MAT 615 PROBLEM SET 4

Homework Policy. This problem set fills in the details of the Stable Reduction Theorem from lecture, used to complete the proof of the Irreducibility Theorem of Deligne and Mumford for geometric fibers of $\overline{\mathcal{M}}_{g,n} \to \text{Spec } \mathbb{Z}$.

Problems.

Problem 0.(Variant of the valuative criterion of properness.) Let B be a Noetherian scheme that is integral, i.e., irreducible and reduced. Let X be a projective B-scheme. Let Z be a nowhere dense closed subset of X (possibly empty). Let U denote the dense open complement of Z. Let V be finite type and separated over B, and let $f : U \to V$ be a proper, surjective morphism. Let $V^o \subset V$ be an open subset that is dense in every B-fiber.

(a) A B-scheme is *generically geometrically connected* if every irreducible component of the B-scheme dominates B and if the fiber of the B-scheme over Spec $\overline{\text{Frac}(B)}$ is connected. Show that V^o is generically geometrically connected over B if and only if V is generically geometrically connected over B. If U is generically geometrically connected over B, prove that V is generically geometrically connected over B, and the converse holds provided that U is generically geometrically connected over V. Finally, prove that U is generically geometrically connected over B if and only if X is generically geometrically connected over B.

(b) If X is a projective B-scheme that is generically geometrically connected over B, use Zariski's Connectedness Theorem to prove that **every** geometric fiber is connected (cf. Section III.12 of Hartshorne's *Algebraic geometry* and exercises there).

(c) For every algebraically closed field k, for every connected, projective k-scheme X_k, for every closed subset Z_k of X_k that is nonempty and proper, prove that there exists at least one irreducible component Y_k of X_k such that $Z_k \cap Y_k$ is a nonempty, proper closed subset of Y_k.

(d) Continuing the notation from the previous part, denote by U_k the open complement of Z_k in X_k. Let V_k be a finite type, separated k-scheme, and let $f_k : U_k \to V_k$ be a proper morphism. Prove that every irreducible component of V_k that intersects $f_k(Y_k \cap U_k)$ is not proper. Conversely, for every irreducible component of V_k that is not proper, for every irreducible component of U_k that surjects to that component (assuming that there is any), prove that irreducible component of U_k is not proper.

(e) Continuing the previous part, also assume that f_k is surjective. Prove that there exists at least one non-proper irreducible component of U_k that dominates an irreducible component of V_k. For every dense open V_k^o of V_k, conclude that there exists at least one non-proper irreducible component of U_k such that $f^{-1}(V_k^o)$ is dense in that irreducible component.

(f) For every algebraically closed field k, for every irreducible projective k-scheme Y, for every nonempty, proper closed subset Z_Y of Y, for every dense open subset

Y^o of $U_Y := Y \setminus Z_Y$, prove that there exists an irreducible, affine k-scheme D that is smooth of dimension 1, a dense open subscheme $D^o \subset D$ with complement $Z_D := D \setminus D^o$, and a k-morphism $h : D \to Y$ such that $h(D^o)$ is contained in Y^o and $h(Z_D) \subset Z_Y$. Denote by $h^o : D^o \to U_Y$ the restriction of h. Conclude that for every surjective, flat morphism $\nu : \widetilde{D} \to D$, denoting $\widetilde{D}^o := \nu^{-1}(D^o)$, the k-morphism $h^o \circ \nu : \widetilde{D}^o \to U_Y$ has no extension to a k-morphism $\widetilde{D} \to U_Y$ (since the unique extension to Y has image intersecting Z_Y).

(g) In the original setting of the problem, assume that U is generically geometrically connected over B. Conclude that V is proper over B if and only if the following variant of the "valuative criterion" holds. point Spec $k \to B$, for the fiber $V \times_B$ Spec k, for every irreducible, affine, smooth k-curve D, for every dense open affine D^o of D, for every k-morphism $h^o : D^o \to V^o \times_B \text{Spec } k$, there exists a finite, flat, surjective morphism $\nu : \widetilde{D} \to D$ and a k-morphism $\widetilde{h} : \widetilde{D} \to V \times_B \text{Spec } k$ whose restriction to $\widetilde{D}^o := \nu^{-1}(D^o)$ equals $h^o \circ \nu$.

