MAT 614 Intersection Theory
Stony Brook University
Problem Set 4

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MAT 614 Problem Set 4

Homework Policy. Please read through all the problems. I will be happy to discuss the solutions during office hours.

Problems.

Problem 1. Let Y be a finite type, separated k-scheme. Let $\mathcal{E}$ be a locally free $\mathcal{O}_Y$-module of rank $r+1$. Let

$$\pi_{\mathcal{E}} : \mathbb{P}_Y(\mathcal{E}) \to Y, \quad \phi : \pi^*\mathcal{E}^\vee \to \mathcal{O}(1)$$

be a universal pair of a morphism to Y together with an invertible quotient of the pullback of $\mathcal{E}^\vee$ (to help calibrate conventions, this is *covariant* in $\mathcal{E}$ with respect to locally split monomorphisms of locally free sheaves). Recall in the proof of the splitting principle, for each $q = 0, \dots, r$ and for each integer $l \in \mathbb{Z}$, we defined the group homomorphism,

$$\widetilde{\pi}^*_q : A_{l-q}(Y) \to A_l(\mathbb{P}_Y(\mathcal{E})), \quad \beta_{l-q} \mapsto c_1(\mathcal{O}(1))^{r-q} \cap \pi^*\beta_{l-q},$$

together with the direct sum of these homomorphisms,

$$\widetilde{\pi}^* : \bigoplus_{q=0}^{r} A_{l-q}(Y) \to A_l(\mathbb{P}(\mathcal{E})).$$

(a) For every morphism $\tau : T \to Y$, prove that the following diagram is Cartesian.

$$\begin{array}{ccc} \mathbb{P}_T(\tau^*\mathcal{E}) & \xrightarrow{\mathbb{P}\tau} & \mathbb{P}_Y(\mathcal{E}) \\ {\scriptstyle \pi_{\tau^*\mathcal{E}}}\downarrow & & \downarrow{\scriptstyle \pi_{\mathcal{E}}} \\ T & \xrightarrow[\tau]{} & Y, \end{array}$$

where $\mathbb{P}\tau$ is the unique Y-morphism pulling back the tautological invertible quotient of $\pi^*_{\mathcal{E}}\mathcal{E}^\vee$ to the tautological invertible quotient of

$$\pi^*_{\tau^*\mathcal{E}}(\tau^*\mathcal{E})^\vee = (\tau \circ \pi_{\tau^*\mathcal{E}})^*\mathcal{E}^\vee.$$

Therefore, for every integer $l \in \mathbb{Z}$, $\widetilde{\pi_{T,\tau^*\mathcal{E}}}^*_q$ is a group homomorphism,

$$(\widetilde{\pi}^*_q)^{l-q}_\tau : A_{l-q}(T) \to A_l(T \times_Y \mathbb{P}_Y(\mathcal{E})).$$

Prove that this defines a bivariant class,

$$\widetilde{\pi}_q^* \in A^{-q}(\mathbb{P}_Y(\mathcal{E}) \xrightarrow{\pi} Y).$$

(b) Recall that for every integer $m = 0, \dots, r$, we defined a group homomorphism

$$\lambda_m : A_l(\mathbb{P}_Y(\mathcal{E})) \to A_{l-m}(Y), \quad \alpha_l \mapsto \pi_*(c_1(\mathcal{O}(1))^m \cap \alpha_l),$$

together with the direct sum of these homomorphisms,

$$\lambda : A_l(\mathbb{P}_Y(\mathcal{E})) \to \bigoplus_{m=0}^{r} A_{l-m}(Y).$$

Recall that we proved that $\widetilde{\pi}^*$ is surjective, and the composition $\lambda \circ \widetilde{\pi}^*$ differs from the identity by a nilpotent operator (of nilpotence degree $\leq r$), so that the composition is invertible. Thus, $\widetilde{\pi}^*$ is an isomorphism, even though λ is not the "true" inverse.

