RESEARCH STATEMENT

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1. Rational Points of Rationally Connected Varieties

Do polynomial equations with coefficients in a field K have solutions in K, and if so, how many? Already for two linear polynomials in two variables, i.e., two intersecting lines in the plane, the answer becomes uniform only if we work with *projective* varieties, or equivalently, homogeneous polynomials. For K algebraically closed, this is settled by Hilbert's Nullstellensatz, but it remains a basic problem for non-algebraically closed fields such as $\mathbb{Q}$, $\mathbb{F}_p(t)$, etc. Many number theorists focus on *nonexistence*, or at least scarcity of rational points, and with good reason: it is useless to search for solutions if they provably do not exist. However, my work focuses on *existence* and density results under simple, testable geometric hypotheses that are special, yet ubiquitous. The following theorem was conjectured by Kollár-Miyaoka-Mori and generalizes the classical theorem of Tsen that a homogeneous polynomial with coefficients in $K = \mathbb{C}(t)$ has a rational point if the degree is less than the number of variables.

Theorem 1.1 (Graber-Harris-Starr, [22])**.** *A projective algebraic variety over $K = \mathbb{C}(t)$ has K-rational points if for a general choice of $t \in \mathbb{C}$ the specialization is rationally connected, and then the K-points are Zariski dense in the $\overline{K}$-points.*

Classically, a system of homogeneous equations is *unirational* if there is a "map" providing the location of each solution, i.e., all general solutions arise from a single multivariable polynomial function whose output for a general choice of inputs is a solution of the system. Because of homogeneity, we can use polynomials, but classically the coordinates of these maps were *rational functions*, i.e., fractions of polynomial functions in some fraction field $\mathbb{C}(t_1, \ldots, t_n)$ of a polynomial ring $\mathbb{C}[t_1, \ldots, t_n]$. Anybody who uses a GPS knows that often a *route* from origin to destination is more convenient than a map. Here, a route, or *rational curve* is the set of outputs of a single variable polynomial function whose outputs are solutions of the system. A projective algebraic variety is *rationally connected* if every pair of points on the variety is contained in a rational curve, i.e., there is always a route from origin to destination. Rational connectedness is often easy to check.

This theorem is robust, there is an extension to positive characteristic by de Jong and myself, [12, de Jong-Starr], and it has numerous applications: the proof of rational connectedness of log-$\mathbb{Q}$-Fano manifolds by Qi Zhang, [50], the proof of Shokurov's Conjecture by Hacon-McKernan, [24], the proof by Hassett-Tschinkel of weak approximation at places of good reduction, [30], the proof by Kebekus-Solá Conde-Toma of the Bogomolov-McQuillan theorem, [32], etc. Moreover, there is a converse theorem by Graber, Harris, Mazur and myself which we used to settle (in the negative) an old problem posed by Serre to Grothendieck (an analogous positive result over finite fields was first proved by Mazur in 1972, [39], and recently generalized by Berthelot-Esnault-Rülling, [4]).

Theorem 1.2 (Graber-Harris-Mazur-Starr, [21])**.** *For every family X of varieties over a quasi-projective parameter variety M, the restriction of the family over every curve in M has a rational section if and only if there is a subfamily $Y \subset X$ whose fiber over a general point of M is rationally connected. In particular, there exists a smooth projective variety X over $K = \mathbb{C}(t)$ with no K-point yet with vanishing $h^q(X, \mathcal{O}_X)$ for every $q > 0$.*

In fact, our X is an Enriques surface. After Graber, Harris, Mazur and I proved this "Converse Theorem", Lafon found an example of an Enriques surface over $\mathbb{C}((t))$ without any local point, [35]. Following this, Esnault asked about the existence of an Enriques surface with local points and with "index 1", i.e., the gcd of the degrees of all closed points equals 1, yet without K-rational points. By elaborating the technique of the "Converse Theorem", I constructed such Enriques surfaces. Thus, at present there is no conjecture about sufficient conditions to guarantee existence of a rational point on an $\mathcal{O}$-acyclic variety such as an Enriques surface (it seems that rationally connected varieties are precisely the correct family of varieties for such results).