Problem 1.(Separatedness of $\overline{\mathcal{M}}_{g,n} \to \text{Spec } \mathbb{Z}$.) Let B be an irreducible, affine, regular, Noetherian scheme (finite type over Spec $\mathbb{Z}$ if you like), and let B^o be a dense open subscheme. For $j = 1, 2$, let

$$\zeta_j = (\pi_j : C_j \to B, (\sigma_{i,j} : B \to C_j)_{i=1,\dots,n}),$$

be an ordered pair of a proper, flat morphism π_j of relative dimension 1, an ordered n-tuple of pairwise disjoint sections $\sigma_{i,j}$ of π_j with image in the smooth locus of π_j such that every geometric fiber is a connected, reduced, at-worst-nodal curve of arithmetic genus g. Further, assume π_j-ampleness of the log relative dualizing sheaf

$$\omega_{\pi_j,\sigma} := \omega_{\pi_j}\left(\sum_{i=1}^{n} \underline{\sigma_{i,j}(B)}\right).$$

Let $\phi : C_1 \times_B B^o \to C_2 \times_B B^o$ be a B^o-isomorphism such that every $\phi \circ \sigma_{i,1}$ equals $\sigma_{i,2}$ as morphisms from B^o to C_2. Also, assume that π_j is smooth over B^o.

(a) For every semistable family over B,

$$\zeta = (\pi : C \to B, (\sigma_i : B \to C)_{i=1,\dots,n}),$$

and for $\alpha : C \to C_1$ and $\beta : C \to C_2$ proper, surjective B-morphisms that restrict over B^o to isomorphisms such that $\phi \circ \alpha$ equals β, prove that every connected component of the α-exceptional locus, resp. of the β-exceptional locus, is either a rational chain terminating in a "leaf" or a chain of rational "bridge", cf. **Problem 4** below.

(b) Conclude that for each corresponding fiber, for every integer $d \geq 1$, every global section of $\omega_\pi^{\otimes d}$ on the component is identically zero on the exceptional rational chain. Thus, for every integer $d \geq 1$, prove that the $\mathcal{O}_B$-modules of global sections of $\omega_{\pi_j,\sigma}^{\otimes d}$ for $j = 1, 2$ are both equal as $\mathcal{O}_B$-submodules of the $\mathcal{O}_B$-module of all rational sections of $\omega_{\pi,\sigma}^{\otimes d}$ via pullback. Show that these identifications are compatible with product of sections, and thus give an isomorphism of section algebras of $\omega_{\pi_j,\sigma}$ for $j = 1, 2$. Taking relative Proj, conclude that ϕ extends uniquely to an isomorphism over all of B.

(c) Read ahead in the next two problems about existence of a semistable degeneration. Apply this beginning with $\overline{C}$ equal to the Zariski closure in $C_1 \times_B C_2$ of the graph of ϕ to conclude that there does exist a semistable family ζ and morphisms α and β as above. Combined with the reductions in the previous exercise (applied to the diagonal morphism), this implies separatedness of the stack of stable families of genus-g, n-pointed, at-worst-nodal curves.

Problem 2.(Reduction of properness of $\overline{\mathcal{M}}_{g,n}$ to the Stable Reduction Theorem.) In Sections 3 and 4 of Chapter 7 of *Geometric Invariant Theory*, David Mumford first constructed for every integer $g \geq 3$ the coarse moduli space M_g as a quasi-projective scheme over Spec $\mathbb{Z}$ (the moduli spaces of hyperelliptic curves are special cases of the Hurwitz schemes discussed in lecture). Moreover, using infinitesimal deformation theory, there exists an open subscheme M_g^o of M_g that is smooth over Spec $\mathbb{Z}$, that is dense in every geometric fiber of $M_g \to \text{Spec }\mathbb{Z}$, and that precisely parameterizes the families of smooth, projective curves of genus g that have only the identity automorphism.

(a) Please read the statement of "Chow's Lemma for Deligne-Mumford Stacks", cf. Corollaire 16.6.1, p. 154 of *Champs algébriques* by Laumon and Moret-Bailly, or some other source. Apply this to the stack $\overline{\mathcal{M}}_g \to \text{Spec }\mathbb{Z}$. This stack is algebraic, even Deligne-Mumford, and smooth over Spec $\mathbb{Z}$ by Artin's axioms and infinitesimal deformation theory. It is quasi-compact by the explicit computation that $\omega_\pi^{\otimes 3}$ is π-relatively very ample (this can be checked fiber-by-fiber, using induction on the number of irreducible components of the fiber). Using the previous exercise, this quasi-compact, smooth, Deligne-Mumford stack is also separated over Spec $\mathbb{Z}$. Thus, conclude that, after base change by a finite, flat, surjective morphism of Dedekind schemes $B \to \text{Spec }\mathbb{Z}$, there exists a representable, proper, surjective, generically étale 1-morphism $f : V \to \overline{\mathcal{M}}_{g,B}$ and a dense, open immersion $V \hookrightarrow X$, where X is a projective B-scheme. Since $\overline{\mathcal{M}}_g$ is generically geometrically connected over Spec $\mathbb{Z}$, conclude that for an appropriate choice of $B \to \text{Spec }\mathbb{Z}$, also X is generically geometrically connected over B.