Using now the properties of the Chern classes, prove that the "true" inverse of $\widetilde{\pi}^*$ is the direct sum μ of the homomorphisms,

$$\mu_m : A_l(\mathbb{P}_Y(\mathcal{E}) \to A_{l-m}(Y)), \quad \alpha_l \mapsto \sum_{n=0}^{m} c_{m-n}(\mathcal{E}) \cap \mu_n(\alpha_l).$$

(c) For every morphism $f : X \to S$, for every integer p, define

$$\widetilde{\pi}^* : \bigoplus_{q=0}^{n} A^{p+q}(X \xrightarrow{f} S) \to A^p(\mathbb{P}_Y(\mathcal{E}) \xrightarrow{f\circ\pi} S),$$

that sends a tuple $a = (a_{p+q})_{0 \leq q \leq r}$ of bivariant classes to the bivariant class,

$$\widetilde{\pi}^*(a_{p+q})_q := \sum_{q=0}^{r} \widetilde{\pi}_q^* \cap a_{p+q},$$

i.e., use the associative product of bivariant classes a_{p+q} with the bivariant classes $\widetilde{\pi}_q^*$ defined above. Similarly, for every integer $m = 0, \dots, r$ and for every integer p, define

$$\mu_m : A^p(\mathbb{P}_Y(\mathcal{E}) \xrightarrow{f\circ\pi} S) \to A^{p+m}(Y \xrightarrow{f} S),$$

sending a bivariant class α_p to

$$\mu_m(\alpha_p) := \sum_{n=0}^{n} c_{m-n}(\mathcal{E}) \cap \pi_*(c_1(\mathcal{O}(1))^n \cap \alpha_p).$$

Check that this is well-defined, i.e., every $\mu_m(\alpha_p)$ is a bivariant class in $A^{p+m}(Y \xrightarrow{f} S)$.

Define μ to be the direct sum of the maps μ_n,

$$\mu : A^p(\mathbb{P}_Y(\mathcal{E}) \xrightarrow{f\circ\pi} S) \to \bigoplus_{m=0}^{r} A^{p+m}(Y \xrightarrow{f} S).$$

Prove that the composition,

$$\mu \circ \widetilde{\pi}^* : \bigoplus_{l=0}^{r} A^{p+l}(Y \xrightarrow{f} S) \to \bigoplus_{m=0}^{r} A^{p+m}(Y \xrightarrow{f} S),$$

is the identity. Conclude that $\widetilde{\pi}^*$ is injective and that μ is surjective.

(d) Consider the composition $\widetilde{\pi}^* \circ \mu$. Recall that there is a short exact sequence of locally free sheaves on $\mathbb{P}_Y(\mathcal{E})$,

$$0 \longrightarrow \Omega^1_\pi(1) \xrightarrow{w_\mathcal{E}} \pi^*\mathcal{E}^\vee \xrightarrow{\phi_\mathcal{E}} \mathcal{O}(1) \longrightarrow 0$$

Thus, on $\mathbb{P}_Y(\mathcal{E}) \times_Y \mathbb{P}_Y(\mathcal{E})$, there is a composition homomorphism,

$$\mathrm{pr}_1^*[\Omega^1_\pi(1)] \xrightarrow{\mathrm{pr}_1^* w_\mathcal{E}} \mathrm{pr}_1^*\pi^*\mathcal{E}^\vee \cong \mathrm{pr}_2^*\pi^*\mathcal{E}^\vee \xrightarrow{\mathrm{pr}_2^*\phi_\mathcal{E}} \mathrm{pr}_2^* O(1),$$

or equivalently, the twist by $\mathrm{pr}_s^*\mathcal{O}(-1)$,

$$\alpha_\mathcal{E} : \mathrm{pr}_1^*[\Omega^1_\pi(1)] \otimes_\mathcal{O} \mathrm{pr}_2^*[\mathcal{O}(-1)] \to \mathcal{O}_{\mathbb{P}\mathcal{E}\times_X\mathbb{P}\mathcal{E}}.$$

Recall that the image sheaf is precisely the ideal sheaf of the image of the diagonal immersion,

$$\Delta : \mathbb{P}_Y(\mathcal{E}) \to \mathbb{P}_Y(\mathcal{E}) \times_Y \mathbb{P}_Y(\mathcal{E}).$$

Define $\mathcal{F}$ to be the dual locally free sheaf of rank r,

$$\mathcal{F} := \left(\mathrm{pr}_1^*[\Omega^1_\pi(1)] \otimes_\mathcal{O} \mathrm{pr}_2^*[\mathcal{O}(-1)]\right)^\vee,$$

and define s to be the global section $\alpha_\mathcal{E}^\dagger$. Associated to $\mathcal{F}$ and s, **soon** we will define a Gysin homomorphism,

$$\Delta^! \in A^r(\mathbb{P}_Y(\mathcal{E}) \xrightarrow{\Delta} \mathbb{P}_Y(\mathcal{E}) \times_Y \mathbb{P}_Y(\mathcal{E})),$$

which is a bivariant class. Assume both of the following properties of this bivariant class.

