MAT 615 PROBLEM SET 2

Homework Policy. This problem set explores various constructions of parameter spaces in algeraic geometry.

Problems.

Problem 0.(**Abelian cones**) You can read more about Abelian cones in the appendix of Fulton's *Intersection theory*. Let $(S, \mathcal{O}_S)$ be a scheme. Let $\mathcal{E}$ be a quasi-coherent $\mathcal{O}_S$-module. You may assume that $\mathcal{E}$ is locally finitely presented. For every S-scheme

$$(f, f^{\#}) : (T, \mathcal{O}_T) \to (S, \mathcal{O}_S),$$

define a **functional** for $\mathcal{E}$ on T to be an $\mathcal{O}_T$-module homomorphism,

$$\chi : f^*\mathcal{E} \to \mathcal{O}_T.$$

For every S-scheme,

$$(\widetilde{f}, \widetilde{f}^{\#}) : (\widetilde{T}, \mathcal{O}_{\widetilde{T}}) \to (S, \mathcal{O}_S),$$

and for every morphism of S-schemes,

$$(g, g^{\#}) : (\widetilde{t}, \mathcal{O}_{\widetilde{T}}) \to (T, \mathcal{O}_T),$$

define the **pullback** of χ by $(g, g^{\#})$ to be

$$g^*\chi : \widetilde{f}^*\mathcal{E} \to \mathcal{O}_{\widetilde{T}},$$

where we identify $\widetilde{f}^*\mathcal{E}$ with $g^*f^*\mathcal{E}$ and where we identify $g^*\mathcal{O}_T$ with $\mathcal{O}_{\widetilde{T}}$ via associativity of pullback.

(a) Prove that these notions define a set-valued, contravariant functor,

$$\mathbb{A}_S(\mathcal{E}) : S - \mathbf{schemes}^{\text{opp}} \to \mathbf{Sets},$$

associating to every S-scheme T the set of functionals of $\mathcal{E}$ on T, and associating to every morphism of S-schemes the pullback map above.

(b) For every morphism of schemes,

$$(v, v^{\#}) : (\widetilde{S}, \mathcal{O}_{\widetilde{S}}) \to (S, \mathcal{O}_S),$$

for the contravariant functor

$$\mathbb{A}_{\widetilde{S}}(v^*\mathcal{E}) : \widetilde{S} - \mathbf{schemes}^{\text{opp}} \to \mathbf{Sets},$$

prove that there exists a unique natural isomorphism of contravariant functors,

$$\mathbb{A}_v(\mathcal{E}) : \mathbb{A}_{\widetilde{S}}(v^*\mathcal{E}) \Rightarrow \mathbb{A}_S(\mathcal{E})_{\widetilde{S}-schemes},$$

from $\mathbb{A}_{\widetilde{S}}(v^*\mathcal{E})$ to the restriction of $\mathbb{A}_S(\mathcal{E})$ to the category of $\widetilde{S}$-schemes considered as a subcategory of the category of S-schemes (typically not a full subcategory). Prove that $\mathbb{A}_V(\mathcal{E})$ is natural in v. Conclude that if $\mathbb{A}_S(\mathcal{E})$ is representable by an S-scheme, then also $\mathbb{A}_{\widetilde{S}}(v^*\mathcal{E})$ is representable by the $\widetilde{S}$-scheme that is the fiber product over S of this S-scheme and of $\widetilde{S}$.

(c) Prove that the functor $\mathbb{A}_S(\mathcal{E})$ is a sheaf for the Zariski topology. For every S-scheme $(T, \mathcal{O}_T)$, for every collection of open immersions,

$$\mathfrak{T} := ((g_\alpha, g_\alpha^\#) : (\widetilde{T}_\alpha, \mathcal{O}_{\widetilde{T}_\alpha}) \to (T, \mathcal{O}_T))_{\alpha \in A},$$

that form a Zariski covering of $(T, \mathcal{O}_T)$, a **$\mathfrak{T}$-descent datum** for $\mathbb{A}_S(\mathcal{E})$ is an A-indexed system

$$(\chi_\alpha : \widetilde{g}_\alpha^* f^* \mathcal{E} \to \mathcal{O}_{\widetilde{T}_\alpha})_{\alpha \in A},$$

of functionals of $\mathcal{E}$ on $(\widetilde{T}_\alpha, \mathcal{O}_{\widetilde{T}_\alpha})$ such that for every ordered pair $(\alpha, \beta) \in A \times A$, the pullbacks to $\widetilde{T}_{\alpha,\beta} := \widetilde{T}_\alpha \times_T \widetilde{T}_\beta$ of χ_α via the first projection and of χ_β via the second projection are equal as functionals of $\mathcal{E}$ on $\widetilde{T}_{\alpha,\beta}$. Prove that for every $\mathfrak{T}$-descent datum, there exists a unique functional χ of $\mathcal{E}$ on T such that every χ_α is the pullback to $\widetilde{T}_\alpha$ of χ. Conclude that the functor $\mathbb{A}_S(\mathcal{E})$ is representable by an S-scheme if there exists a Zariski open covering of S,

$$\mathfrak{S} := ((v_\alpha, v_\alpha^\#) : (\widetilde{S}_\alpha, \mathcal{O}_{\widetilde{S}_\alpha}) \to (S, \mathcal{O}_S))_{\alpha \in A},$$

such that for every $\alpha \in A$, the functor $\mathbb{A}_{\widetilde{S}_\alpha}(v_\alpha^* \mathcal{E})$ is representable by an $\widetilde{S}_\alpha$-scheme.

