Name: ______________________ Core Competency Exam A

You do not need to show work. Answer on the line.

5. **Problem 1** The equation $|x+3| = 2$ has: **(1)** no solutions, **(2)** a unique solution, **(3)** one positive and one negative solution, **(4)** two positive solutions, or **(5)** two negative solutions. $x=-1$ $x=-5$

2. **Problem 2** The set of all real numbers where $|2x-4| < 2$ is:

(1) $[1,3]$, **(2)** $(1,3)$, **(3)** $(-\infty,1]\cup[3,\infty)$, **(4)** $(-\infty,1)\cup(3,\infty)$, or **(5)** $(2/2, 6/2]$.

1. **Problem 3** The reflection through the x-axis of the graph of $y = f(x)$ is the graph of:

(1) $y = -f(x)$, **(2)** $y = f(-x)$, or **(3)** $y = -f(-x)$.

4. **Problem 4** For the function $f(x) = 1/(-x+3)$, $x \neq 3$, the value $f(f(2))$ equals: $f(2)=1, f(1)=\frac{1}{2}$

(1) 1, **(2)** $-1/2$, **(3)** -1, **(4)** $1/2$, or **(5)** undefined.

4. **Problem 5** For the functions $f(x) = 1 + (\frac{1}{x})$, $x \neq 0$, and $g(x) = \frac{1}{2x-1}$, $x \neq 1/2$, the composite function $f(g(x))$, $x \neq 1/2$, equals

$$\textbf{(1)}\ \frac{x+1}{x},\quad \textbf{(2)}\ \frac{2x}{2x-1},\quad \textbf{(3)}\ 1-\frac{1}{1/(2x-1)},\quad \textbf{(4)}\ 2x.$$

3. **Problem 6** The equation of the line with slope -2 containing the point $(x,y) = (1,2)$ is:

(1) $y = -2x+2$, **(2)** $y-1 = -2(x-2)$, **(3)** $y = -2x+4$, or **(4)** $y-2 = 2(x-1)$.

4. **Problem 7** The line containing the two points $(x,y) = (1,0)$ and $(x,y) = (2,3)$ has equation $m=\frac{3-0}{2-1}=3$

$$\textbf{(1)}\ y-0 = \frac{2-1}{3-0}(x-1),\quad \textbf{(2)}\ y-1 = \frac{3-0}{2-1}(x-0),\quad \textbf{(3)}\ y = 3x,\ \text{or}\quad \textbf{(4)}\ y = 3x-3.$$

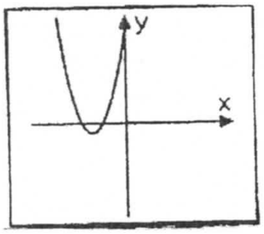
1. **Problem 8** The perpendicular line to $y = 2x+1$, containing the point $(1,2)$ has equation $m=-\frac{1}{2}$

(1) $y = (-1/2)x + (5/2)$, **(2)** $y = 2x$, **(3)** $y = (1/2)x + (3/2)$, or **(4)** $y = (-1/3)x + 2$.

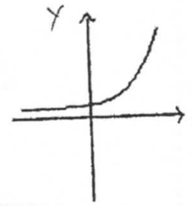
1. **Problem 9** The solutions of the quadratic equation $x^2 + 3x = 0$ are **(1)** $x = 0$ and $x = -3$, **(2)** $x = 0$ and $x = 3$, **(3)** $x = -3$ and $x = +3$, or **(4)** undefined.

3. **Problem 10** The parabola with equation $y = x^2 - 5x + 6$ satisfies $y > 0$ for x in

(1) $(2,3)$, **(2)** $[2,3]$, **(3)** $(-\infty,2)\cup(3,\infty)$, **(4)** $(-\infty,2]\cup[3,\infty)$.



PROBLEM 1



PROBLEM 9

Name: ______________________ Core Competency Exam B

1 . **Problem 1.** The displayed conic section is a:
(1) parabola, (2) circle, (3) ellipse, or (4) hyperbola.

2 . **Problem 2.** The number $(8)^{1/2}/\sqrt{4}$ equals: $2\sqrt{2}/2 = \sqrt{2}$

(1) $2\sqrt{2}$, (2) $\sqrt{2}$, (3) 2, (4) $2^{-1/2}$, or (5) 1.