Property 1. The composition $\Delta_*\Delta^!$ equals $c_r(\mathcal{F})$ in $A^r(\mathbb{P}_Y(\mathcal{E}) \times_Y \mathbb{P}_Y(\mathcal{E}))$.

Property 2. The composition $\Delta^!\mathrm{pr}_2^*$ in $A^0(\mathbb{P}_Y(\mathcal{E}) \xrightarrow{\mathrm{Id}} \mathbb{P}_Y(\mathcal{E})$ is the identity homomorphism.

Via the "standard techniques", the general case of each property reduces to the case that Y equals Spec$k$ and $\mathbb{P}_Y(\mathcal{E})$ is $\mathbb{P}^r$, where they can be proved by hand. Assuming both of these properties, prove that Δ_* equals $c_r(\mathcal{F})\mathrm{pr}_2^*$. In particular, since $\mathrm{pr}_{1,*}\Delta_*$ is the identity homomorphism, prove that $\mathrm{pr}_{1,*}c_r(\mathcal{F})\mathrm{pr}_2^*$ is the identity homomorphism. Use this to prove that $\widetilde{\pi}^* \circ \mu$ is the identity.

(e) In particular, conclude that, as a graded module over the graded ring $A^*(Y \xrightarrow{\mathrm{Id}} Y)$, $A^*(\mathbb{P}_Y(\mathcal{E}) \xrightarrow{\pi} Y)$ is isomorphic to the free graded module,

$$\bigoplus_{q=0}^{r} A^{*+q}(Y \xrightarrow{\mathrm{Id}} Y).$$

Use this to prove that, via the pullback algebra homomorphism,

$$\pi^* : A^*(Y \xrightarrow{\mathrm{Id}} Y) \to A^*(\mathbb{P}_Y(\mathcal{E}) \xrightarrow{\mathrm{Id}} \mathbb{P}_Y(\mathcal{E})),$$

the second ring is generated over the first ring by the element

$$t = c_1(\mathcal{O}(1))\cap,$$

subject to the monic equation,

$$t^{r+1} + c_1(\mathcal{E})t^r + \cdots + c_q(\mathcal{E})t^{r+1-m} + \cdots + c_r(\mathcal{E})t + c_{r+1}(\mathcal{E}) = 0.$$

Problem 2. Let $\mathcal{E}$ and $\mathcal{F}$ be locally free sheaves on Y of respective ranks $r+1$ and r. Let

$$\Sigma : 0 \longrightarrow \mathcal{F} \xrightarrow{\ u\ } \mathcal{E} \xrightarrow{\ v\ } \mathcal{O}_Y \longrightarrow 0,$$

be a short exact sequence of locally free sheaves. Recall from lecture that this gives rise to a commutative diagram

$$\begin{array}{ccc} \mathbb{P}_Y(\mathcal{F}) & \xrightarrow{\mathbb{P}_Y(u)} & \mathbb{P}_Y(\mathcal{E}) \\ {\scriptstyle \pi_{\mathcal{F}}}\downarrow & & \downarrow{\scriptstyle \pi_{\mathcal{E}}} \\ Y & \xrightarrow[\mathrm{Id}]{} & Y \end{array}$$

such that the image of $\mathbb{P}_Y(u)$ is a Cartier divisor class $[\mathbb{P}_Y(\mathcal{F})]$ on $\mathbb{P}_Y(\mathcal{E})$ representing $\mathcal{O}(1)$.