(d) For every surjective $\mathcal{O}_S$-module homomorphism,

$$\psi : \widetilde{\mathcal{E}} \to \mathcal{E},$$

for every functional χ of $\mathcal{E}$ on T, define the **ψ-associated** functional of $\widetilde{\mathcal{E}}$ on T to be

$$\chi \circ f^* \psi : f^* \widetilde{\mathcal{E}} \to \mathcal{O}_T.$$

Prove that this defines a natural transformation of contravariant functors,

$$\mathbb{A}_S(\psi) : \mathbb{A}_S(\mathcal{E}) \to \mathbb{A}_S(\widetilde{\mathcal{E}}).$$

Prove that $\mathbb{A}_S(\psi)$ is natural in ψ. Prove that $\mathbb{A}_S(\psi)$ satisfies the obvious compatibility with $\mathbb{A}_v(\mathcal{E})$ and $\mathbb{A}_v(\widetilde{\mathcal{E}})$. Also prove that $\mathbb{A}_S(\psi)$ is compatible with Zariski open covers of S. Assume now that $\mathbb{A}_S(\widetilde{\mathcal{E}})$ is representable by an S-scheme. Let

$$(e, e^\#) : (\widetilde{G}, \mathcal{O}_{\widetilde{G}}) \to (S, \mathcal{O}_S), \quad \widetilde{\chi} : e^* \widetilde{\mathcal{E}} \to \mathcal{F},$$

be a representing object. Assume also that the kernel $\text{Ker}(\psi)$ is globally generated. Define

$$(g, g^\#) : (G, \mathcal{O}_G) \to (\widetilde{G}, \mathcal{O}_{\widetilde{G}}),$$

to be the common zero scheme in $(\widetilde{G}, \mathcal{O}_{\widetilde{G}})$ of the following collection of global sections of $\mathcal{O}_{\widetilde{G}}$,

$$\{e^* \mathcal{O}_S \xrightarrow{e^* k} e^* \widetilde{\mathcal{E}} \xrightarrow{\widetilde{\chi}} \mathcal{O}_{\widetilde{G}} | k \in \text{Ker}(\psi)(S)\}.$$

Prove that $g^* \widetilde{\chi}$ factors uniquely through $g^* e^* \psi$, say

$$g^* \widetilde{\chi} = \chi \circ g^* e^* \psi, \quad \chi : g^* e^* \mathcal{E} \to \mathcal{O}_G.$$

Conclude that

$$(g, g^\#) \circ (e, e^\#) : (G, \mathcal{O}_G) \to (S, \mathcal{O}_S), \quad \chi : g^* e^* \mathcal{E} \to \mathcal{O}_G,$$

represents the functor $\mathbb{A}_S(\mathcal{E})$. Since $\text{Ker}(\psi)$ is quasi-coherent, thus globally generated when restricted to the opens $\widetilde{S}_\alpha$ of a Zariski open covering $\mathfrak{S}$ of S, conclude that $\mathbb{A}_S(\mathcal{E})$ is representable by an S-scheme whenever $\mathbb{A}_S(\widetilde{\mathcal{E}})$ is representable by an S-scheme. Combining the parts above, conclude representability of $\mathbb{A}_S(\mathcal{E})$ for every scheme $(S, \mathcal{O}_S)$ and for every quasi-coherent $\mathcal{O}_S$-module $\mathcal{E}$ assuming representability just for the free $\mathcal{O}_S$-modules $\mathcal{O}_S^{\oplus I}$ where I is an arbitrary set (possibly infinite).

(e) For every set I, denote by $\mathbb{A}^I_{\mathbb{Z}}$ the scheme associated to the commutative ring $\mathbb{Z}[\{t_i : i \in I\}]$ that is a polynomial $\mathbb{Z}$-algebra in variables t_i indexed by I. For every scheme $(S, \mathcal{O}_S)$, denote by $\mathbb{A}^I_S$ the fiber product over Spec $\mathbb{Z}$ of $\mathbb{A}^I_{\mathbb{Z}}$ and of S considered as an S-scheme. Every element t_i defines a global section χ_i of the structure sheaf of $\mathbb{A}^I_{\mathbb{Z}}$. Thus, the I-system $(\chi_i)_{i\in I}$ defines a homomorphism of quasi-coherent sheaves,

$$\chi : \mathcal{O}^{\oplus I}_{\mathbb{A}^I_S} \to \mathcal{O}_{\mathbb{A}^I_S},$$

i.e., a functional of $\mathcal{O}_S^{\oplus I}$ on $\mathbb{A}^I_S$. Prove that this functional represents the functor $\mathbb{A}_S(\mathcal{O}_S^{\oplus I})$. Combined with the previous parts, prove that the functor $\mathbb{A}_S(\mathcal{E})$ is representable for every quasi-coherent $\mathcal{O}_S$-module $\mathcal{E}$. Since this functor is representable, we will denote any representing object also by $(\mathbb{A}_S(\mathcal{E}), \chi)$ (this is an abuse of notation, but it rarely causes confusion).

(f) For the special case $\mathcal{E} = \mathcal{O}_S$, conclude that $\mathbb{A}_S(\mathcal{O}_S)$ is the usual affine S-scheme $\mathbb{A}^1_S$ with its structure as a ring object in the category of S-schemes. In particular, the maximal open subscheme $\mathbb{G}_{m,S}$ of $\mathbb{A}^1_S$ on which χ is an isomorphism inherits a structure of commutative S-group scheme via multiplication. For every quasi-coherent $\mathcal{O}_S$-module $\mathcal{E}$, prove that the $\mathcal{O}_S$-module structure endows $\mathbb{A}_S(\mathcal{E})$ with a structure of $\mathbb{A}^1_S$-module object in the category of affine S-schemes. In particular, $\mathbb{A}_S(\mathcal{E})$ is an affine S-scheme with integral geometric fibers, it is a commutative S-group scheme under addition, it has a compatible action of $\mathbb{G}_{m,S}$ by scaling that distributes with respect to addition, and the induced scaling action of $\mathbb{G}_{m,S}$ on the conormal sheaf of the zero section of $\mathbb{A}_S(\mathcal{E})$ acts through the tautological character of $\mathbb{G}_{m,S}$. An affine, commutative S-group scheme (group law written as addition) having integral geometric fibers, with a distributive scaling action of $\mathbb{G}_{m,S}$ such that the induced action on the conormal sheaf of the zero section acts through the tautological character of $\mathbb{G}_{m,S}$ is called an **Abelian cone** over S. Prove that the rule $\mathcal{E} \mapsto \mathbb{A}_S(\mathcal{E})$ defines an equivalence of Abelian categories between the Abelian category of quasi-coherent $\mathcal{O}_S$-modules and the Abelian category of Abelian cones over S. Prove that the Abelian cone is locally finitely presented, resp. smooth over S, if and only if the $\mathcal{O}_S$-module $\mathcal{E}$ is locally finitely presented, resp. locally free.