4 . **Problem 3.** The function $(x^2 - x)/\sqrt{x^3}$, $x > 0$, equals: $\frac{x^{4/2} - x^{2/2}}{x^{3/2}} = \frac{x^{4/2}}{x^{3/2}} - \frac{x^{2/2}}{x^{3/2}} = x^{\frac{1}{2}} - x^{-\frac{1}{2}}$

(1) $(1-x)/\sqrt{x}$, (2) $\sqrt{(x^4 - x^2)/x^3}$, (3) $\sqrt{(x^4/x^3)} - \sqrt{(x/x^3)}$, (4) $x^{1/2} - x^{-1/2}$, or (5) $x^{1/2} - x$.

2 . **Problem 4.** The function $(xy^3)^2/(xy)^3$, $x > 0$, $y > 0$, equals: $\frac{x^2y^6}{x^3y^3} = x^{-1}y^3$

(1) $(x^2y^3)/(x^3y)$, (2) $x^{-1}y^3$, (3) x^{-1}, (4) y^3, or (5) $(2xy^3)/(3xy)$.

3 . **Problem 5.** The function $(x+1)^2 + (x-1)^2$ equals $(x^2+2x+1) + (x^2-2x+1) = 2x^2+2$

(1) $(x^2+2x+1)-(x^2-2x-1)$, (2) $((x+1)+(x-1))((x+1)-(x-1))$, (3) $2x^2+2$, (4) $2x^2+4x$, or (5) $2x^2$.

2 . **Problem 6.** The solution x of the equation $2(3^x) = 54$ is: $3^x = 54/2 = 27$, $x=3$

(1) $\log_3(54)/\log_3(2)$, (2) 3, (3) 27, (4) 4, or (5) $\log_5(54) - \log_5(2)$.

1 . **Problem 7.** The function $\log_3(9x^2)$, $x > 0$, equals: $\log_3(9) + \log_3(x^2) = 2 + 2\log_3(x)$

(1) $2 + 2\log_3(x)$, (2) $2 + \log_5(x^2)$, (3) $2\log_3(9x)$, (4) $2\log_5(3x)$, or (5) $(\log_3(3x))^2$.

4 . **Problem 8.** The solution $x > 0$ of the equation $\log_2(8x^2) = 7$ is: $2^3x^2 = 2^7$, $x^2 = 2^4$, $x = 2^2 = 4$

(1) 16, (2) $2^7/8$, (3) 2, (4) 4, or (5) $\sqrt{2^7/2^8}$.

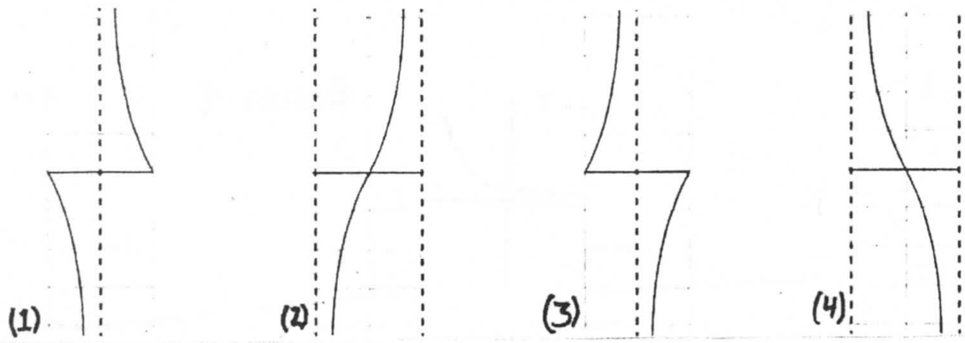
1 . **Problem 9.** The displayed graph might be the graph of the function $f(x) =$

(1) $5(2^x)$, (2) $3(1/2)^x$, (3) $-2(3^x)$, or (4) $2\log_3(x)$.

3 . **Problem 10.** For the unique real numbers $x > 0$, $y > 0$ with $\log_2(x) = 3$ and $\log_2(y) = 1/2$, the expression $\log_2(xy^3)$ equals:

(1) $\sqrt{512}$, (2) $3\log_2(8\sqrt{2})$, (3) $9/2$, (4) $\log_5(x) + 3\log_5(y)$, or (5) $\log_5(16\sqrt{2})$.

$\log_2|x| + 3\log_2|y|$
$= 3 + 3\cdot\frac{1}{2} = \frac{9}{2}$

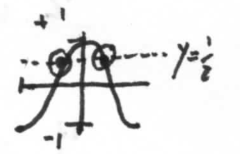


Name: ______________________________ Core Competency Exam C

All angle measures are in radians.

3___. **Problem 1.** Which of the displayed graphs is $\tan(x)$ on $0 \le x \le \pi$?
(1) Graph 1, (2) Graph 2, (3) Graph 3, or (4) Graph 4.

