MAT 614 Problem Set 2

Homework Policy. Read through and carefully consider all of the following problems.

Problems.

Problem 1. Let k be a field, let V be a k-vector space of finite dimension n, and let r be an integer with $1 \leq r \leq n-1$. Denote by $(G, V^\vee \otimes_k \mathcal{O}_V \twoheadrightarrow S^\vee)$ a universal pair of a k-scheme G and a rank-r, locally free quotient of free $\mathcal{O}_G$-module $V^\vee \otimes_k \mathcal{O}_V$, i.e., G is a Grassmannian $\text{Grass}_k(r, V) = \text{Grass}_k(\mathbb{P}^{r-1}, \mathbb{P}V)$. Fix a complete flag $U_\bullet = (U_0, U_1, \dots, U_{n-1}, U_n)$ of k-linear subspaces of V, i.e., each U_ℓ is an ℓ-dimensional linear subspace of V and each U_ℓ is a subset of $U_{\ell+1}$. Necessarily U_0 is the zero subspace and U_n equals V. For each r-dimensional linear subspace W of V, for each integer $q = 1, \dots, r$, define $a_q(U_\bullet, W)$ to be the maximum nonnegative integer a such that the k-subspace $W \cap U_{n+q-r-a}$ of V has k-vector space dimension at least q.

(a) Prove that these integers form a nondecreasing sequence,

$$0 \leq a_r \leq a_{r-1} \leq \cdots \leq a_1 \leq n-r.$$

(b). For every nondecreasing sequence of integers $a_\bullet = (a_1, \dots, a_r)$ as above, define the **Schubert cell** $\Sigma^o_{a_\bullet}(U_\bullet)$, resp. the **Schubert cycle** $\Sigma_{a_\bullet}(U_\bullet)$, to be the subset of G parameterizing all k-linear subspaces W such that for every $q = 1, \dots, r$, the integer $a_q(U_\bullet, W)$ equals a_q, respectively is at least as positive as a_q. Prove that $\Sigma^o_{a_\bullet}(U_\bullet)$ is an irreducible locally closed subset whose underlying reduced scheme is isomorphic to an affine space over k, and compute the dimension. Prove that $\Sigma_{a_\bullet}(U_\bullet)$ is an irreducible closed subset that equals the closure of this affine space in G.

(c). For $U_\bullet$ held fixed, as $a_\bullet$ varies over all possible nondecreasing r-tuples of integers between 1 and $n-r$, prove that the corresponding Schubert cells form a locally closed partition of G into affine spaces. Conclude that the Chow groups of G are free Abelian groups with additive basis equal to the cycle classes $\sigma_{a_\bullet}$ of the associated Schubert cycles (which are independent of the choice of $U_\bullet$). Use this to compute the Betti numbers and Euler characteristics for several choices of the pair of integers (n, r).

(d). Show that the Chern classes of $S^\vee$ are equal to Schubert classes $\sigma_{a_\bullet}$ for special choices of $a_\bullet$. Determine the correspondence between the Chern classes and these "special Schubert classes".

(e). As a commutative $\mathbb{Z}$-algebra, the Chow ring of G is generated by the Chern classes of $S^\vee$ with relations coming from the degree $n+1-r$ component of the power series $c(V^\vee \otimes_k \mathcal{O}_G)/c(S^\vee)$. For several choices of (n, r), work out what are these relations explicitly, and double check that the associated graded pieces of the Chow ring have rank equal to that computed in part (c) above.

(f). For several choices of (n, r), compute the explicit polynomial in the special Schubert classes that gives $\sigma_{a_\bullet}$ for every choice of $a_\bullet$. In particular, if you write out each product of a Schubert class $\sigma_{a_\bullet}$ by one of the special Schubert classes as a linear combination of Schubert classes, what pattern do you notice for the coefficients?

(f). In one of the recommended textbooks, read about Pieri's Rule and Giambelli's Formula. Together, these give one "presentation" of the Chow ring of G. Use this presentation to repeat some of the simple enumerative computations, such as the number of lines in projective 3-space that intersect each of four specified (yet general) lines.

Problem 2. Let X be a separated, finite type k-scheme (the broadest class of schemes for which we have defined Chow groups, etc., in this course). Let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of rank r.

(a). For every short exact sequence of locally free $\mathcal{O}_X$-modules,

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0,$$

describe a corresponding filtration on the d-fold tensor product $\bigotimes_{\mathcal{O}_X}^d \mathcal{E}$, on the exterior power $\bigwedge_{\mathcal{O}_X}^d \mathcal{E}$, and on the symmetric power $\text{Sym}_{\mathcal{O}_X}^d \mathcal{E}$. Deduce that these operations extend to the Grothendieck ring $K^0(X)$ of locally free $\mathcal{O}_X$-modules of finite rank.

(b) Use the Splitting Principle to prove that for every integer d, there exists a power series $t_d(c_1, \dots, c_r)$ in the Chow ring of X, resp. $e_d(c_1, \dots, c_r)$, $s_d(c_1, \dots, c_r)$, where each variable c_i is homogeneous of degree-i and each graded piece of the power series is homogeneous in these variables, such that the total Chern class of $\bigotimes_{\mathcal{O}_x}^d \mathcal{E}$, resp. of $\bigwedge_{\mathcal{O}_X}^d \mathcal{E}$, of $\text{Sym}_{\mathcal{O}_X}^d \mathcal{E}$, equals the specified power series evaluated on the Chern classes of $\mathcal{E}$.

(c). Compute these polynomials when $r = 2$ and $d = 1, 2, 3, 4, 5$. Combine this with the previous problem to give another method of computing the number of lines on a smooth cubic surface in projective 3-space, resp. on a sufficiently general quintic threefold in projective 4-space.

Problem 3. How many lines in projective 3-space are tangent to each of four specified (yet general) hypersurfaces of respective degrees d_1, d_2, d_3, and d_4?

Problem 4. For two sufficiently general smooth, connected curves C_1, respectively C_3, that span a projective 3-space and have degree and genus d_1 and g_1, respectively d_2 and g_2, how many lines in projective 3-space are simultaneously secant lines to both C_1 and C_2?

Problem 5. On the Grassmannian G of Problem 1, what is the degree of the top self-intersection of the unique Schubert cycle that is a hypersurface?