Problem 1.(**Grassmannians**) You can read more about Grassmannians in Griffiths-Harris, *Principles of algebraic geometry*. Let $(S, \mathcal{O}_S)$ be a scheme. Let $\mathcal{E}$ be a quasi-coherent $\mathcal{O}_S$-module. You may assume that $\mathcal{E}$ is locally finitely presented. Let r be a nonnegative integer (some authors define the Grassmannian to be empty if $r < 0$ by convention). For every S-scheme,

$$(f, f^\#) : (T, \mathcal{O}_T) \to (S, \mathcal{O}_S),$$

a **locally free quotient of rank** r for $\mathcal{E}$ is a pair,

$$(\mathcal{F}, \phi : f^*\mathcal{E} \to \mathcal{F}),$$

where $\mathcal{F}$ is a locally free $\mathcal{O}_T$-module of constant rank r, and where ϕ is a surjective homomorphism of $\mathcal{O}_T$-modules. Such a pair is uniquely determined by its kernel, since $\mathcal{F}$ is the quotient of $f^*\mathcal{E}$ by the kernel (since the category of $\mathcal{O}_T$-modules is an Abelian category). Two pairs are **isomorphic** if they have equal kernels as

$\mathcal{O}_T$-submodules of $f^*\mathcal{E}$ (some authors formulate the Grassmannian functor in terms of kernels instead of quotients, which is fine, but introduces subtleties elsewhere). For every S-scheme,

$$(\widetilde{f}, \widetilde{f}^\#) : (\widetilde{T}, \mathcal{O}_{\widetilde{T}}) \to (S, \mathcal{O}_S),$$

and for every morphism of S-schemes,

$$(g, g^\#) : (\widetilde{t}, \mathcal{O}_{\widetilde{T}}) \to (T, \mathcal{O}_T),$$

define the **pullback** of $(\mathcal{F}, \phi)$ by $(g, g^\#)$ to be the isomorphism class of

$$(g^*\mathcal{F}, g^*\phi : g^*f^*\mathcal{E} \to g^*\mathcal{F}),$$

where we use associativity of pullback to identify $g^*f^*\mathcal{E}$ with $\widetilde{f}^*\mathcal{E}$.

(a) Prove that these notions define a set-valued, contravariant functor,

$$\underline{G}_S(\mathcal{E}, r) : S - \textbf{schemes}^{\text{opp}} \to \textbf{Sets},$$

associating to every S-scheme the set of isomorphism classes of locally free quotients of rank r of $\mathcal{E}$ over that S-scheme, and associating to every morphism of S-schemes the pullback map between the sets of isomorphism classes.

(b) For every morphism of schemes,

$$(v, v^\#) : (\widetilde{S}, \mathcal{O}_{\widetilde{S}}) \to (S, \mathcal{O}_S),$$

for the contravariant functor

$$G_{\widetilde{S}}(v^*\mathcal{E}, r) : \widetilde{S} - \textbf{schemes}^{\text{opp}} \to \textbf{Sets},$$

prove that there exists a unique natural isomorphism of contravariant functors,

$$G_v(\mathcal{E}, r) : G_{\widetilde{S}}(v^*\mathcal{E}, r) \Rightarrow G_S(\mathcal{E}, r)_{\widetilde{S}-schemes},$$

from $G_{\widetilde{S}}(v^*\mathcal{E}, r)$ to the restriction of $G_S(\mathcal{E}, r)$ to the category of $\widetilde{S}$-schemes considered as a subcategory of the category of S-schemes (typically not a full subcategory). Prove that $G_V(\mathcal{E}, r)$ is natural in v. Conclude that if $G_S(\mathcal{E}, r)$ is representable by an S-scheme, then also $G_{\widetilde{S}}(v^*\mathcal{E}, r)$ is representable by the $\widetilde{S}$-scheme that is the fiber product over S of this S-scheme and of $\widetilde{S}$.

(c) Prove that the functor $G_S(\mathcal{E}, r)$ is a sheaf for the Zariski topology. For every S-scheme $(T, \mathcal{O}_T)$, for every collection of open immersions,

$$\mathfrak{T} := ((g_\alpha, g_\alpha^\#) : (\widetilde{T}_\alpha, \mathcal{O}_{\widetilde{T}_\alpha}) \to (T, \mathcal{O}_T))_{\alpha \in A},$$

that form a Zariski covering of $(T, \mathcal{O}_T)$, a **$\mathfrak{T}$-descent datum** is an A-indexed system

$$(\mathcal{F}_\alpha, \phi_\alpha : \widetilde{g}_\alpha^* f^*\mathcal{E} \to \mathcal{F}_\alpha)_{\alpha \in A},$$

of locally free quotients of rank r of $\mathcal{E}$ over $(\widetilde{T}_\alpha, \mathcal{O}_{\widetilde{T}_\alpha})$ such that for every ordered pair $(\alpha, \beta) \in A \times A$, the pullbacks to $\widetilde{T}_{\alpha,\beta} := \widetilde{T}_\alpha \times_T \widetilde{T}_\beta$ of $(\mathcal{F}_\alpha, \phi_\alpha)$ via the first projection and of $(\mathcal{F}_\beta, \phi_\beta)$ via the second projection are isomorphic as locally free quotients of rank r of $\mathcal{E}$ on $\widetilde{T}_{\alpha,\beta}$. Prove that for every $\mathfrak{T}$-descent datum, there exists a locally free quotient of rank r of $\mathcal{E}$ on T, $(\mathcal{F}, \phi)$, such that every $(\mathcal{F}_\alpha, \phi_\alpha)$ is isomorphic to the pullback to $\widetilde{T}_\alpha$ of $(\mathcal{F}, \phi)$. Also prove that $(\mathcal{F}, \phi)$ is unique up to isomorphism. Conclude that the functor $G_S(\mathcal{E}, r)$ is representable if there exists a Zariski open covering of S,

$$\mathfrak{S} := ((v_\alpha, v_\alpha^\#) : (\widetilde{S}_\alpha, \mathcal{O}_{\widetilde{S}_\alpha}) \to (S, \mathcal{O}_S))_{\alpha \in A},$$

such that for every $\alpha \in A$, the functor $G_{\widetilde{S}_\alpha}(v_\alpha^*\mathcal{E}, r)$ is representable by an $\widetilde{S}_\alpha$-scheme.