Theorem 1.3 (Starr, [47]). *There exists an Enriques surface over $K = \mathbb{C}(t)$ that has local points everywhere, that has index 1, and yet the surface has no K-rational point.*

In a similar vein, for Abelian varieties, Graber and I sharpened the "Converse Theorem", demonstrating that sections of Abelian varieties are a "one-dimensional phenomenon".

Theorem 1.4 (Graber-Starr, [23]). *For every smooth, quasi-projective complex variety M, there is an explicit family of curves on M (the "triangle curves"), such that for every normal, projective variety X and for every surjective morphism $X \to M$ whose geometric generic fiber is an Abelian variety, for a very general curve $C \subset M$ in this family, the restriction $Sections(X/M) \to Sections(X_C/C)$ is an isomorphism.*

2. Rationally Simply Connected Varieties and Rational Points

Despite the many answers that flow from the Graber-Harris-Starr theorem, it raises just as many new questions. In particular, what is the analogue for other fields such as $\mathbb{C}(s,t)$ or $\mathbb{F}_p(t)$? We have one answer that settles the geometric case of a conjecture of Serre, formulated in 1963, [6]. Existence of points depends on the vanishing of the *elementary obstruction* of Colliot-Thélène and Sansuc, which has a simple description in the case of Picard rank one. A projective variety together with a specified embedding in projective space, $X \subset \mathbb{P}^N_K$, has vanishing "elementary obstruction in Picard rank one" if, even after extending from K to the algebraic closure $\overline{K}$, every projective embedding of X is obtained from this one by Veronese re-embedding followed by linear projection.

Theorem 2.1 (de Jong - He - Starr, [11]). *For $K = \mathbb{C}(s,t)$, or more generally the function field of a complex surface, for an embedded projective variety with vanishing elementary obstruction in Picard rank one, if the base change to $\overline{K}$ is* rationally simply connected *and if the codimension 1 specializations are irreducible, then there exists a K-rational point.*

A codimension 1 specialization is a substitution of the parameters such as $t = s$ that reduces from two free parameters to one parameter, and irreducibility means that, after removing the singular set from the specialized subvariety of $\mathbb{P}^N$, the smooth locus is still connected. Since existence of rational sections can be checked after specialization, it suffices to check this transversality condition for a sufficiently general deformation, and this is usually easy. The condition on rational simple connectedness roughly means that the parameter spaces of rational curves containing two general points of the variety are themselves rationally connected. This is analogous to simple connectedness of a manifold, i.e., path connectedness of the loop spaces. In particular, a complete intersection in $\mathbb{P}^n$ of sufficiently general hypersurfaces of degrees $d_1, \dots, d_c$ is rationally simply connected if and only if $d_1^2 + \cdots + d_c^2 \leq n$, [16, DeLand]. This inequality means that not only is the first Chern class of the tangent bundle positive, i.e., $d \leq n$ as in Tsen's theorem, but also we have positivity of the second graded piece of the Chern character, $\mathrm{ch}_2 = (c_1(T_X)^2 - 2c_2(T_X))/2$.

One special case of the theorem settles the split case of Serre's 1963 "Conjecture II" for function fields of complex surfaces, [43, Section III.3], the last unproved case (following tremendous earlier results by Merkurjev and Suslin, Bayer-Fluckiger and Parimala, Chernousov, Gille, and Colliot-Thélène).

Corollary 2.2 (Geometric case of Serre's "Conjecture II", de Jong – He – Starr, [11])**.** *For K the function field of a surface over $\mathbb{C}$, or over any algebraically closed field, for every semisimple, simply connected algebraic group over K, every principal homogeneous space for this group over K has a K-rational point.*

One key step is an argument lifting from positive characteristic, e.g., $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, to characteristic 0, e.g., $\mathbb{C}$. Combining this "discriminant avoidance" technique with the theorem above and seminal work of Esnault, [17] and [18], and Esnault-Xu, [19], on rational points of rationally connected varieties defined over finite fields, Chenyang Xu and I proved the following.