(a) Combined with the previous problem, conclude that for every integer l,

$$(\mathbb{P}_Y(u)_*, \pi_{\mathcal{E}}^*) : A_l(\mathbb{P}_Y(\mathcal{F})) \oplus A_{l-r}(Y) \to A_l(\mathbb{P}_Y(\mathcal{E})),$$

is an isomorphism. In particular, conclude that $\mathbb{P}_Y(u)_*$ is a split monomorphism.

(b) Denote by $j : \mathrm{Spl}(\Sigma) \to \mathbb{P}_Y(\mathcal{E})$ the open complement of the closed image of $\mathbb{P}_Y(u)$. Prove that the following composition is an isomorphism of locally free sheaves on $\mathrm{Spl}(\Sigma)$,

$$j^*\pi^*\mathcal{O}_Y^\vee \xrightarrow{j^*\pi^*v^\dagger} j^*\pi^*\mathcal{E}^\vee \xrightarrow{j^*\phi} j^*\mathcal{O}(1).$$

Conclude that there exists a unique splitting

$$\sigma : \jmath^*\pi^*\mathcal{O}_Y \to j^*\pi^*\mathcal{E}$$

of the pullback $(\pi \circ j)^*\Sigma$ of Σ such that $\sigma^\dagger$ equals $j^*\phi$ as invertible quotients.

Prove that the pair

$$((\pi \circ j) : \mathrm{Spl}(\Sigma) \to Y, \sigma)$$

represents the functor on Y-schemes of splittings of the pullback of Σ. Use this to define a natural Y-action on $\mathrm{Spl}(\Sigma)$ of the (additive) Y-group scheme $\mathbb{V}(\mathcal{F})$ representing sections of the pullback of $\mathcal{F}$, or equivalently, representing homomorphisms $\mathcal{F}^\vee \to \mathcal{O}$ (this is the formulation that extends more readily to coherent sheaves). Prove that this action makes $\mathrm{Spl}(\Sigma)$ into a Zariski local torsor for $\mathbb{V}(\mathcal{F})$.

(c) Use **(a)** to prove that the open-closed sequence is a split short exact sequence,

$$0 \longrightarrow A_l(\mathbb{P}_Y(\mathcal{F})) \xrightarrow{\mathbb{P}_Y(u)_*} A_l(\mathbb{P}_Y(\mathcal{E}) \xrightarrow{j^*} A_l(\mathrm{Spl}(\Sigma)) \longrightarrow 0\cdot$$

Moreover, conclude that the flat pullback homomorphism,

$$(\pi_{\mathcal{E}} \circ j)^* : A_{l-r}(Y) \to A_l(\mathrm{Spl}(\Sigma)),$$

is an isomorphism. For every splitting s of Σ on Y, denoting by $\widetilde{s} : Y \to \mathrm{Spl}(\Sigma)$ the induced Y-morphism, define

$$\widetilde{s}^! : A_l(\mathrm{Spl}(\Sigma)) \to A_{l-r}(Y)$$

to be the inverse isomorphism.

Problem 3. Let $(X, \mathcal{O}_X)$ be a locally ringed space. For every open subset $U \subset X$, define $S_X(U) \subset \mathcal{O}_X(U)$ to be the subset of elements f such that for every $p \in U$, the germ f_p is a nonzerodivisor in the local ring $\mathcal{O}_{X,p}$.

(a) Prove that the subset $S_X(U)$ contains all invertible elements in $\mathcal{O}_X(U)$ and is multiplicatively closed, i.e., if both f and g are in $S_X(U)$, then also $f \cdot g$ is in $S_X(U)$. Denote by

$$i_X^{\mathrm{pre}}(U) : \mathcal{O}_X(U) \to \mathcal{K}_X^{\mathrm{pre}}(U)$$

the homomorphism to the associated ring of fractions $\mathcal{O}_X(U)[S_X(U)^{-1}]$.

(b) For every inclusion of open subsets, $V \subset U \subset X$, denoting by

$$\rho_V^U : \mathcal{O}_X(U) \to \mathcal{O}_X(V)$$

the restriction homomorphism, prove that $\rho_V^U(S_X(U))$ is contained in $S_X(V)$. Conclude that the composition $i_X^{\mathrm{pre}}(V) \circ \rho_V^U$ maps $S_X(U)$ to units in $\mathcal{K}_X^{\mathrm{pre}}(V)$. Therefore there exists a unique ring homomorphism,

$$\widetilde{\rho}_V^U : \mathcal{K}_X^{\mathrm{pre}}(U) \to \mathcal{K}_X^{\mathrm{pre}}(V),$$

such that $\widetilde{\rho}_V^U \circ i_X^{\text{pre}}(U)$ equals $i_X^{\text{pre}}(V) \circ \rho_V^U$.