(d) For every surjective $\mathcal{O}_S$-module homomorphism,

$$\psi : \widetilde{\mathcal{E}} \to \mathcal{E},$$

for every locally free quotient of rank r of $\mathcal{E}$ on T, $(\mathcal{F}, \phi)$ define the ψ-**associated** locally free quotient of rank r of $\widetilde{\mathcal{E}}$ on T to be

$$(\mathcal{F}, \phi \circ f^*\psi : f^*\widetilde{\mathcal{E}} \to \mathcal{F}).$$

Prove that this defines a natural transformation of contravariant functors,

$$G_S(\psi, r) : G_S(\mathcal{E}, r) \to G_S(\widetilde{\mathcal{E}}, r).$$

Prove that $G_S(\psi, r)$ is natural in ψ. Prove that $G_S(\psi, r)$ satisfies the obvious compatibility with $G_v(\mathcal{E}, r)$ and $G_v(\widetilde{\mathcal{E}}, r)$. Also prove that $G_S(\psi, r)$ is compatible with Zariski open covers of S. Assume now that $G_S(\widetilde{\mathcal{E}}, r)$ is representable by an S-scheme, say

$$(e, e^\#) : (G, \mathcal{O}_G) \to (S, \mathcal{O}_S), \quad (\mathcal{F}, \phi : e^*\widetilde{\mathcal{E}} \to \mathcal{F}),$$

is a representing object. Assume also that the kernel $\mathrm{Ker}(\psi)$ is globally generated. Define

$$(g, g^\#) : (\widetilde{G}, \mathcal{O}_{\widetilde{G}}) \to (G, \mathcal{O}_G),$$

to be the common zero scheme in G of the following collection of global sections of $\mathcal{F}$,

$$\{\mathcal{O}_G = e^*\mathcal{O}_S \xrightarrow{e^*k} e^*\widetilde{\mathcal{E}} \xrightarrow{\phi} \mathcal{F} | k \in \mathrm{Ker}(\psi)(S)\}.$$

Prove that $g^*\phi$ factors uniquely through $g^*e^*\psi$, say

$$g^*\phi = \widetilde{\phi} \circ g^*e^*\psi, \quad \widetilde{\phi} : g^*e^*\mathcal{E} \to g^*\mathcal{F}.$$

Conclude that

$$(g, g^\#) \circ (e, e^\#) : (\widetilde{G}, \mathcal{O}_{\widetilde{G}}) \to (S, \mathcal{O}_S), \quad (g^*\mathcal{F}, \widetilde{\phi} : g^*e^*\mathcal{E} \to g^*\mathcal{F}),$$

represents the functor $G_S(\mathcal{E}, r)$. Since $\mathrm{Ker}(\psi)$ is quasi-coherent, thus globally generated when restricted to the opens $\widetilde{S}_\alpha$ of a Zariski open covering $\mathfrak{S}$ of S, conclude that $G_S(\mathcal{E}, r)$ is representable whenever $G_S(\widetilde{\mathcal{E}}, r)$ is representable. Combining the parts above, conclude representability of $G_S(\mathcal{E}, r)$ for every scheme $(S, \mathcal{O}_S)$ and for every quasi-coherent $\mathcal{O}_S$-module $\mathcal{E}$ assuming representability just for the free $\mathcal{O}_S$-modules $\mathcal{O}_S^{\oplus I}$ where I is an arbitrary set (possibly infinite).

(e) For every locally free $\mathcal{O}_S$-module $\mathcal{M}$ of rank r, for every short exact sequence of $\mathcal{O}_S$-modules,

$$\Sigma : 0 \to \mathcal{M} \xrightarrow{\mu} \mathcal{E} \xrightarrow{\nu} \mathcal{N} \to 0,$$

that admits a splitting,

$$\sigma : \mathcal{N} \to \mathcal{E}, \quad \nu \circ \sigma = \mathrm{Id}_{\mathcal{N}},$$

define $G_S(\mathcal{E}, r)_\mu$ to be the subfunctor of $G_S(\mathcal{E}, r)$ of those locally free quotients $(\mathcal{F}, \phi)$ of rank r of $\mathcal{E}$ on T such that the composition,

$$\phi \circ f^*\mu : \mathcal{M} \to \mathcal{F},$$

is an isomorphism of locally free $\mathcal{O}_T$-modules. Denote by

$$\phi_{\Sigma,\sigma} : f^*\mathcal{N} \to f^*\mathcal{M},$$

the unique $\mathcal{O}_T$-module homomorphism such that

$$(\phi \circ f^*\mu) \circ \phi_{\Sigma,\sigma} = \phi \circ f^*\sigma.$$

Via adjointness of Hom and tensor product, interpret $\phi_{\Sigma,\sigma}$ as a functional for $\mathit{Hom}_{\mathcal{O}_S}(\mathcal{M},\mathcal{N})$ on T,

$$\widehat{\phi}_{\Sigma,\sigma} : f^*\mathit{Hom}_{\mathcal{O}_S}(\mathcal{M},\mathcal{N}) \to \mathcal{O}_T.$$

Prove that the rule associating to every $(\mathcal{F},\phi)$ in $G_S(\mathcal{E},r)_\mu$ the functional $\widetilde{\phi}_{\Sigma,\sigma}$ defines a natural isomorphism of set-valued contravariant functors on S-schemes,

$$i_{\Sigma,\sigma} : G_S(\mathcal{E},r)_\mu \Rightarrow \mathbb{A}_S(\mathit{Hom}_{\mathcal{O}_S}(\mathcal{M},\mathcal{N})).$$

Using the previous problem, conclude that $G_S(\mathcal{E},r)_\mu)$ is representable by an affine S-scheme, sometimes called a **distinguished open affine**. Also show that the representing S-scheme is locally finitely presented, resp. smooth over S, if and only if the $\mathcal{O}_S$-module $\mathcal{E}$ is locally finitely presented, resp. locally free.

(f) Finally, in the special case that $\mathcal{E} = \mathcal{O}_S^{\oplus I}$, by varying over the subsets $M \subset I$ of cardinality r, and by defining $\mu : \mathcal{M} \to \mathcal{E}$ to be the corresponding sum of summands indexed by M, prove that the corresponding functors $G_S(\mathcal{E},r)_\mu$ give a Zariski open covering of the functor $G_S(\mathcal{E},r)$. Sometimes this is called the **distinguished open affine cover**. Conclude that $G_S(\mathcal{E},r)$ is representable by an S-scheme. Prove that this representing S-scheme is locally finitely presented if and only if it is smooth over S if and only if I is a finite set. Using the previous parts, prove that for every quasi-coherent $\mathcal{O}_S$-module $\mathcal{E}$, the functor $G_S(\mathcal{E},r)$ is representable by an S-scheme. Moreover, if $\mathcal{E}$ is locally finitely presented, resp. locally free, prove that $G_S(\mathcal{E},r)$ is a locally finitely presented S-scheme, resp. a smooth S-scheme.