3___. **Problem 2.** The number of intersections of $y = \cos(x)$ and $y = 1/2$ with $-\pi < x < \pi$ is:
(1) no intersections, (2) one, (3) two, (4) three, or (5) infinitely many intersections.



4___. **Problem 3.** The function $\sin(-\theta)$ equals:

(1) $\cos(\theta)$, (2) $-\cos(\theta)$, (3) $\sin(\theta)$, (4) $-\sin(\theta)$, or (5) $\tan(\theta)$.

3___. **Problem 4.** The function $\cos(x + \pi)$, equals:

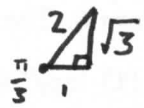
(1) $-\sin(x)$, (2) $\sin(x)$, (3) $-\cos(x)$, (4) $\cos(x)$, or (5) $\tan(x)$.

2___. **Problem 5.** The function $\cos(2x)$ equals

(1) $2\sin(x)\cos(x)$, (2) $(\cos(x))^2 - (\sin(x))^2$, (3) $(\cos(x))^2 + (\sin(x))^2$, or (4) $2\cos(x)$,

1___. **Problem 6.** The value $\sin(\pi/3)$ equals

(1) $\sqrt{3}/2$, (2) $1/\sqrt{2}$, (3) $1/2$, (4) 0, or (5) 1.



2___. **Problem 7.** The value $\sin(5\pi/2)$ equals

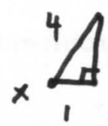
(1) $\sqrt{3}/2$, (2) 1, (3) 0, (4) -1, or (5) $1/\sqrt{2}$.

5___. **Problem 8.** The expression $(\sin(x))^2 \cot(x) \csc(x)$ equals

$\frac{\sin^2(x)}{1} \cdot \frac{\cos(x)}{\sin(x)} \cdot \frac{1}{\sin(x)} = \cos(x)$

(1) $\sin(x)$, (2) $\sec(x)$, (3) $\tan(x)$, (4) $\sin^2(x)$, or (5) $\cos(x)$.

2___. **Problem 9.** For the angle $0 < x < \pi$ with $\cos(x) = 1/4$, the value $\sin(x)$ equals



$\sqrt{4^2-1^2} = \sqrt{15}$

(1) 4, (2) $\sqrt{15}/4$, (3) $\sqrt{3}/2$, or (4) $1/\sqrt{2}$.

2___. **Problem 10.** For the function $f(x) = \sin(2x)$, the value $f(\pi/8)$ equals

$2 \cdot \frac{\pi}{8} = \frac{\pi}{4}, \ \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

(1) 1, (2) $1/\sqrt{2}$, (3) $1/2$, (4) $\sqrt{3}/2$, or (5) 0.

Mastery Exam. Show all Work.

Name: ______________________ **Problem 1:** __________ /30

Mastery Problem 1(30 points) For all parts of this problem, $f(x)$ equals $3\sin(2x-(\pi/4))$. Show all work.

(a)(5 points) Find the range of $f(x)$. Express your answer in interval notation $[y_{\min}, y_{\max}]$ for the maximum possible value $y_{\max}$ of $f(x)$ and the minimum possible value $y_{\min}$ of $f(x)$.

$-1 \le \sin(\theta) \le +1$

$-3 \le 3\sin(\theta) \le +3$, Range $= [-3, +3]$

(b)(5 points) Find the smallest **positive** real number s such that $f(s)$ is the maximum value $y_{\max}$. Also, find the smallest **positive** real number t such that $f(t)$ is the minimum value $y_{\min}$.

$3\sin(2x-\frac{\pi}{4}) = 3$

$\sin(2x-\frac{\pi}{4}) = 1$

$2x-\frac{\pi}{4} = \frac{\pi}{2} \pm 2n\pi$

$2x = \frac{3\pi}{4} \pm 2n\pi$

$x = 3\pi/8 \pm n\pi$

$3\sin(2x-\frac{\pi}{4}) = -3$

$\sin(2x-\frac{\pi}{4}) = -1$

$2x-\frac{\pi}{4} = \frac{3\pi}{2} \pm 2n\pi$

$2x = \frac{7\pi}{4} \pm 2n\pi$

$x = \frac{7\pi}{8} \pm n\pi$

$$\boxed{s = \frac{3\pi}{8},\quad t = \frac{7\pi}{8}}$$

(c)(5 points) Restrict the domain of $f(x)$ to $[s, t]$ for s and t as above. For the inverse function f^{-1} of f on this domain, find the domain and range of f^{-1}.

