MAT 614 Problem Set 3

Homework Policy. Read through and carefully consider all of the following problems.

Problems.

Problem 1. For a closed subscheme X of $\mathbb{P}^n_k$ that has pure dimension equal to $m \geq 0$ and degree equal to d, inside the Grassmannian parameterizing $n-m-1$-planes in $\mathbb{P}^n_K$, consider the locus $\text{Chow}_{X,\mathbb{P}^n_k}$ parameterizing $n-m-1$-planes that have nonempty intersection with X.

(a) Show that $\text{Chow}_{X,\mathbb{P}^n_k}$ is a closed subset of the Grassmannian.

(b) Show that this closed subset has pure codimension 1.

(c) Use det and Div, or any other method, to assign an appropriate multiplicity to each irreducible component of this hypersurface in $\mathbb{P}^n_k$. For your definition of the associated Cartier divisor, prove that $\text{Chow}_{X,\mathbb{P}^n_k}$ varies algebraically, for a flat family of closed subschemes X over an arbitrary base. What is the total degree of $\text{Chow}_{X,\mathbb{P}^n_k}$ relative to the Plücker embedding?

(d) Show that the underlying point set of X can be recovered as an appropriate Fano scheme parameterizing those closed subschemes of $\text{Chow}_{X,\mathbb{P}^n_k}$ that are Schubert varieties isomorphic to the Grassmannian of $n-m-2$-planes in a projective space $\mathbb{P}^{n-1}$.

(e). Use the theory of "flattening stratifications" to prove that inside the projective space parameterizing hypersurface of the appropriate degree, the locus of those arising as $\text{Chow}_{X,\mathbb{P}^n_k}$ for some closed subscheme X of pure dimension equal to m and degree equal to d is a closed subset. This closed subset is the **Chow variety** of degree-d, effective m-cycles in $\mathbb{P}^n_k$.

Problem 2. For X a degree-d hypersurface in $\mathbb{P}^n_k$, prove that the associated Chow variety equals the Hilbert scheme, i.e., the usual projective space of global sections on $\mathbb{P}^n_k$ of $\mathcal{O}_{\mathbb{P}^n}(d)$.

Problem 3. For X a degree-d, 0-dimensional closed subscheme, prove that the associated Chow variety is the symmetric power $\text{Sym}^d_k(\mathbb{P}^n_k) = (\mathbb{P}^n_k \times_{\text{Spec } k} \cdots \times_{\text{Spec } k} \mathbb{P}^n_k)/\mathfrak{S}_d$.

Problem 4. For every integer m, denote by $\text{HP}_{\mathbb{P}^m}(t)$ the Hilbert polynomial of $\mathbb{P}^m_k$ with respect to $\mathcal{O}_{\mathbb{P}^m}(1)$. For every integer $m \geq 0$, for every integer $n \geq m$, prove that the Hilbert scheme parameterizing closed subschemes of $\mathbb{P}^n$ with Hilbert polynomial $\text{HP}(t) = \text{HP}_{\mathbb{P}^m}(t)$ is the Grassmannian k-scheme $\text{Grass}_k(\mathbb{P}^m_k, \mathbb{P}^n_k)$. For every integer $m \geq 0$, for every integer $n \geq m$, for every integer $d \geq 1$, prove that the Hilbert scheme parameterizing closed subschemes of $\mathbb{P}^n$ with Hilbert polynomial $\text{HP}(t) = \text{HP}_{\mathbb{P}^m}(t) - \text{HP}_{\mathbb{P}^m}(t-d)$ is a Zariski locally trivial projective space bundle over the Grassmannian k-scheme $\mathrm{Grass}_k(\mathbb{P}^m_k, \mathbb{P}^n_k)$ with fiber isomorphic to the Hilbert scheme / Chow variety of degree-d hypersurfaces in $\mathbb{P}^m_k$.

Problem 5. In the previous problem, set $m = 2$ and $d = 2$, so that the Hilbert scheme is a $\mathbb{P}^5$-bundle over the Grassmannian $\mathrm{Grass}_k(\mathbb{P}^2_k, \mathbb{P}^n_k)$. Prove that the parameter space of "complete conic curves" in $\mathbb{P}^n$ is a blowing up of the Hilbert scheme along the (Zariski locally constant) subvariety parameterizing double-lines, i.e., the parameter space is a Zariski locally trivial fiber bundle over the Grassmannian whose fiber is the blowing up of $\mathbb{P}^5$ along a Veronese image of $\mathbb{P}^2$.

Problem 6. Consider the "standard generators" of the Picard group of the space of complete conics: the pullback of an ample generator of the Picard group of the Grassmannian, the pullback of the locus of complete conics intersecting a specified codimension-2 linear subvariety of $\mathbb{P}^n_k$, and the class of the exceptional divisor of the blowing up. With respect to these generators, compute the nef and effective cones of the Hilbert scheme and of the space of complete conic curves. In particular, compute the extremal face of the nef cone of the Hilbert scheme that equals the pullback of the nef cone of the Chow variety, and compute the extremal face of the nef cone of the space of complete conic curves that equals the pullback of the nef cone of the Hilbert scheme.

Problem 7. For the description of the Hilbert scheme as a $\mathbb{P}^5$-bundle over the Grassmannian, consider the dual $\mathbb{P}^5$-bundle over the Grassmannian. Recall that on the dense open subscheme of the Hilbert scheme parameterizing smooth conic curves, there is a "dual conics" morphism to the dual $\mathbb{P}^5$-bundle (compatible with the projections to the Grassmannian). The space of complete conic curves equals the minimal blowing up of the Hilbert scheme on which this rational transformation extends to an everywhere regular morphism. There is an involution on the space of complete conics that permutes a conic with its dual conic.

(a) Under this involution, what happens to the three standard generators of the Picard group?

(b) Under this involution, what happens to the extremal face of the nef cone corresponding to the pullback of the nef cone of the Hilbert scheme, resp. the pullback of the nef cone of the Chow variety?

Problem 8. Are there "modular" interpretations of the contractions of the parameter space of complete conic curves corresponding to the extremal faces in the previous problem?

Problem 9. When n equals 3, what is the Chow ring of the Hilbert scheme of plane conics in $\mathbb{P}^3$, and what is the Chow ring of the Hilbert scheme of complete conic curves in $\mathbb{P}^3$. What part of the Chow ring is "pulled back" from the Chow variety?

Problem 10. How many smooth plane conics in $\mathbb{P}^3_k$ are tangent to each of the hyperplanes in a general 8-tuple of hyperplanes in $\mathbb{P}^3_k$?