Theorem 2.3 (Starr-Xu, in preparation)**.** *A projective variety over $K = \mathbb{F}_p(t)$ or any global function field has a K-rational point if it is the specialization from characteristic 0 of a rationally simply connected variety with vanishing Picard rank one elementary obstruction.*

In particular, this reproves for these fields the split case of Serre's Conjecture II (the full conjecture for $\mathbb{F}_p(t)$ was proved by Harder, [26]), the Brauer-Hasse-Noether theorem, the Tsen-Lang theorem, [49], [36], but also it gives new theorems for complete intersections in Grassmannian varieties when combined with the PhD thesis of my former advisee Robert Findley, [20].

Although it is tempting to seek an analogue over number fields, it has been known since Terjanian's disproof of the Artin conjecture, [48], that a theorem as above cannot hold for all number fields. However, Ax and Kochen did prove an analogue of Artin's conjecture for "most" local fields, [1]. This is tied to Ax's theory of PAC fields as arise in the proof of the Ax-Kochen theorem. I have generalized the Graber-Harris-Starr theorem to PAC fields.

Theorem 2.4 (Starr, [46])**.** *For every perfect PAC field K that contains the algebraic closure of its prime subfield, for every specialization over K of a family of projective varieties over a DVR whose geometric generic fiber is separably rationally connected, the variety has a K-point.*

In characteristic 0 this was proved by a completely different method by Kollár and then Hogadi-Xu, following Kollár's proof of the characteristic 0 Ax conjecture, [33], [31]. Since in characteristic p the theorem is also known for all subfields k of the algebraic closure of the prime field by work of Esnault and Esnault-Xu,[17], [18], [19], I am quite hopeful about the Ax conjecture in all characteristics.

3. Spaces of Rational Curves on Varieties

The spaces of rational curves on a rationally connected variety are algebro-geometric analogues of path and loop spaces of a path-connected manifold. A key innovation was Kontsevich's introduction of his *spaces of stable maps*, [34], following earlier work of Gromov and Witten. These spaces, together with the Behrend-Fantechi theory of virtual fundamental classes in algebraic geometry based on perfect obstruction theories, [3], lead to a robust theory of Gromov-Witten invariants in algebraic geometry. These are inspired by the classical enumerative problem of counting the number of rational curves of a given homology class in a specified projective variety that intersect given general linear spaces. Nonetheless, there are few cases where these Gromov-Witten invariants are *proved* to agree with these enumerative curve counts, i.e., where the Gromov-Witten invariants are *enumerative*. Following earlier work of Kontsevich when the target is a projective homogeneous variety, [34], one of the first such results for inhomogeneous varieties was the following.

Theorem 3.1 (Harris-Roth-Starr, Coskun-Starr, [27], [10])**.** *For every integer $d \leq (n+1)/2$, for every sufficiently general hypersurface $X \subset \mathbb{CP}^n$ of degree d, for every integer $e \geq 1$, the Kontsevich space $\overline{M}_{0,n}(X, e)$ is irreducible and reduced of the expected dimension. In particular, the Gromov-Witten invariants are enumerative.*

This is proved by induction on e, using a combination of the Bend-and-Break technique applied so powerfully by Mori and a careful study of the boundary of the Kontsevich space. A better argument for the base case was the key to the recent theorem of Eric Riedl and David Yang, who proved the theorem above whenever $d \leq n-2$, [42]. This is the optimal range for the Kontsevich spaces to be irreducible of the expected dimension.

It was hoped that the Kontsevich spaces of a rationally connected variety might be "almost" rationally connected, and, in particular, uniruled. This would provide an inductive strategy to prove that every rationally connected variety is unirational. However, Kontsevich spaces need not be uniruled, and now most experts believe that there exist non-unirational Fano manifolds – this is one of the main open problems in this area. The first examples of non-uniruled Kontsevich spaces were discovered in joint work with de Jong. The target Fano manifold is a smooth cubic hypersurface $X \subset \mathbb{CP}^4$. This has a unique $(3,1)$-Hodge class, up to scaling.