(g) Read about the r-fold tensor product functor on the category of $\mathcal{O}_S$-modules, resp. symmetric power and exterior power functors. Prove that there is a natural transformation of functors,

$$\mathrm{Pl}_S(\mathcal{E},r) : G_S(\mathcal{E},r) \to G_S(\bigwedge_{\mathcal{O}_S}^{r} \mathcal{E},1),$$

associating to every locally free quotient (E,ϕ) of rank r the induced locally free quotient of rank 1,

$$\bigwedge^{r} : f^*\bigwedge_{\mathcal{O}_S}^{r}\mathcal{E} \to \bigwedge_{\mathcal{O}_T}^{r}\mathcal{F}.$$

Prove that the induced morphism of representing S-schemes is a closed immersion, the **Plücker embedding**. When $\mathcal{E}$ equals $\mathcal{O}_S^{\oplus I}$ for a finite set I, find minimal homogeneous generators for the homogeneous ideal of this closed immersion in the homogeneous coordinate ring of the projective space $G_S(\bigwedge_{\mathcal{O}_S}^r \mathcal{E},1)$ over S (these generators are usually called **Plücker relations**).

Problem 2.(**Geometric quotients of finite group actions.**) Let $(\Gamma, m : \Gamma \times \Gamma \to \Gamma)$ be a finite group. Let $(S,\mathcal{O}_S)$ be a scheme. Let

$$(f,f^\#) : (T,\mathcal{O}_T) \to (S,\mathcal{O}_S)$$

be a locally ringed space over S. A **Γ-action** on T via S-morphisms is a group homomorphism,

$$\rho : \Gamma \to \mathrm{Aut}_{S-\mathbf{LRS}}(T,\mathcal{O}_T).$$

For every $\gamma \in \Gamma$, denote by

$$\rho_\gamma : (T, \mathcal{O}_T) \to (T, \mathcal{O}_T)$$

the associated automorphism of locally ringed spaces over S. An S-morphism,

$$p : (T, \mathcal{O}_T) \to (R, \mathcal{O}_R),$$

is Γ-**invariant** if every composition $p \circ \rho_\gamma$ equals p. A Γ-invariant S-morphism q is a **categorical quotient** if every Γ-invariant S-morphism factors uniquely through q.

(a) Prove that there exists a categorical quotient of the underlying topological spaces,

$$q : T \to T/\Gamma,$$

in the category of topological spaces together with a continuous map to the (Zariski) topological space of S.

(b) Prove that there is an induced action of Γ on the pushforward sheaf $q_*\mathcal{O}_T$. Define $\mathcal{O}_{T/\Gamma}$ to be the subsheaf of $q_*\mathcal{O}_T$ consisting of the Γ-invariant sections. Prove that this is a sheaf of subrings of $q_*\mathcal{O}_T$. Together this defines a ringed space $(T/\Gamma, \mathcal{O}_{T/\Gamma})$ over $(S, \mathcal{O}_S)$ and a Γ-invariant morphism,

$$(q, q^\#) : (T, \mathcal{O}_T) \to (T/\Gamma, \mathcal{O}_{T/\Gamma}),$$

in the category of ringed spaces over S.

(c) Now assume that $(T, \mathcal{O}_T)$ is a scheme, and assume that every Γ-orbit in $(T, \mathcal{O}_T)$ has a neighborhood basis by Γ-invariant open affine subschemes U. Prove that there exists a unique open subset U/Γ of T/Γ whose inverse image under q equals U, and prove that the ringed space structure on U/Γ induced by $(T/\Gamma, \mathcal{O}_{T/\Gamma})$ is Spec $\mathcal{O}_T(U)^\Gamma$. Conclude that $(T/\Gamma, \mathcal{O}_{T/\Gamma})$ is an S-scheme, and q is a categorical quotient in the category of S-schemes. Prove that q is a quasi-finite, affine morphism whose induced homomorphisms of local rings are integral ring homomorphisms. If $(T, \mathcal{O}_T)$ is locally finitely presented over $(S, \mathcal{O}_S)$, prove that q is even a finite morphism.

(d) If $(T, \mathcal{O}_T)$ is a quasi-projective S-scheme, resp. a projective S-scheme, prove that every Γ-orbit has a neighborhood basis of Γ-invariant open, affine subschemes. Show that the tensor power of the Serre twisting sheaf corresponding to the integer $n = \#\Gamma$ has a natural Γ-linearization. Conclude that T/Γ is a quasi-projective S-scheme, resp. a projective S-scheme.

(e) With hypotheses as in (c), let

$$(v, v^\#) : (\widetilde{S}, \mathcal{O}_{\widetilde{S}}) \to (S, \mathcal{O}_S)$$

be a morphism of S-schemes. Prove that the induced Γ-action on the fiber product $\widetilde{S} \times_S T$ is an action via $\widetilde{S}$-morphisms, and that also this action satisfies the hypotheses of (c). Conclude that there is a unique induced morphism of $\widetilde{S}$-schemes,

$$(\widetilde{S} \times_S T)/\Gamma \to \widetilde{S} \times_S (T/\Gamma).$$

If $(v, v^\#)$ is flat, or if the integer n is invertible in $\mathcal{O}_T(T)$, prove that this morphism of $\widetilde{S}$-schemes is an isomorphism. Thus, q is a **uniform categorical quotient**, and it is even a **universal categorical quotient** if Γ is **tame**. Find an example of a non-tame action of a finite group on an S-scheme such that q is not a universal categorical quotient.

Problem 3.(**Symmetric products**) For an $(S,\mathcal{O}_S)$-scheme $(T,\mathcal{O}_T)$, for every integer r, denote by $(T^r,\mathcal{O}_{T^r})$ the r-fold self-fiber-product of $(T,\mathcal{O}_T)$ over $(S,\mathcal{O}_S)$. For every $i=1,\dots,r$, denote by

$$\text{pr}_i : (T^r,\mathcal{O}_{T^r}) \to (T,\mathcal{O}_T)$$

the corresponding projection S-morphism. Denote by $\mathfrak{S}_r$ the symmetric group on n letters having $n=r!$ elements. For every $\gamma\in\mathfrak{S}_r$, denote by

$$\rho_\gamma : (T^r,\mathcal{O}_{T^r}) \to (T^r,\mathcal{O}_{T^r})$$

the unique S-morphism such that for every $i=1,\dots,r$, the composition $\text{pr}_i\circ\rho_\gamma$ equals $\text{pr}_{\gamma^{-1}(i)}$.