Domain f : $[\frac{3\pi}{8}, \frac{7\pi}{8}]$ → Range f^{-1} : $[\frac{3\pi}{8}, \frac{7\pi}{8}]$

Range f : $[-3, +3]$ → Domain f^{-1} : $[-3, +3]$

(d)(10 points) Find a formula for the inverse function f^{-1} above. Your answer should involve a standard inverse trigonometric function such as $\arcsin(\theta) = \sin^{-1}(\theta)$ with range $[-\pi/2, \pi/2]$ or $\arccos(\theta) = \cos^{-1}(\theta)$ with range $[0, \pi]$. Please double-check that your formula has the same domain and range as in (c).

$y = 3\sin(2x-\frac{\pi}{4})$, $\frac{\pi}{2} \le 2x-\frac{\pi}{4} \le \frac{3\pi}{2}$

$\frac{y}{3} = \sin(2x-\frac{\pi}{4})$, $\frac{\pi}{2} \le 2x-\frac{\pi}{4} \le \frac{3\pi}{2}$

$-\frac{\pi}{2} \le \sin^{-1}(\frac{y}{3}) \le \frac{\pi}{2}$, $\frac{\pi}{2} \le 2x-\frac{\pi}{4} \le \frac{3\pi}{2}$

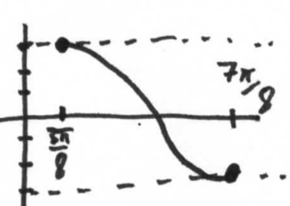
$\pi - (2x-\frac{\pi}{4}) = \sin^{-1}(\frac{y}{3})$