Theorem 3.2 (de Jong-Starr, [13])**.** *If X is sufficiently general, then the $(3,1)$-form induces a holomorphic $(2,0)$-form on every desingularization of $\overline{M}_{0,0}(X,e)$, and the form is nondegenerate if e is odd and ≥ 5, resp. has 1-dimensional kernel if e is even and ≥ 6. In particular, $\overline{M}_{0,0}(X,e)$ is non-uniruled, resp. is either non-uniruled or a conic bundle over a non-uniruled variety.*

The $(2,0)$-form discovered by de Jong and me has recently been studied further by Lehn-Lehn-Sorger-van Straten who prove that, for $e=3$, there is a smooth birational model of the rational quotient of $\overline{M}_{0,0}(X,3)$ on which the $(2,0)$-form is everywhere nondegenerate, i.e., the model is a hyperkähler sixfold, [38]. By combining the de Jong-Starr method with a quite different method of Roya Beheshti, Beheshti and I proved the following *strong* non-uniruledness result for spaces of curves of small degree.

Theorem 3.3 (Beheshti-Starr, [2])**.** *For every $n \geq 5$, for every smooth, degree n hypersurface $X \subset \mathbb{P}^n$, every sufficiently general point of X is contained in no rational surface that is fibered by rational curves that are (generically) smooth and $(n-1)$-normal.*

If in this theorem we could remove the hypothesis that the generic rational curve is smooth and $(n-1)$-normal, then we would prove that these Fano hypersurfaces are not unirational, thus settling the open problem of unirational of Fano manifolds.

In the positive direction, when $\mathrm{ch}_2(T_X)$ is non-negative, then the Kontsevich spaces often are rationally connected. The first such result was proved by Harris and me, [28, Harris-Starr], and the optimal result, *weak rational simple connectedness*, was proved by de Jong and me. This is relevant thanks to a *weak approximation* theorem of Hassett (generalized by me from the *strong* to the *weak* form of rational simple connectedness), [29]: for every fibration over a curve, as in the Graber-Harris-Starr theorem, if one fiber is weakly rationally simply connected, then every finite collection of formal sections of the fibration (at disjoint points of the base curve) can be approximated to arbitrary (contact) order by rational sections, i.e., *weak approximation* holds. The following theorem was incorporated into the PhD thesis of one of de Jong's advisee Matt DeLand, [16], who proved an even stronger result that X is strongly rationally simply connected whenever $\mathrm{ch}_2(T_X)$ is strictly positive.

Theorem 3.4 (de Jong-Starr, [14])**.** *For every smooth complete intersection $X \subset \mathbb{P}^n$, if $ch_2(T_X)$ is non-negative (excluding quadric surfaces), then for every $e \gg 0$, the Kontsevich space $\overline{M}_{0,2}(X,e)$ is rationally connected, and even the geometric generic fiber of the evaluation morphism to $X \times X$ is rationally connected (weak rational simple connectedness). In particular, these varieties satisfy weak approximation.*

An analogous theorem has been proved for complete intersections in Grassmannian varieties in the PhD thesis of my advisee, Robert Findley, [20]. His theorem shattered the naive guesses for complete intersections in projective homogeneous varieties.

Question 3.5 (Starr)**.** For a projective homogeneous space G/P of Picard number 1 such as an orthogonal or Lagrangian Grassmannian, which smooth complete intersections X in G/P are rationally simply connected?

Although smooth complete intersections in $\mathbb{P}^n$, in Grassmannians, and in projective homogeneous spaces are special among all Fano manifold, de Jong and I do have a partial result in the general case.

Theorem 3.6 (de Jong-Starr, [15])**.** *Let X be a complex Fano manifold with $ch_2(T_X)$ nef and with pseudo-index ≥ 3. For every curve class β such that $\overline{M}_{0,0}(X,\beta)$ is irreducible and parametrizes at least one free rational curve, $\overline{M}_{0,0}(X,\beta)$ is uniruled.*

This is proved by an analysis for the Kontsevich spaces of the virtual canonical bundles associated to the Behrend-Fantechi perfect obstruction theory, together with a "virtual" extension of Mori's fundamental Bend-and-Break approach to proving uniruledness. A key role is played by a special birational contraction of the Kontsevich space discovered by Coskun, Harris and myself as part of our general investigation of the ample and effective cones of the Kontsevich spaces with target $\mathbb{P}^n$.