(a) Prove that this defines an action of $\mathfrak{S}_r$ on $(T^r,\mathcal{O}_{T^r})$ via S-morphisms. Prove that this action is natural in $(T,\mathcal{O}_T)$.

(b) Under the hypothesis that $(T,\mathcal{O}_T)$ is quasi-projective, prove that there exists a uniform categorical quotient of this action that is a quasi-projective S-scheme. This is the r^{th} **symmetric product** of $(T,\mathcal{O}_T)$ over $(S,\mathcal{O}_S)$,

$$(q,q^{\#}) : (T^r,\mathcal{O}_{T^r}) \to (\text{Sym}^r_S(T),\mathcal{O}_{\text{Sym}^r(T)}).$$

In the special case that T equals $\mathbb{A}^m_S$ for some nonnegative integer m, describe the Γ-orbits on $T^r\cong\mathbb{A}^{mr}_S$ in terms of multi-diagonals, and setup the computation of the ring of invariants for the corresponding local rings of $\text{Sym}^r_S(\mathbb{A}^m_S)$.

(c) In the special case that T equals $\mathbb{A}^1_S$, use the elementary symmetric polynomials to define an isomorphism of $\text{Sym}^r_S(\mathbb{A}^1_S)$ with $\mathbb{A}^r_S$. In particular, conclude that q is a finite, flat morphism between smooth S-schemes.

(d) Use the fact that the symmetric product is functorial in T and is a uniform categorical quotient to conclude that whenever T is étale locally isomorphism to $\mathbb{A}^1_S$ as an S-scheme, then also $\text{Sym}^r_S(T)$ is a smooth S-scheme of relative dimension r, and q is a finite, flat morphism. Conclude that for every smooth, quasi-projective S-scheme T of relative dimension 1, then also $\text{Sym}^r_S(T)$ is a smooth, quasi-projective S-scheme of relative dimension r.

(e) If T is a flat, quasi-projective S-scheme with geometrically connected fibers, show that also T^r is a flat, quasi-projective S-scheme with geometrically connected fibers. Conclude that also $\text{Sym}^r_S(T)$ is a quasi-projective S-scheme with geometrically connected fibers. Altogether, if T is a smooth, quasi-projective S-scheme of relative dimension 1 having geometrically connected fibers, then also $\text{Sym}^r_S(T)$ is a smooth, quasi-projective S-scheme of relative dimension r having geometrically connected fibers. If T is also projective over T, then prove that $\text{Sym}^r_S(T)$ is projective over T.

(f) For a smooth, quasi-projective S-scheme T of relative dimension 1, show that the image Δ of the diagonal morphism $T\to T\times_S T$ is an effective Cartier divisor in T. Moreover, the projection from Δ to the first factor,

$$\text{pr}_1 : \Delta \to T,$$

is an isomorphism, hence flat. For every $i = 1, \dots, r$, denote by Δ_i the inverse image effective Cartier divisor in $T^r \times_S T$ under the projection,

$$\mathrm{pr}_{i,r+1} : T^r \times_S T = T^{r+1} \to T \times_S T.$$

Show that the projection,

$$\mathrm{pr}_{1,\dots,r} : \Delta_i \to T^r,$$

is an isomorphism, hence flat. Denote by $\Delta_{1,\dots,r}$ the sum from $i = 1, \to, r$ of the effective Cartier divisor Δ_i. Prove that this is flat over T^r. Prove that there exists a unique effective Cartier divisor $\mathcal{D} \subset \mathrm{Sym}^r_S(T) \times_S T$ such that $\Delta_{1,\dots,r}$ equals the pullback of $\mathcal{D}$ via the S-morphism,

$$q \times \mathrm{Id}_T : T^r \times_S T \to \mathrm{Sym}^r_S(T) \times_S T,$$

and prove that $\mathcal{D}$ is flat over $\mathrm{Sym}^r_S(T)$.

(g) With hypotheses as above, show that $\mathrm{Sym}^r_S(T)$ and the closed subscheme $\mathcal{D} \subset \mathrm{Sym}^r_S(T) \times_S T$ represents the Hilbert functor on S-schemes of closed subschemes of T that are finite and flat of constant length r over the base. Denote by $\mathcal{O}(\underline{\mathcal{D}})$ the associated invertible sheaf on $\mathrm{Sym}^r_S(T) \times_S T$. If T is projective, if every geometric fiber is connected of arithmetic genus g, and if $r \geq 2g - 1$, prove that the higher direct images of $\mathcal{O}(\underline{\mathcal{D}})$ are all zero, and the pushforward of $\mathcal{O}(\underline{\mathcal{D}})$ is locally free of rank $r + 1 - g$ compatible with arbitrary base change. Denote the pushforward by $\mathcal{G}_r^\vee$, i.e., the dual of a locally free sheaf $\mathcal{G}_r$ on $\mathrm{Sym}^r_S(T)$ of rank $r + 1 - g$.

Problem 4.(**The Hom scheme from a curve to a projective scheme**) Let $(S, \mathcal{O}_S)$ be a connected scheme. Let

$$(f, f^\#) : (C, \mathcal{O}_C) \to (S, \mathcal{O}_S)$$

be a smooth, projective S-scheme of relative dimension 1 having connected geometric fibers. Denote by g the arithmetic genus of every fiber, i.e., the derived pushforward $R^1 f_* \mathcal{O}_C$ is locally free of rank g, compatible with arbitrary base change of S.

Let $\widetilde{\mathcal{E}}$ be a locally free $\mathcal{O}_S$-module of rank $r + 1 > 1$ so that $G_S(\widetilde{\mathcal{E}}, 1)$ is Zariski locally over S isomorphic to $\mathbb{P}^r_S$. Denote $G_S(\widetilde{\mathcal{E}}, 1)$ by $\mathbb{P}$. Denote by $\mathcal{O}_{\mathbb{P}}(1)$ the relative Serre twisting sheaf on $\mathbb{P}$.

Let $e \geq 2g - 1$ be an integer. For every S-scheme T, define a S-**morphism** from C to $\mathbb{P}$ of relative degree e over T is an S-morphism,

$$u : T \times_S C \to \mathbb{P},$$

such that the pullback $u^* \mathcal{O}_{\mathbb{P}}(1)$ is an invertible sheaf of constant relative degree e over T. For every S-scheme $\widetilde{T}$ and for every morphism of S-schemes,

$$(g, g^\#) : (\widetilde{T}, \mathcal{O}_{\widetilde{T}}) \to (T, \mathcal{O}_T),$$

define the **pullback** of u by $(g, g^\#)$ to be the composition

$$\widetilde{T} \times_S C \xrightarrow{g \times \mathrm{Id}_C} T \times_S C \xrightarrow{u} \mathbb{P}.$$

(a) Prove that these notions define a set-valued, contravariant functor,

$$\mathrm{Hom}^e_S(C, \mathbb{P}) : S - \mathbf{schemes}^{\mathrm{opp}} \to \mathbf{Sets},$$

associating to every S-scheme T the set of S-morphisms from C to $\mathbb{P}$ of relative degree e over T and associating to every morphism of S-schemes the pullback map above.

