sin^2(x)$, (b) $2\cos^2(x) - 1$, (c) $2\sin(x)\cos(x)$, (d) $2\tan(x)/(1-\tan^2(x))$.

(iii) $\tan(2x)$

(a) $\cos^2(x) - \sin^2(x)$, (b) $2\cos^2(x) - 1$, (c) $2\sin(x)\cos(x)$, (d) $2\tan(x)/(1-\tan^2(x))$.

Solution to (7) These are the answers: **(i)** (c), **(ii)** (b), **(iii)** (d).

(8) In each of the following cases, compute $f(a)$.

(i) $f(x) = \sin(2x)$, $a = \pi/12$, $f(a) = ?$

$$(a)\ 2\sin(\pi/6)\cos(\pi/6),\quad (b)\ 1/2,\quad (c)\ \sqrt{3}/2,\quad (d)\ \sqrt{2-\sqrt{3}}/2.$$

(ii) $f(x) = \cos(-2x)$, $a = \pi/8$, $f(a) = ?$

$$(a)\ \cos^2(\pi/4) - \sin^2(\pi/4),\quad (b)\ 1/\sqrt{2},\quad (c)\ -1/\sqrt{2},\quad (d)\ \sqrt{2+\sqrt{2}}/2.$$

(iii) $f(x) = \tan(3x)$, $a = \pi/4$, $f(a) = ?$

$$(a)\ \tan(\pi/3)(3-\tan^2(\pi/3))/(1-3\tan^2(\pi/3)),\quad (b)\ 1,\quad (c)\ -1,\quad (d)\ \sqrt{3}.$$

Solution to (8) (i) Since $2\pi/12$ equals $\pi/6$, $f(a)$ equals $\sin(\pi/6)$. Therefore $f(a)$ equals $1/2$; the answer is (b).

(ii) Since $-2\pi/8$ equals $-\pi/4$, $f(a)$ equals $\cos(-\pi/4)$. Therefore $f(a)$ equals $\sqrt{2}/2$; the answer is (b).

(iii) Since $\sin(3\pi/4)$ equals $\sqrt{2}/2$, and since $\cos(3\pi/4)$ equals $-\sqrt{2}/2$, also $\tan(3\pi/4)$ equals -1. Therefore $f(a)$ equals -1; the answer is (c).

(9) In each case, find the number of intersection points of the given graphs over the specified interval of the x-axis.

(i) $f(x) = \sin(x)$, $g(x) = \cos(x)$, $0 < x < \pi/2$.

(a) no intersections, (b) one , (c) two , (d) three , (e) infinite intersections.

(ii) $f(x) = \sin(x)$, $g(x) = \tan(x)$, $-\pi/2 < x < \pi/2$.

(a) no intersections, (b) one , (c) two , (d) three , (e) infinite intersections.

(iii) $f(x) = \cos(x)$, $g(x) = 1/2$, $-\pi/2 < x < \pi/2$.

(a) no intersections, (b) one , (c) two , (d) three , (e) infinite intersections.

(iv) $f(x) = \tan(x)$, $g(x) = x$, $-\pi/2 < x < \pi/2$.

(a) no intersections, (b) one , (c) two , (d) four , (e) infinite intersections.

(v) $f(x) = \sin(x)$, $g(x) = x/2$, $-\pi/2 < x < \pi/2$.

(a) no intersections, (b) one, (c) two, (d) three, (e) infinite intersections.

Solution to (9) (i) If $\sin(x)$ equals $\cos(x)$, then $\tan(x)$ equals 1. Since $\tan(x)$ is increasing from 0 to $\pi/2$, ranging from 0 to $+\infty$, there is precisely one point of intersection; the answer is (b). In fact, that intersection is at $x = \pi/4$.

(ii) If $\sin(x)$ equals $\sin(x)/\cos(x)$, then $\cos(x)$ equals 1. The unique angle between $-\pi/2$ and $\pi/2$ where $\cos(x)$ equals its maximum value 1 is $x = 0$. Therefore there is precisely one point of intersection; the answer is (b).

(iii) The function $\cos(x)$ is increasing for $\pi/2 \leq x \leq 0$, ranging from 0 to 1. Similarly, the function $\cos(x)$ is decreasing for $0 \leq x \leq \pi/2$, ranging from 1 to 0. Thus there are precisely two intersections; the answer is (c). One intersection is $x = -\pi/3$, and the other is $x = \pi/3$.

(iv) This is definitely a challenging practice problem, certainly more challenging than an actual exam problem. For $0 < x < \pi/2$, $\tan(x)$ is always greater than x. This can be seen geometrically by forming the arc of the unit circle of length x from $(1, 0)$ to $(\cos(x), \sin(x))$. The corresponding sector of the unit circle is inscribed in the right angle with one leg from $(0, 0)$ to $(1, 0)$ and the second leg from $(1, 0)$ to $(1, \tan(x))$. Visually, it is clear that "wrapping" the second leg of the triangle around the unit circle more than covers the arc. Because $\tan(x)$ is greater than x, the unique intersection point for $0 \leq x < \pi/2$ is at $x = 0$. Similarly, the unique intersection point for $-\pi/2 < x \leq 0$ is at $x = 0$. Hence, there is precisely one intersection point at $x = 0$; the answer is (b).

(v) Again, this is a challenging exercise. As in the previous part, consider the sector of the unit circle bounded by the line segment from $(0, 0)$ to $(1, 0)$ and the line segment from $(0, 0)$ to

$(\cos(x), \sin(x))$. The area of this sector is $x/2$. Now consider the rectangle containing this sector with base equal to the line segment from $(0, 0)$ to $(1, 0)$ and with height $\sin(x)$. The area of this rectangle is $\sin(x)$. Because the rectangle contains the sector, $\sin(x) > x/2$ for $0 < x < \pi/2$. Thus, the unique intersection point for $0 \leq z < \pi/2$ is at $x = 0$. Similarly, the unique intersection point for $-\pi/2 < x \leq 0$ is at $x = 0$. Hence, there is precisely one intersection point at $x = 0$; the answer is (b).