(b) Again by infinitesimal deformation theory, the open subscheme M_g^o is dense in every fiber of $\overline{\mathcal{M}}_g$ (this is essentially the statement that the boundary is a divisor in every fiber – even a simple normal crossings divisor). Conclude that the hypotheses of **Problem 0** hold.

(c) Use **Problem 0** to conclude that $\overline{\mathcal{M}}_g$ is proper over Spec $\mathbb{Z}$ if and only if, for every geometric point Spec $k \to \text{Spec }\mathbb{Z}$ and for every k-morphism $h^o : D^o \to W \times_{\text{Spec }\mathbb{Z}} \text{Spec } k$ from a dense open D^o of an irreducible, affine, smooth k-curve D, there exists a finite, flat, surjective morphism $\nu : \widetilde{D} \to D$ and an extension $\widetilde{h} : \widetilde{D} \to \overline{\mathcal{M}}_g$ whose restriction to $\widetilde{D}^o$ equals $h^o \circ \nu$.

Problem 3.(Semistable Reduction for Families of Smooth Curves) Let g, n be non-negative integers. Let k be an algebraically closed field. Let D be an irreducible, smooth k-curve. Let $D^o \subset D$ be a dense open affine. Let

$$\zeta^o = (\pi^o : C^o \to D^o, (\sigma_i^o : D^o \to C^o)_{i=1,\dots,n}),$$

be an ordered pair of a proper, smooth morphism of schemes, π^o, and an ordered n-tuple of pairwise disjoint sections σ_i^o of π^o such that every geometric fiber is a connected, smooth curve of genus g with n distinct marked points.

(a) Work through Exercise III.5.8, p. 232 of Hartshorne's *Algebraic geometry*, or some equivalent exercise that proves that the generic fiber of π^o has an ample effective divisor. Since C^o is regular, prove that the Zariski closure in C^o of this closed subset of the generic fiber of π^o is an effective Cartier divisor. Use the exercise to prove that this effective Cartier divisor is π^o-ample. Combined with Proposition II.7.5, p. 154, conclude that there exists a π^o-very ample invertible sheaf on C^o. Thus, there exists a locally closed immersion of D-schemes, $C^o \hookrightarrow \mathbb{P}^N_D$, for some integer N.

(b) After taking Zariski closures and normalizing (using the Noether normalization theorem to preserve projectivity), conclude that there exists a factorization of π^o,

$$C^o \hookrightarrow \overline{C} \xrightarrow{\overline{\pi}} D,$$

where $\overline{C}$ is a normal, 2-dimensional k-scheme, and where $\overline{\pi}$ is projective and flat.

(c) Read about the "Albanese method" for reducing resolution of surface singularities to the case of double points as mentioned in lecture. One excellent resource is Sections 2.5 and 2.6 of Kollár's *Lectures on resolution of singularities*. Thus, up to a birational modification, assume that $\overline{C}$ is a smooth k-scheme.

(d) Now consider the divisor in $\overline{C}$ that is the union of the scheme-theoretic fibers of $\overline{\pi}$ over the points of $D \setminus D^o$ together with the closures of the images $\sigma_i^o(D^o)$. Read about embedded resolution of curves in a surface. Conclude that after further blowing up of $\overline{C}$ over $D \setminus D^o$, the sections σ_i^o extend to pairwise disjoint sections that are contained in the smooth locus of $\overline{\pi}$, and every fiber of $\overline{\pi}$ with its reduced structure is a normal crossings divisor in $\overline{C}$.

(e) Read about ramification in codimension 1 and Krasner's Lemma. Conclude that for every sufficiently positive and divisible integer e, for every finite flat morphism $\nu : \widetilde{D} \to D$ that is totally ramified over every point of $D \setminus D^o$ with multiplicity e, for the normalization $(\overline{C} \times_D \widetilde{D})^{\text{nor}}$, every fiber over $\widetilde{D}$ is generically smooth.