$\frac{5\pi}{4} - 2x = \sin^{-1}(\frac{y}{3})$

$2x = \frac{5\pi}{4} - \sin^{-1}(\frac{y}{3})$

$$\boxed{x = \frac{5\pi}{8} - \frac{1}{2}\sin^{-1}\left(\frac{y}{3}\right)}$$

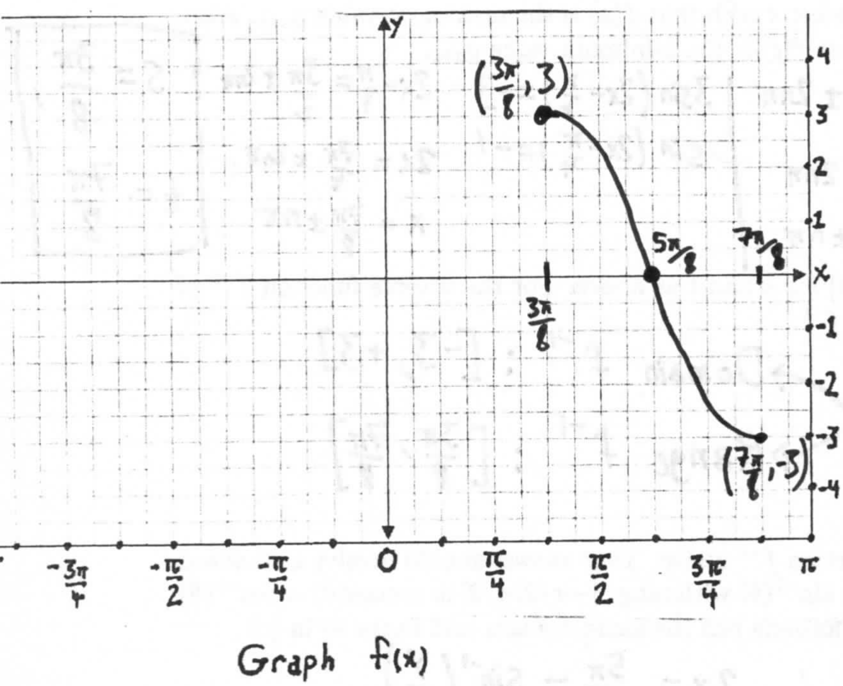
OR

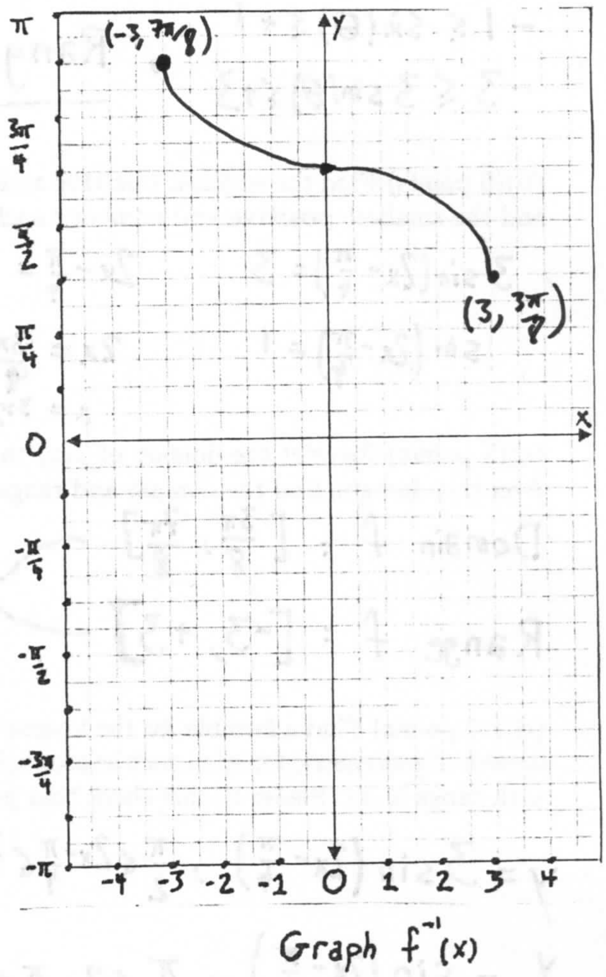


$$\boxed{x = \frac{3\pi}{8} + \frac{1}{2}\cos^{-1}\left(\frac{y}{3}\right)}$$

Name: ______________________________ **Problem 1 continued.**

(e)(5 points) Please graph below the function f on the specified domain $[s, t]$ and the inverse function f with its domain. Please label the coordinates of the endpoints of each graph.





Name: ______________________________ **Problem 2:** ____________ /35

Mastery Problem 2(35 points) A mass of radioactive material decays from 480 tons at time $t = 0$ yearst to 60 tons at time $t = 150$ years. Assume that the mass decays following an exponential decay model. Do all of the following. You may follow whatever order you prefer, but please show all work and indicate clearly your answers to each part.

(a)(15 points) Write the formula for the mass $a(t)$ of the mass after time t. Please include appropriate units in your answer (the same units used above), and leave no undefined constants in your final answer.

(b)(10 points) Find the half-life for the radioactive material.

(c)(10 points) Find the time t at which the mass of radioactive material is 7.5 tons.

$a(t) = a_0 \cdot 2^{-t/h}$, $a_0 = \text{initial mass} = 480 \text{ tons}$, $h = \text{half-Life}$

$a(150\text{ yrs}) = \frac{1}{8} a_0 = 2^{-3} \cdot a_0$, $a(150\text{ yrs}) = a_0 \cdot 2^{-150 \text{yrs}/h}$,

$3 = 150\text{ yrs}/h$, $\boxed{h =}\ 150\text{ yrs}/3 = \boxed{50\text{ yrs}}$

$$\boxed{a(t) = 480 \text{ tons} \cdot 2^{-t/50\text{yrs}}}$$

$480 \text{ tons} \cdot 2^{-t/50\text{ yrs}} = 7.5 \text{ tons} = 480 \text{ tons} \cdot \frac{1}{64} = 480 \text{ tons} \cdot 2^{-6}$

$\frac{t}{50\text{ yrs}} = 6$, $t = 6 \times 50\text{ yrs} = \boxed{300\text{ years}}$

Name: ______________________________ **Problem 3:** ____________ /30

Mastery Problem 3(30 points) For the following expression,

$$f(x) = \frac{2x^2 - 3x + 1}{x^2 - 1},$$

do all of the following.

(a)(5 points) Find the maximal domain on which the expression is defined. Please write your answer in interval notation.

Undefined when

$x^2 - 1 = (X-1)(X+1) = 0$, $x = -1$, $x = +1$

Domain: $(-\infty, -1) \cup (-1, +1) \cup (+1, \infty)$

(b)(5 points) Find the coordinates (x, y) of each point where the graph crosses the x-axis.

$2x^2 - 3x + 1 = (2x-1)(x-1) = 0$

$x = 1$, $x = \frac{1}{2}$. $f(x)$ undefined at $x=1$.

$f(\frac{1}{2}) = \frac{0}{-3/4} = 0$.

Graph crosses only at $(x,y) = \boxed{(\frac{1}{2}, 0)}$.

(c)(5 points) Find the coordinates (x, y) of each point where the graph crosses the y-axis.

$f(0) = \frac{2 \cdot 0^2 - 3 \cdot 0 + 1}{0^2 - 1} = \frac{1}{-1} = -1$. Graph crosses at $(x,y) = \boxed{(0, -1)}$.