(b) Repeat the parts of Problem 0, concluding that $\text{Hom}_S^e(C,\mathbb{P})$ is representable by an S-scheme if and only if it is locally representable for the Zariski topology.

(c) Let x be a global section of $\mathcal{O}_{\mathbb{P}}(1)$. Denote by $\text{Hom}_S^e(C,\mathbb{P})_x$ the subfunctor of $\text{Hom}_S^e(C,\mathbb{P})$ of S-morphisms u from C to $\mathbb{P}$ of relative degree e over T such that u^*x restricts on every fiber of $T\times_S C\to T$ as a regular section of the pullback of $\mathcal{O}_{\mathbb{P}}(1)$, i.e., the zero scheme of u^*x contains no geometric fiber. Prove that the zero scheme of u^*x is a closed subscheme of $T\times_S C$ that is finite and flat over T of relative degree e. Conclude that there is a natural transformation,

$$\text{Div}_x : \text{Hom}_S^e(C,\mathbb{P})_x \to \text{Sym}_S^e(C).$$

(d) Now assume that x is the first term in a free basis $(x,x_1,\dots,x_r)$ of the locally free, rank-$(r+1)$ pushforward of $\mathcal{O}_{\mathbb{P}}(1)$ to S. Show that for each $i=1,\dots,r$, the section x_i defines a functional for the pullback $\text{Div}_x^*\mathcal{G}_r$ on T. Thus, $(x_1,\dots,x_r)$ defines a lifting of Div_x to a natural transformation from $\text{Hom}_S^e(C,\mathbb{P})_x$ to the smooth Abelian cone,

$$p:\mathbb{A}\to\mathbb{P}_{\mathbb{C}},$$

associated to the locally free sheaf $\mathcal{G}_e^{\oplus r}$ on $\text{Sym}_S^e(C)$. Prove that this natural transformation is an isomorphism from $\text{Hom}_S^e(C,\mathbb{P})_x$ to the maximal open subscheme U of $\mathbb{A}$ parameterizing ordered r-tuples of linear functionals $(\chi_1,\dots,\chi_r)$ such that the common zero scheme of $\chi_1,\dots,\chi_r$ and $\mathcal{D}$ is empty in $U\times_S C$. Thus, conclude that $\text{Hom}_S^e(C,\mathbb{P})_x$ is representable by an S-scheme that is quasi-projective.

(e) Now use the fact that Zariski locally on S, the $\mathcal{O}_S$-module $\widetilde{\mathcal{E}}$ has a free basis $(x_0,x_1,\dots,x_r)$ to conclude that, Zariski locally on S, $\text{Hom}_S^e(C,\mathbb{P})$ is representable by an S-scheme that is finitely presented, covered by $r+1$ quasi-projective open subschemes $\text{Hom}_S^e(C,\mathbb{P})_{x_i}$ for $i=0,\dots,r$. Using the previous parts, conclude that $\text{Hom}_S^e(C,\mathbb{P})$ is representable by an S-scheme that is finitely presented. Finally, use the valuative criterion of separatedness to prove that $\text{Hom}_S^e(C,\mathbb{P})$ is separated over S. (In fact, it is even quasi-projective over S, equal to an open subscheme of one of the projective connected components of the relative Hilbert scheme of $C\times_S\mathbb{P}$ over S.)

(f) For every locally closed immersion,

$$i:X\hookrightarrow\mathbb{P},$$

show that the induced natural transformation,

$$\text{Hom}_S(C,X)\to\text{Hom}_S(C,\mathbb{P}),$$

is relatively representable by locally closed immersions. Since $\text{Hom}_S^e(C,\mathbb{P})$ is representable by separated, finite type S-schemes, conclude that also the inverse image of $\text{Hom}_S^e(C,\mathbb{P})$ in $\text{Hom}_S(C,X)$ is representable by separated, finite type S-schemes.

(g) For every positive integer e, not necessarily with $e\geq 2g-1$, for every integer $d\geq 1$, consider the Veronese d-uple embedding

$$v_d:\mathbb{P}\hookrightarrow\mathbb{P}_d,$$

that pulls back $\mathcal{O}_{\mathbb{P}_d}(1)$ to $\mathcal{O}_{\mathbb{P}}(d)$. If the product integer ed is $\geq 2g-1$, then $\text{Hom}_S^{ed}(C, \mathbb{P}_d)$ is representable by separated, finite type S-schemes. Use the previous part to conclude that also $\text{Hom}_S^e(C, \mathbb{P})$ is representable by separated, finite type S-schemes.

(h) Read again the previous problem set to review the infinitesimal deformation theory of $\text{Hom}_S(C, X)$ over S.

Problem 5(The quotient of Hom$^e(\mathbb{P}^1, \mathbb{P}^1)$ by Aut$(\mathbb{P}^1)$.) There are many ways to construct the quotient of $\text{Hom}^e(\mathbb{P}^1, \mathbb{P}^1)$ by the affine group scheme $\text{Aut}(\mathbb{P}^1)$. This problem outlines one classical construction, going back at least to Hurwitz, via branch divisors. The construction extends to all characteristics that do not divide the degree of the morphism

Let $e \geq 1$ be an integer. Let $(S, \mathcal{O}_S)$ be the relative Spec of $\mathbb{Z}[1/e]$. Let C be $\mathbb{P}^1_S$ and let $\mathbb{P}$ be $\mathbb{P}^1_S$. Then for every integer $e \geq 1$, there is a separated, finite type scheme $\text{Hom}^e_S(\mathbb{P}^1_S, \mathbb{P}^1_S)$ parameterizing S-morphisms from $\mathbb{P}^1_S$ to $\mathbb{P}^1_S$ of relative degree e. By associating to every morphism u the graph of u as a Cartier divisor in $\mathbb{P}^1_S \times_S \mathbb{P}^1_S$ in the complete linear series of $\mathcal{O}(1, e) := \text{pr}_1^*\mathcal{O}(1) \otimes \text{pr}_2^*\mathcal{O}(e)$, the scheme $\text{Hom}^e_S(\mathbb{P}^1_S, \mathbb{P}^1_S)$ is even a dense open subscheme of the projective space $\mathbb{P}_S H^0(\mathbb{P}^1_S \times_S \mathbb{P}^1_S, \mathcal{O}(1, e))$. In particular, $\text{Hom}^e_S(\mathbb{P}^1_S, \mathbb{P}^1_S)$ is a smooth S-scheme of relative dimension $2e+1$ whose geometric fibers over S are integral (even dense opens in projective space).

