MAT 614 Problem Set 1

Homework Policy. Read through and carefully consider all of the following problems.

Problems.

Problem 1. (a). For a given quasi-coherent sheaf, what is the parameter space for locally free quotient sheaves of specified rank?

(b). What is the parameter space for flags of locally free quotient sheaves of specified rank?

(c). Under what conditions on the quasi-coherent sheaf is the parameter space quasi-compact over the base scheme?

(d). When is the parameter space separated?

(e). When the parameter space is quasi-compact over the base scheme, is it proper over the base scheme?

(f). When the parameter space is proper over the base scheme, is it projective over the base scheme?

(g). When the parameter space is projective over the base scheme, what is one choice of relatively ample invertible sheaf?

(h). Is there a choice of relatively ample invertible sheaf that generates the Picard group of each geometric fiber?

(i). Under what conditions on the quasi-coherent sheaf is the parameter space smooth over the base scheme?

(j). When the parameter space is smooth over the base scheme, what is the relative dualizing sheaf (written in terms of the relatively ample invertible sheaf above and pullbacks of invertible sheaves from the base scheme)?

(k). When the parameter space is smooth over the base scheme, what is the sheaf of relative differentials?

(l). Can you write the sheaf of relative differentials as a tensor product of locally free sheaves of smaller rank? If so, then the first Chern class of your ample generator of the Picard group, inside the first sheaf cohomology of the sheaf of relative differentials, can be interpreted also as an element of an Ext group between the two "tensor components".

(m). What is the corresponding short exact sequence of locally free sheaves on your parameter space?

Problem 2. (a). What is a Grassmannian?

(b). Is there a covering of the Grassmannian by open affine subsets, each isomorphic to an affine space, such that each closed complement generates the divisor class group?

(c). What is the least number of such open affines that are necessary to cover the Grassmannian?

(d). Does there exist a partition of the Grassmannian into finitely many locally closed subvarieties, each of which is isomorphic to an affine space (of varying degrees)?

(e). How many such open affines do you need?

(f). When the base field is $\mathbb{C}$ and we consider the associated complex manifold of the Grassmannian with its "classical" topology (also often called the "analytic" or "Hausdorff" topology), what are the Betti numbers? What is the Euler characteristic?

Problem 3. (a). How many conics in the projective plane contain each of five general specified points?

(b). How many conics in the projective plane contain each of four general specified points and are tangent to one general line?

(c). How many conics in the projective plane contain each of three general specified points and are tangent to two specified general lines?

Problem 4. Assume that the characteristic of the base field is not 2. Then every smooth conic in the projective plane is the zero scheme of a quadratic form in three variables that is equivalent to a symmetric three-by-three matrix that is invertible. The inverse matrix is also a symmetric three-by-three matrix.

(a). What is the interpretation of the associated smooth conic in the dual projective plane?

(b). How many such "dual conics" contain each of five general specified points?

(c). How many dual conics contain each of four general specified points and are tangent to a general line?

(d). How many dual conics contain each of three general specified points and are tangent to two specified general lines?

Problem 5. Given two vector spaces of finite dimension and a linear isomorphism between them, inside the direct sum of the two vector spaces, consider the three half-dimensional subspaces consisting of the two summands as well as the graph of the linear isomorphism. These give three points in a Grassmannian parameterizing half-dimensional subspaces of the direct sum vector space (or equivalently, the half-dimensional quotient spaces).

(a). What is the minimal degree of a connected curve in this Grassmannian that contains the three points?

(b). Can you describe such a curve explicitly?

(c). The curve parameterizes half-dimensional subspaces whose associated projective spaces are linear subvarieties of the projective space of the direct sum. What is the smallest closed subvariety of the projective space that contains all of these linear subvarieties?

(d). What is the degree of this closed subvariety of projective space? Can you describe it explicitly?

(e). What are all the linear subvarieties of this closed subvariety? Are there linear subvarieties that are not contained in those coming from the half-dimensional subspaces parameterized by the curve in the Grassmannian?

Problem 6. Specialize the previous problem to the case that the two vector spaces each have dimension two. Then the smallest closed subvariety containing the corresponding lines in the projective three-space is a hypersurface.

(a). What is the degree of this hypersurface?

(b). Can you describe the hypersurface explicitly?

(c). What are all the lines in this hypersurface?

(d). Are there more lines than those parameterized by the curve in the Grassmannian?

(e). For a general line in the projective three-space, in how many points does that general line intersect the hypersurface?

(f). How many lines in projective three-space intersect each of the three original lines as well as the new general line?

Problem 7. Assume that the ground field is algebraically closed of characteristic different from 2 or 3. Let x, y and z be homogeneous coordinates on the projective plane. Is every smooth plane cubic projectively equivalent to a member of the pencil of plane cubics spanned by the zero sets of $x^3 + y^3 + z^3$ and the zero set of xyz? If so, how many members of this pencil are projectively equivalent to a general specified smooth plane cubic?

Problem 8. For a general pencil of plane curves of degree d, how many members are singular? For each such member, what type of singularity does each member have? If you remove the singular points from that member, is the resulting quasi-projective variety still connected? Can there be a pencil of plane curves of degree d where there are precisely two members that are singular? If so, what type of singularity does each member have?

Problem 9. For a smooth, projective, geometrically connected curve in projective three-space that is contained in no hyperplane, for a general point of the curve, for the unique line in projective three-space that has maximal order of contact with the curve (i.e., the scheme-theoretic intersection is a divisor on the line of maximal possible degree), does that "tangent line" intersect the curve in a second point? What if the base field has positive characteristic?

Problem 10. In projective n-space, for $n + 2$ specified general points, what is the minimal degree of a connected curve that contains all of the points? For one additional specified point, is there a curve that contains all $n+3$ points. Now consider the curve with the ordered $(n+3)$-tuple of points, considered as a "marked curve". As we vary the additional point, does the isomorphism class of the marked curve also vary?