(d)(5 points) Determine whether or not $f(x)$ has a well-defined limit as x approaches $+\infty$ or $-\infty$. If the limit does exist, find the limit and write the equation of the corresponding horizontal asymptote in the form $y = a$ for some real number a.

$f(x) = \frac{\cancel{x^2}(2 - \frac{3}{x} + \frac{1}{x^2})}{\cancel{x^2}(1 - \frac{1}{x^2})} = \frac{2 - \cancel{\frac{3}{x}}^{\to 0} + \cancel{\frac{1}{x^2}}^{\to 0}}{1 - \cancel{\frac{1}{x^2}}^{\to 0}} \longrightarrow \frac{2}{1} = 2.$

Horizontal Asymptote $\boxed{y = 2.}$

(e)(5 points) Determine whether or not there is a vertical asymptote. If so, for each vertical asymptote, find the equation in the form $x = b$ for a real number b.

Near $x = -1$

$f(x) = \frac{(2x-1)(x-1)}{(x+1)(x-1)} \to \frac{(-3)(-2)}{0} \pm\infty$

Vertical asymptote at $\boxed{x = -1}$.

Near $x = +1$

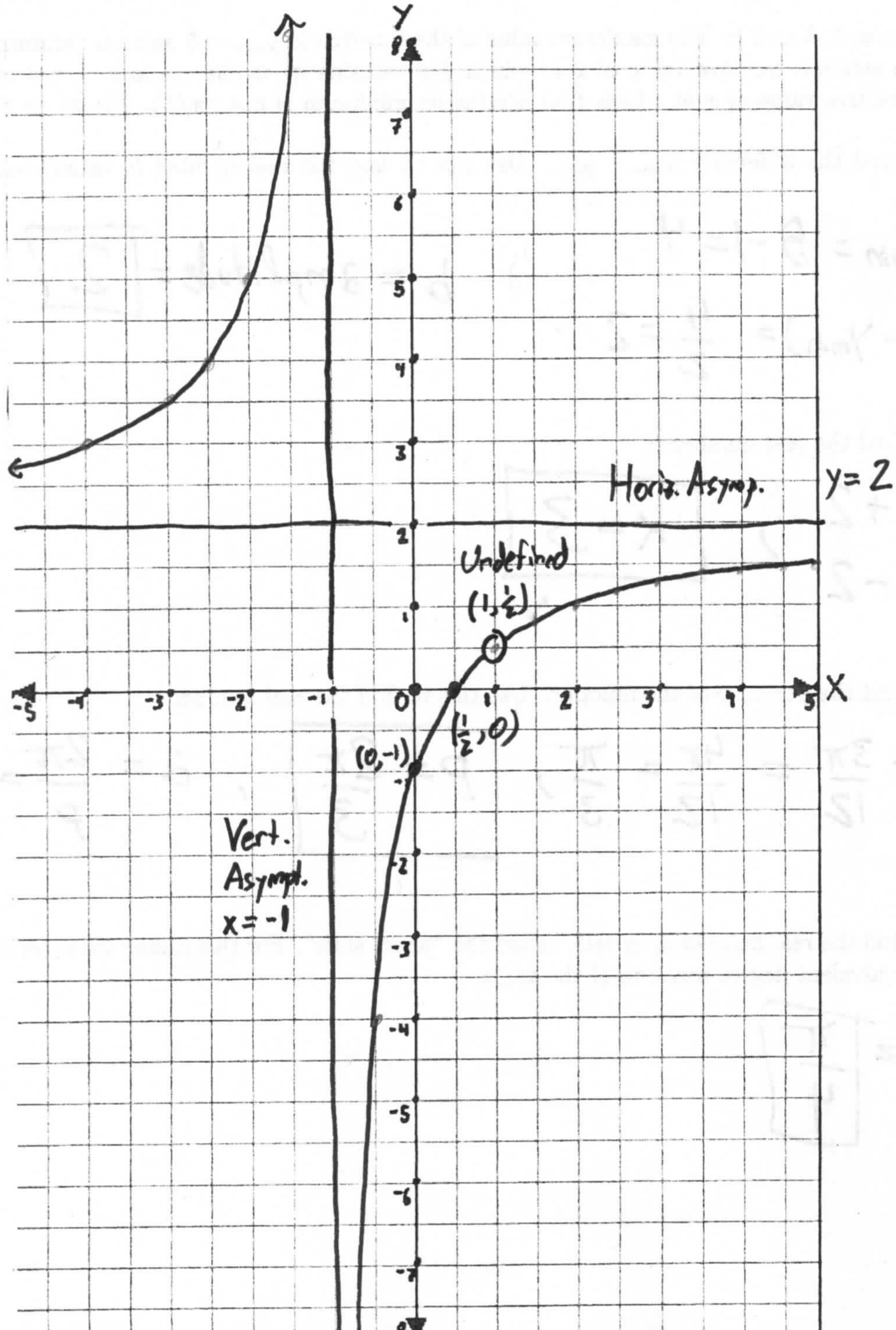
$f(x) = \frac{(2x-1)(x-1)}{(x+1)(x-1)} = \frac{2x-1}{x+1}$, $x \neq 1$