(a) Read the statement and proof of the Noether Normalization Theorem. Let $(S, \mathcal{O}_S)$ be a normal scheme that is either a locally finite type scheme over a field or a locally finite type scheme over the ring of integers of a number field (most generally, finite type over an excellent scheme). Let

$$(f, f^\#) : (T, \mathcal{O}_T) \to (S, \mathcal{O}_S),$$

be a separated, finite type morphism from a normal, integral scheme. Prove that the integral closure $\mathcal{A}$ of $\mathcal{O}_S$ in $f_*\mathcal{O}_T$ is a locally finitely presented, quasi-coherent $\mathcal{O}_S$-module that is also an $\mathcal{O}_S$-algebra. Use the universal property of the relative Spec to prove that there is a factorization of f as a composition,

$$(g, g^\#) : (T, \mathcal{O}_T) \to (\text{Spec }_S\mathcal{A}, \mathcal{O}_{\text{Spec }\mathcal{A}}), \quad (h, h^\#) : (\text{Spec }_S\mathcal{A}, \mathcal{O}_{\text{Spec }\mathcal{A}}) \to (S, \mathcal{O}_S),$$

where $(g, g^\#)$ is a dominant morphism with irreducible geometric generic fiber, and where $(h, h^\#)$ is a finite, surjective morphism between normal schemes.

(b) Denote by H_e the relative Hom scheme $\text{Hom}^e_S(\mathbb{P}^1_S, \mathbb{P}^1_S)$. Consider the morphism

$$(\text{pr}_1, u) : H_e \times_S \mathbb{P}^1_S \to H_e \times_S \mathbb{P}^1_S$$

of smooth, projective curves over H_e. Via contravariance of H_e-relative differentials, there is a pullback map,

$$(\text{pr}_1, u)^* : u^*\Omega_{\mathbb{P}^1_S/S} \to \text{pr}_2^*\Omega_{\mathbb{P}^1_S/S}.$$

The image equals $\text{pr}_2^*\Omega_{\mathbb{P}^1_S/S} \otimes \mathcal{I}_R$, where $\mathcal{I}_R \subset \mathcal{O}_{H_e \times \mathbb{P}^1}$ is an ideal sheaf of a closed subscheme R of $H_e \times_S \mathbb{P}^1_S$. Use the hypothesis that $1/e$ is invertible in S to conclude that R is an effective Cartier divisor that is finite and flat over H_e of relative degree $2e-2$. (To see why the invertibility of e is relevant, consider for instance, the Frobenius morphism on $\mathbb{P}^1_{\mathbb{F}_p}$ of degree p for which R is all of $\mathbb{P}^1_{\mathbb{F}_p}$.)

(c) Read about traces and norms for finite flat morphisms of schemes. In particular, associated to the finite, flat morphism (pr_1, u), there is a pushforward effective Cartier divisor

$$B = (\mathrm{pr}_1, u)_*(R)$$

in $H_e \times_S \mathbb{P}^1_S$ that is also finite and flat over H_e of relative degree $2e-2$. By the universal property of the Hilbert scheme, there is an induced S-morphism,

$$\mathrm{br}_e : \mathrm{Hom}^e_S(\mathbb{P}^1_S, \mathbb{P}^1_S) \to \mathrm{Sym}^{2e-2}_S(\mathbb{P}^1_S).$$

Prove that this **branch morphism** is invariant for the natural action of $\mathrm{Aut}_S(\mathbb{P}^1_S)$ on $\mathrm{Hom}^e_S(\mathbb{P}^1_S, \mathbb{P}^1_S)$ by precomposing on the domain.

(d) Review from the last problem set about infinitesimal deformation theory. For a field k whose characteristic does not divide e, for a k-morphism $u : \mathbb{P}^1_k \to \mathbb{P}^1_k$ with **geometrically reduced** ramification divisor $R \subset \mathbb{P}^1_k$ of degree $2e-2$, prove that $u^*T_{\mathbb{P}^1_k/k}(-\underline{R})$ is canonically isomorphic to $T_{\mathbb{P}^1_k/k}$. Conclude that the Zariski tangent space of the fiber of br_e is precisely the 3-dimensional Lie algebra $H^0(\mathbb{P}^1_k, T_{\mathbb{P}^1_k/k})$ of $\mathrm{Aut}_k(\mathbb{P}^1_k)$. By comparing dimensions, conclude that br_e is dominant.

(e) Denote by

$$(g, g^{\#}) : \mathrm{Home}^e_S(\mathbb{P}^1_S, \mathbb{P}^1_S) \to \mathrm{Spec}\ \mathcal{A}, \quad (h, h^{\#}) : \mathrm{Spec}\ \mathcal{A} \to \mathrm{Sym}^{2e-2}_S(\mathbb{P}^1_S),$$

the factorization of br_e from the first part above. Using the previous part, prove that $(h, h^{\#})$ is generically étale. Over $\mathbb{C}$, prove that every fiber of $(g, g^{\#})$ is a finite union of $\mathrm{Aut}_{\mathbb{C}}(\mathbb{P}^1_{\mathbb{C}})$-orbits by using the Riemann existence theorem to reconstruct a branched cover of $\mathbb{CP}^1$ in terms of the branch divisor B and the representation of the fundamental group of $\mathbb{CP}^1 \setminus B$ in the symmetric group $\mathfrak{S}_e$. In fact, by specialization for tamely ramified covers (cf. SGA 1), this also holds in every characteristic p that does not divide e. Thus, every fiber of $(g, g^{\#})$ is a union of finitely many $\mathrm{Aut}_S(\mathbb{P}^1_S)$-orbits. In particular, br_e is a dominant morphism between smooth S-schemes such that every geometric fiber has pure dimension 3. It follows that the image of $(g, g^{\#})$ is an open subscheme $\mathrm{M}_{0,0}(\mathbb{P}^1_S/S, e)$.