Theorem 3.7 (Coskun-Harris-Starr, [8], [7])**.** *For every $n \geq 0$, for every $m \geq 0$, and for every $e \geq 0$, there is an explicit description of the nef cone, resp. basepointfree cone, ample cone, of the Kontsevich space of $\overline{M}_{0,m}(\mathbb{P}^n,e)$ in terms of the same cone for the moduli space of $\overline{M}_{0,m+e}/\mathfrak{S}_e$ of genus 0 curves with m labeled points and e unlabeled points. For every integer $e = 1,\dots,n$, there is an explicit simplicial description of the effective cone of $\overline{M}_{0,0}(\mathbb{P}^n,e)$, and this equals the pseudo-effective cone.*

This has been generalized to Grassmannians by Coskun and me, [9, Coskun-Starr]. The Segal Conjecture, sharpened by Cohen-Jones-Segal, [5], is an overarching conjecture in this area predicting the topology of moduli spaces of rational curves on Fano manifolds. Although it is wide open, Zhiyu Tian and I were recently able to establish the "Picard group" version for Fano manifolds of small degree. Our proof combines the inductive technique of the Harris-Roth-Starr theorem, an analysis I carried out for singularities of low degree Kontsevich spaces, and the technique from the Greer-Li-Tian theorem on Picard groups of K3 surfaces (with the Coskun-Harris-Starr birational contraction above playing the role of the "GIT moduli space" from Greer-Li-Tian).

Theorem 3.8 (Starr – Zhiyu Tian, in preparation)**.** *For $d \leq n-(1/2)-\sqrt{n-(33/4)}$, for $X \subset \mathbb{CP}^n$ a general degree d hypersurface, for every $e \geq 1$, $\overline{M}_{0,0}(X,e)$ is algebraically simply connected, and the pullback map $Pic(\overline{M}_{0,0}(\mathbb{CP}^n,e))_{\mathbb{Q}} \to Pic(\overline{M}_{0,0}(X,e))_{\mathbb{Q}}$ is a bijection. This bijection preserves the nef cone and the basepointfree cone.*

4. Properties of Algebraic Stacks

In addition to Geometric Invariant Theory, the other modern approach to moduli spaces is the theory of *algebraic stacks* developed by Deligne-Mumford and Michael Artin. Roughly, algebraic stacks are the objects obtained by gluing together quotients of schemes by algebraic group actions that are not necessarily free actions. Although many of the basic results for schemes were proved for algebraic stacks, cf. [37, Laumon – Moret-Bailly] and [44, The Stacks Project], many questions remain open. Olsson and I settled one of these problems in our work on Hilbert schemes and Quot schemes of stacks.

Theorem 4.1 (Olsson - Starr, [40]). *For every proper Deligne-Mumford stack over a field, the Hilbert functor and the Quot functor are represented by algebraic spaces that are separated and locally finitely presented over the field. If the stack is a global quotient stack that has a projective coarse moduli space, then the Hilbert and Quot functors (with specified Hilbert polynomial) are projective schemes over the field.*

More important than the theorem were the techniques we developed: a generalization of devissage to stacks that leverages the Chow lemma of Laumon – Moret-Bailly, the notion of "generating sheaves" as a tool to understand the Abelian category of coherent sheaves on a global quotient stack, and a new notion of a "two-step obstruction theory". In a subsequent preprint [45, Starr], I investigated further the idea of a "two-step obstruction theory", proving that Artin's axioms for an algebraic stack are compatible with compositions (roughly, the functorial formulation of a multi-step obstruction theory). All of these notions were vastly generalized in separate works of Olsson [41], who extended these results to Artin stacks with finite diagonal, and Hall - Rydh, [25], who proved the ultimate generalization. Nonetheless, there are elementary open questions even for the simplest case of a smooth, two-dimensional Deligne-Mumford stack X over $\mathbb{C}$ that is a global quotient stack, that has quasi-projective coarse moduli space, and that is stacky at only finitely many points.

Question 4.2 (Li Li – Starr). For a stack X as above, what are the Betti numbers, resp. Hodge numbers, classes in the Grothendieck group, of the associated smooth Hilbert schemes $\mathrm{Hilb}^{P(t)}_{X/\mathbb{C}}$ parameterizing 0-dimensional closed substacks with specified Hilbert polynomial $P(t)$?

Many special cases follow from work of Gusein-Zade, Luengo and Melle-Hernandez, but the general case is open.