$\to \frac{2 \cdot 1 - 1}{1 + 1} = \frac{1}{2} \neq \pm\infty$.

No vertical asymptote at $x = 1$.

Name: ______________________ **Problem 3 continued.**

(f)(5 points) On the axes below, please give a rough sketch of the graph of $f(x)$. Please label every vertical or horizontal asymptote, every intersection point with the x-axis or y-axis, and every point at which the function is not defined.



Name: ____________________ **Problem 4:** __________ /30

Mastery Problem 4(30 points) A function $f(x)$ has the form

$$y = f(x) = k + b\cos((x-h)/a)$$

for real numbers a, b, h and k. The maximum value of the function is $y_{\max} = 5$ and has minimum value $y_{\max} = 1$. The smallest positive value of x at which $f(x)$ attains its maximum is $s = \pi/4 = 3\pi/12$. The smallest positive value of x at which $f(x)$ attains its minimum is $t = 7\pi/12$. Please do all of the following.

(a)(5 points) Find the difference $y_{\max} - y_{\min}$. Use this to find the real number b, usually called the "amplitude".

$y_{max} - y_{min} = 5 - 1 = 4$

$\frac{1}{2}(y_{max} - y_{min}) = \frac{4}{2} = 2$

$b = \text{amplitude} = \boxed{2}$

(b)(5 points) Find the real number k.

$5 = 3 + 2$

$1 = 3 - 2$, $\boxed{k = 3}$

(c)(5 points) Find the period p of the function. Use this to find the real number a.

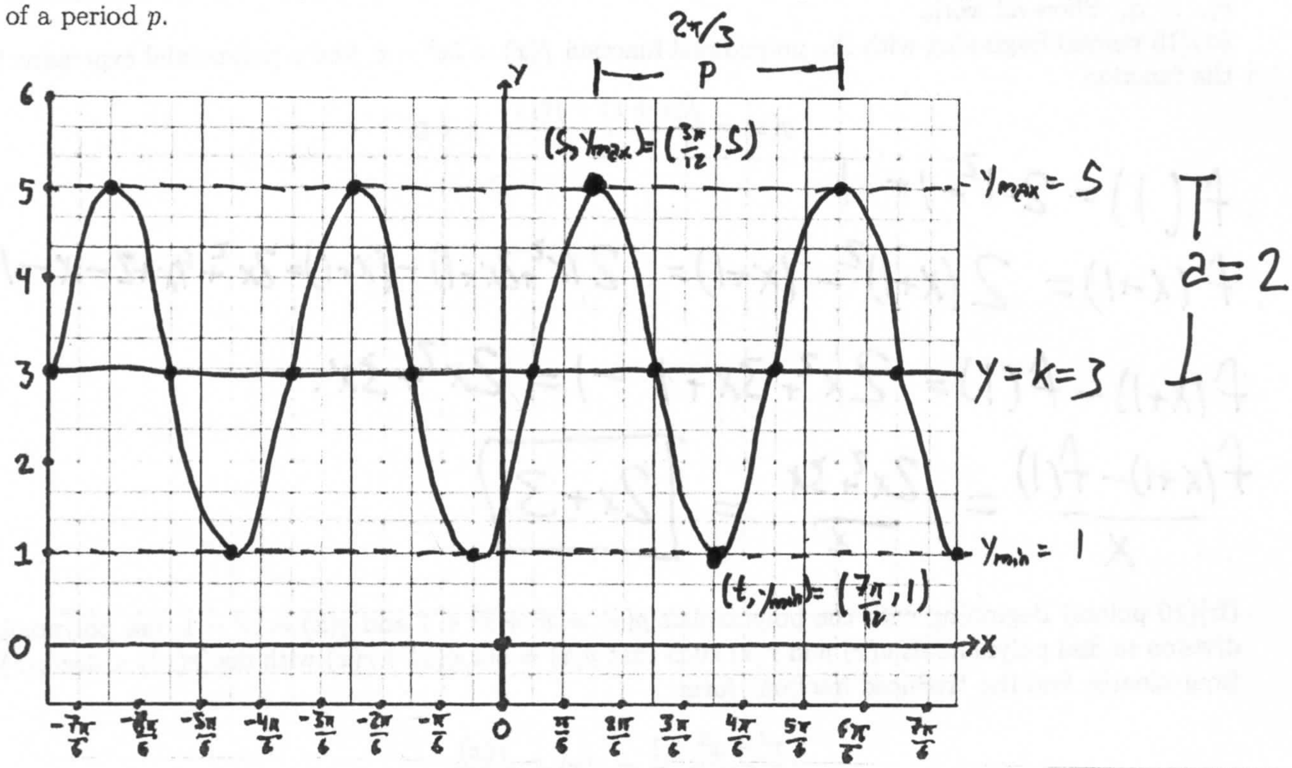
$\frac{1}{2}p = \frac{7\pi}{12} - \frac{3\pi}{12} = \frac{4\pi}{12} = \frac{\pi}{3}$, $\boxed{p = \frac{2\pi}{3}}$, $a = \frac{2\pi}{p} = \frac{2\pi}{2\pi/3} = \boxed{3}$