(f) At every double point of the reduced structure on a fiber of $\overline{\pi}$, prove that there is a system of parameters r, s and positive integers a, b such that $r^a s^b$ equals a unit times the pullback of t^c for a uniformizing parameter t on D and a positive integer c. Read about Hirzebruch-Jung desingularization and resolution of cyclic quotient singularities, e.g., in Sections 2.3 and 2.4 of Kollár's *Lectures on resolution of singularities*. Conclude that if e is sufficiently divisible, then the normalization $(\overline{C} \times_D \widetilde{D})^{\text{nor}}$ has a double point singularity at the points lying over this double point, and the fiber of the corresponding point of $\widetilde{D}$ has a double point (i.e., ordinary node curve singularity).

(g) Putting the pieces together, conclude that after passing to a finite, flat, surjective cover $\nu : \widetilde{D} \to D$ that is normal (hence smooth, since $\widetilde{D}$ is a curve over an algebraically closed field), there exists an extension of ζ over $\widetilde{D}$ where the geometric generic fibers are projective, connected, at-worst-nodal curves with n distinct points contained in the smooth locus. This is a *semistable reduction* of ζ.

Problem 4.(Stabilization) Continuing the problem above, up to replacing D by $\widetilde{D}$, let D be an irreducible, smooth, 1-dimension scheme over an algebraically closed field, and let

$$\zeta^{\text{pre}} = (\pi^{\text{pre}} : C^{\text{pre}} \to D, (\sigma_i^{\text{pre}} : D \to C^{\text{pre}})_{i=1,\dots,n}),$$

be an ordered pair of a proper, flat morphism π^{pre} of pure relative dimension 1 and an ordered n-tuple of pairwise disjoint sections of π^{pre} with image in the smooth locus of π^{pre} such that every geometric fiber is a connected, reduced, at-worst-nodal curve of arithmetic genus g. Assume, further, that there exists a dense open subscheme D^o of D such that π is smooth over D^o. The only remaining issue is that there may be non-stable geometric fibers of π^{pre}, necessarily over points of $D \setminus D^o$. These are fibers that have an irreducible component that is smooth of genus 0 and contains < 3 "special points", i.e., points that are either nodes of the fiber or marked points. For such a component, the degree of the relative log dualizing sheaf is non-positive.

(a) Let C_k be a fiber of π^{pre} over a closed point, and let $C_{i,k} \subset C_k$ be a genus 0 "leaf", i.e., an irreducible component of C that is smooth of genus 0 and that contains only one node of C. If C^{pre} is smooth at the node of $C_{i,k}$, conclude that $C_{i,k}$ is a Cartier divisor in C^{pre} with self-intersection -1. Thus, for every π^{pre}-very ample divisor A on C^{pre}, conclude that

$$A' := A + (A.C_{i,k})_{C^{\text{pre}}} C_{i,k},$$

is also a Cartier divisor, and the intersection number of A' with $C_{i,k}$ equals 0. Use the method of Castelnuovo's Contraction Theorem, Theorem V.5.7, p. 414 of Hartshorne's *Algebraic geometry*, to prove that this divisor class is π^{pre}-globally generated and defines a contraction to another projective, flat D-scheme that contracts $C_{i,k}$ to a **smooth point** of the target. Conclude that there exists a model of ζ^{pre} that equals ζ over D^o yet has one fewer leaf on the fiber C_k. Prove that the sections of ζ^{pre} that intersect $C_{i,k}$ are precisely the sections of the new family that contain the smooth point. Thus, if $C_{i,k}$ is unstable, i.e., it intersects at most one section, then the new family is still semistable.

(b) In the argument above, if $C_{i,k}$ has two nodes of C_k, called a "bridge", conclude that $C_{i,k}$ has self-intersection -2. Conclude that $A' = 2A + (A \cdot C_{i,k})C_{i,k}$ has intersection number 0 with $C_{i,k}$, is π^{pre}-globally generated, and the corresponding contraction contracts $C_{i,k}$ to an ordinary double point singularity. Assuming there were no sections of ζ^{pre}, prove that the new family is still semistable.

(c) Under the hypothesis that C^{pre} is smooth, show that first there is an iterative sequence of contraction of rational leaves that gives a modified family where every fiber of π^{pre} has no rational leaves. If also every chain of rational bridges has length 1, prove that there is also a further modification so that there are no rational leaves or rational bridges, i.e., the family is stable. This completes the stable reduction in this case.