References

[1] J. Ax and S. Kochen. Diophantine problems over local fields. I. *Amer. J. Math.*, 87:605–630, 1965.
[2] R. Beheshti and J. M. Starr. Rational surfaces in index-one Fano hypersurfaces. *J. Algebraic Geom.*, 17(2):255–274, 2008.
[3] K. Behrend and B. Fantechi. The intrinsic normal cone. *Invent. Math.*, 128(1):45–88, 1997.
[4] P. Berthelot, H. Esnault, and K. Rülling. Rational points over finite fields for regular models of algebraic varieties of Hodge type ≥ 1. *Ann. of Math. (2)*, 176(1):413–508, 2012.
[5] R. L. Cohen, J. D. S. Jones, and G. B. Segal. Stability for holomorphic spheres and Morse theory. In *Geometry and topology: Aarhus (1998)*, volume 258 of *Contemp. Math.*, pages 87–106. Amer. Math. Soc., Providence, RI, 2000.
[6] P. Colmez and J.-P. Serre, editors. *Correspondance Grothendieck-Serre*. Documents Mathématiques (Paris) [Mathematical Documents (Paris)], 2. Société Mathématique de France, Paris, 2001.
[7] I. Coskun, J. Harris, and J. Starr. The effective cone of the Kontsevich moduli space. *Canad. Math. Bull.*, 51(4):519–534, 2008.
[8] I. Coskun, J. Harris, and J. Starr. The ample cone of the Kontsevich moduli space. *Canad. J. Math.*, 61(1):109–123, 2009.
[9] I. Coskun and J. Starr. Divisors on the space of maps to Grassmannians. *Int. Math. Res. Not.*, pages Art. ID 35273, 25, 2006.
[10] I. Coskun and J. Starr. Rational curves on smooth cubic hypersurfaces. *Int. Math. Res. Not. IMRN*, (24):4626–4641, 2009.
[11] A. J. de Jong, X. He, and J. Starr. Families of rationally simply connected varieties over surfaces and torsors for semisimple groups. *Publ. Math. Inst. Hautes Études Sci.*, 114(1):1–85, 2008.
[12] A. J. de Jong and J. Starr. Every rationally connected variety over the function field of a curve has a rational point. *Amer. J. Math.*, 125(3):567–580, 2003.
[13] A. J. de Jong and J. Starr. Cubic fourfolds and spaces of rational curves. *Illinois J. Math.*, 48(2):415–450, 2004.
[14] A. J. de Jong and J. Starr. Low degree complete intersections are rationally simply connected. 2006.
[15] A. J. de Jong and J. Starr. Higher Fano manifolds and rational surfaces. *Duke Math. J.*, 139(1):173–183, 2007.
[16] M. DeLand. Relatively very free curves and rational simple connectedness. *J. Reine Angew. Math.*, 699:1–33, 2015.

[17] H. Esnault. Varieties over a finite field with trivial Chow group of 0-cycles have a rational point. *Invent. Math.*, 151(1):187–191, 2003.

[18] H. Esnault. Deligne's integrality theorem in unequal characteristic and rational points over finite fields. *Ann. of Math. (2)*, 164(2):715–730, 2006. With an appendix by Pierre Deligne and Esnault.

[19] H. Esnault and C. Xu. Congruence for rational points over finite fields and coniveau over local fields. *Trans. Amer. Math. Soc.*, 361(5):2679–2688, 2009.

[20] R. A. Findley. *Rational curves in low degree hypersurfaces of Grassmannian varieties*. ProQuest LLC, Ann Arbor, MI, 2010. Thesis (Ph.D.)–State University of New York at Stony Brook.

[21] T. Graber, J. Harris, B. Mazur, and J. Starr. Rational connectivity and sections of families over curves. *Ann. Sci. École Norm. Sup. (4)*, 38:671–692, 2005.

[22] T. Graber, J. Harris, and J. Starr. Families of rationally connected varieties. *J. Amer. Math. Soc.*, 16(1):57–67 (electronic), 2003.

[23] T. Graber and J. M. Starr. Restriction of sections for families of abelian varieties. In *A celebration of algebraic geometry*, volume 18 of *Clay Math. Proc.*, pages 311–327. Amer. Math. Soc., Providence, RI, 2013.

[24] C. D. Hacon and J. Mckernan. On Shokurov's rational connectedness conjecture. *Duke Math. J.*, 138(1):119–136, 2007.