(d)(5 points) Find the real number h, usually called the "phase shift". For this radian measure h, please also write the equivalent degree measure of the angle.

$h = s = \boxed{\frac{\pi}{4}}$

Name: ______________________________ **Problem 4 continued.**

(e)(5 points) On the axes below, give a rough sketch of at least two periods of the function. Carefully label point $(s, y_{\max})$, the point $(t, y_{\min})$, the horizontal line $y = k$, the length of the amplitude a, and the length of a period p.



(f)(5 points) Use the angle addition formulas to rewrite $f(x)$ in the form

$$y = f(x) = k + u\cos(x/a) + v\sin(x/a)$$

for some choice of real numbers u and v that you compute.

$$f(x) = 3 + 2\cos\left(\frac{x}{3} - \frac{\pi}{12}\right) = 3 + 2\left(\cos\left(\frac{\pi}{12}\right)\cos\left(\frac{x}{3}\right) + \sin\left(\frac{\pi}{12}\right)\sin\left(\frac{x}{3}\right)\right)$$

$$= \boxed{3 + \left(2\cos\left(\frac{\pi}{12}\right)\right)\cos\left(\frac{x}{3}\right) + \left(2\sin\left(\frac{\pi}{12}\right)\right)\sin\left(\frac{x}{3}\right)}$$

Name: ______________________ **Problem 5:** __________ /25

Mastery Problem 5(25 points) Perform the following polynomial computations. Express your polynomials in the form $c_0 + c_1x + c_2x^2 + \cdots + c_nx^n$ for positive whole numbers n and real numbers c_0, c_1, $c_2, \ldots, c_n$. Show all work.

(a)(15 points) Beginning with the polynomial function $f(x) = 2x^2 - x$, find a polynomial expression for the function

$$g(x) = \frac{f(x+1) - f(1)}{x}, \quad x \neq 0.$$

$f(1) = 2 \cdot 1^2 - 1 = 1$

$f(x+1) = 2(x+1)^2 - (x+1) = 2(x^2+2x+1) - (x+1) = 2x^2 + 4x + 2 - x - 1 = 2x^2 + 3x + 1$

$f(x+1) - f(1) = 2x^2 + 3x + 1 - 1 = 2x^2 + 3x.$

$$\frac{f(x+1) - f(1)}{x} = \frac{2x^2+3x}{x} = \boxed{2x+3}$$

(b)(10 points) Beginning with the polynomials $p(x) = x^3 + x^2 + 1$ and $q(x) = x^2 - 1$, use polynomial division to find polynomials $a(x)$ and $r(x)$ such that $p(x) = a(x)q(x) + r(x)$ with $\deg(r(x)) < \deg(q(x))$. Equivalently, find the "reduced fraction" form

$$\frac{x^3 + x^2 + 1}{x^2 - 1} = a(x) + \frac{r(x)}{x^2 - 1}.$$

$$\begin{array}{r l}
 & 1x + 1 \;R\; 1x+2 \\
1x^2 + 0x - 1 \;\big)\!\! & 1x^3 + 1x^2 + 0x + 1 \\
 & 1x^3 + 0x^2 - 1x \\
\hline
 & 1x^2 + 1x + 1 \\
 & 1x^2 + 0x - 1 \\
\hline
 & 1x + 2
\end{array}$$

, $$\frac{x^3+x^2+1}{x^2-1} = \boxed{(x+1) + \frac{x+2}{x^2-1}}$$