(d) Infinitesimal deformation theory implies smoothness of the stack of at-worst-nodal curves and it implies that the boundary is a simple normal crossings divisor. Thus, after replacing D by a further finite, flat base change by a smooth curve, the family ζ^{pre} over D is the pullback by a morphism $D \to T$ of a family over a smooth k-scheme T,

$$\zeta_T = (\pi_T : C_T \to T, (\sigma_{i,T} : T \to C_T)_{i=1,\dots,n}),$$

that is everywhere versal and where T^o is a dense open subscheme whose complement is a simple normal crossings divisor. For a specified leaf of a fiber of π^{pre}, we can also assume that there exists an irreducible component Δ_i of the simple normal crossings divisor such that C_k is the pullback of a fiber over Δ_i and such that there exists a rational leaf $C_{T,i}$ for the restriction of π_T over Δ_i. Repeat the argument from (a) to construct a π_T-globally generated divisor class that precisely contracts $C_{T,i}$, resulting in a new semistable family over T. By taking the pullback by $D \to T$, conclude that there exists a contraction of C^{pre} that contracts rational leaves including $C_{i,k}$, resulting in a new semistable family. Working étale locally, prove that there exists a single finite, flat, surjective base change of D after which every rational leaf is contractible. Repeating this argument finitely often (equal to the longest length of a genus-0 chain of components ending in a leaf), prove that there exists a finite, flat, surjective base change of D and a contraction of the pullback of ζ^{pre} so that there are no rational leaves of the new family.

(e) Repeat the argument above for rational bridges to conclude that after a further finite, flat, surjective base change of D, there is also a contraction of the pullback semistable family so that the new family also has no rational bridges. Thus, there exists a finite, flat, surjective morphism $\nu : \widetilde{D} \to D$ with $\widetilde{D}$ a smooth curve and there exists a contraction of $C^{\text{pre}} \times_D \widetilde{D}$ to a semistable family over $\widetilde{D}$,

$$\widetilde{\zeta} : (\widetilde{\pi} : \widetilde{C} \to \widetilde{D}, (\widetilde{\sigma}_i : \widetilde{D} \to \widetilde{C})_{i=1,\dots,n}),$$

that equals the pullback of ζ^{pre} over $\widetilde{D}^o$ and such that $\widetilde{\zeta}$ is stable. This is a *stable reduction* of ζ^{pre}.

Problem 5.(A first example.) Let k be a field of characteristic different from 2 or 3. Let D be $\mathbb{A}^1_k = \text{Spec } k[t]$, let $\mathbb{P}^2_k$ be Proj $k[q, r, s]$, and let $\overline{C}$ be the closed subscheme of $\mathbb{A}^1_k \times_{\text{Spec } k} \mathbb{P}^2_k$ that is the zero scheme of $qs^2 - r^3 - tq^2 s$. Let $\sigma : D \to \overline{C}$ be the section whose image is the zero scheme of q and r. Explicit compute a semistable reduction, and then compute a stable reduction as a family of genus-1, 1-pointed curves. What is the j-invariant of the stable reduction for the singular fiber over $t = 0$? For more practice with semistable reduction and stable reduction, please read Section C of Chapter 3 of Harris-Morrison, *Moduli of curves*.

Problem 6(Properness and the general Stable Reduction Theorem.) Finally, combine the previous exercises to conclude that the quasi-compact, smooth, separated Deligne-Mumford stack $\overline{\mathcal{M}}_g \to \text{Spec } \mathbb{Z}$ is also proper. Combined with relative representability, and even strong projectivity, of each forgetful morphism $\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$, conclude that for every integer $g \geq 2$ and every integer $n \geq 0$, the smooth Deligne-Mumford stack $\overline{\mathcal{M}}_{g,n} \to \text{Spec } \mathbb{Z}$ is proper. Use this to formulate and prove the general form of the Stable Reduction Theorem, without any hypotheses on the general fiber being smooth, etc.

Finally, also prove the result for $\overline{\mathcal{M}}_{0,n}$, $n \geq 3$, and for $\overline{\mathcal{M}}_{1,n}$, $n \geq 1$, by the "trick" of transforming an n-pointed curve of genus g into a curve of genus $g+n$ by attaching a nodal plane cubic (irreducible, nodal curve of arithmetic genus 1 and geometric genus 0) at each of the n-marked points. Prove that induced 1-morphism from $\overline{\mathcal{M}}_{g,n}$ to $\overline{\mathcal{M}}_{g+n}$ is finite. Thus, properness of $\overline{\mathcal{M}}_{g+n}$ implies properness of $\overline{\mathcal{M}}_{g,n}$.