[25] J. Hall and D. Rydh. General Hilbert stacks and Quot schemes. *Michigan Math. J.*, 64(2):335–347, 2015.

[26] G. Harder. Über die Galoiskohomologie halbeinfacher algebraischer Gruppen. III. *J. Reine Angew. Math.*, 274/275:125–138, 1975. Collection of articles dedicated to Helmut Hasse on his seventy-fifth birthday, III.

[27] J. Harris, M. Roth, and J. Starr. Rational curves on hypersurfaces of low degree. *J. Reine Angew. Math.*, 571:73–106, 2004.

[28] J. Harris and J. Starr. Rational curves on hypersurfaces of low degree. II. *Compos. Math.*, 141(1):35–92, 2005.

[29] B. Hassett. Weak approximation and rationally connected varieties over function fields of curves. In *Variétés rationnellement connexes: aspects géométriques et arithmétiques*, volume 31 of *Panor. Synthèses*, pages 115–153. Soc. Math. France, Paris, 2010.

[30] B. Hassett and Y. Tschinkel. Weak approximation over function fields. *Invent. Math.*, 163(1):171–190, 2006.

[31] A. Hogadi and C. Xu. Degenerations of rationally connected varieties. *Trans. Amer. Math. Soc.*, 361(7):3931–3949, 2009.

[32] S. Kebekus, L. Solá Conde, and M. Toma. Rationally connected foliations after Bogomolov and McQuillan. *J. Algebraic Geom.*, 16(1):65–81, 2007.

[33] J. Kollár. A conjecture of Ax and degenerations of Fano varieties. *Israel J. Math.*, 162:235–251, 2007.

[34] M. Kontsevich. Enumeration of rational curves via torus actions. In *The moduli space of curves (Texel Island, 1994)*, volume 129 of *Progr. Math.*, pages 335–368. Birkhäuser Boston, Boston, MA, 1995.

[35] G. Lafon. Une surface d'Enriques sans point sur $\mathbb{C}((t))$. *C. R. Math. Acad. Sci. Paris*, 338(1):51–54, 2004.

[36] S. Lang. On quasi algebraic closure. *Ann. of Math. (2)*, 55:373–390, 1952.

[37] G. Laumon and L. Moret-Bailly. *Champs algébriques*, volume 39 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2000.

[38] C. Lehn, M. Lehn, C. Sorger, and D. van Straten.

[39] B. Mazur. Frobenius and the Hodge filtration. *Bull. Amer. Math. Soc.*, 78:653–667, 1972.

[40] M. Olsson and J. Starr. Quot functors for Deligne-Mumford stacks. *Comm. Algebra*, 31(8):4069–4096, 2003. Special issue in honor of Steven L. Kleiman.

[41] M. C. Olsson. Hom-stacks and restriction of scalars. *Duke Math. J.*, 134(1):139–164, 2006.

[42] E. Riedl and D. Yang. Kontsevich spaces of rational curves on fano hypersurfaces. 2014.

[43] J.-P. Serre. *Galois cohomology*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, english edition, 2002. Translated from the French by Patrick Ion and revised by the author.

[44] T. Stacks Project Authors. *stacks project. http://stacks.math.columbia.edu, 2015.*

[45] J. Starr. Artin's axioms, composition and moduli spaces. preprint, 2006.

[46] J. Starr. Degenerations of rationally connected varieties and PAC fields. In A celebration of algebraic geometry*, volume 18 of* Clay Math. Proc.*, pages 577–589. Amer. Math. Soc., Providence, RI, 2013.*

[47] J. M. Starr. A pencil of Enriques surfaces of index one with no section. Algebra Number Theory*, 3(6):637–652, 2009.*

[48] G. Terjanian. Un contre-exemple à une conjecture d'Artin. C. R. Acad. Sci. Paris Sér. A-B*, 262:A612, 1966.*

[49] C. Tsen. Quasi-algebraische-abgeschlossene Funktionenkörper. J. Chinese Math.*, 1:81–92, 1936.*

[50] Q. Zhang. Rational connectedness of log $\mathbf{Q}$*-Fano varieties.* J. Reine Angew. Math.*, 590:131–142, 2006.*