(c) Prove that this rule, $((\mathcal{K}_X^{\text{pre}}(U))_U, (\widetilde{\rho}_V^U)_{U,V})$, is a presheaf of commutative, unital rings. Also prove that $(i_X^{\text{pre}}(U))_U$ is a homomorphism of sheaves of rings. Denote by

$$\theta_X : \mathcal{K}_X^{\text{pre}} \to \mathcal{K}_X$$

the associated sheaf. Prove that $\mathcal{K}_X$ has a unique structure of sheaf of rings such that θ_X is a homomorphism of preshveaves of rings. Prove that $\mathcal{K}_X$ is a sheaf of commutative, unital rings, and prove that $i_X := \theta_X \circ i_X^{\text{pre}}$ is a homomorphism of sheaves of commutative, unital rings.

For every point $p \in X$, prove that the associated homomorphism of stalks,

$$i_{X,p}^{\text{pre}} : \mathcal{O}_{X,p} \to \mathcal{K}_{X,p}^{\text{pre}},$$

is equivalent to the total ring of fractions of $\mathcal{O}_{X,p}$ as an $\mathcal{O}_{X,p}$-algebra. In particular, if p is a generic point of X, prove that $i_{X,p}^{\text{pre}}$ is an isomorphism.

(d) If $(X, \mathcal{O}_X)$ is a scheme that is integral, resp. covered by open affines U that have a unique associated point η, prove that $\mathcal{K}_X$ is a flasque sheaf that is constant, resp. locally constant, with value $\mathcal{O}_{X,\eta}$.

(e) Prove that $(\mathcal{K}_X, i_X : \mathcal{O}_X \hookrightarrow \mathcal{K}_X)$ is final among pairs $(\mathcal{R}, j : \mathcal{O}_X \hookrightarrow \mathcal{R})$ of a sheaf $\mathcal{R}$ of commutative, unital rings and an injective homomorphism of sheave of commutative, unital rings such that for every point p, the injective homomorphism of stalks j_p is equivalent to a ring of fractions $\mathcal{O}_{X,p}[T^{-1}]$ for some multiplicative subset T of $\mathcal{O}_{X,p}$. To prove this, consider the induced homomorphism of sheaves of $\mathcal{O}_X$-algebra,

$$j \otimes \text{Id} : \mathcal{K}_{\mathcal{X}} \to \mathcal{R} \otimes_{\mathcal{O}_X} \mathcal{K}_X.$$

Prove that for every point p, the stalk $(j \otimes \text{Id})_p$ is an isomorphism. Conclude that $j \otimes \text{Id}$ is an isomorphism. Next, show that

$$\text{Id} \otimes i_X : \mathcal{R} \to \mathcal{R} \otimes_{\mathcal{O}_X} \mathcal{K}_X$$

is injective on stalks, hence injective. Conclude that there exists a unique homomorphism of sheaves of unital rings,

$$h : \mathcal{R} \to \mathcal{K}_X$$

such that $h \circ j$ equals i_X.

Problem 4. Let R be a commutative, unital ring. For every element a in R, for every A-module M, denote by $L_{M,a}$ the A-module homomorphism,

$$L_{M,a} : M \to M, \quad m \mapsto a \cdot m.$$

As above, denote by S the subset of elements a of R such that for every prime ideal $\mathfrak{p}$, the image of a in $R_\mathfrak{p}$ is a nonzerodivisor, i.e., $\text{Ker}(L_{R_\mathfrak{p},a})$ is zero. Denote by S' the set of all nonzerodivisors of A, i.e., $\text{Ker}(L_{R,a})$ is zero.

(a) Use the fact that localization is exact to conclude that S' is contained in S.

(b) Let a be an element of R that is not in S', i.e., there exists nonzero b such that ab equals 0. Since b is nonzero, 1 is not in the ideal $\text{Ker}(L_{R,b})$. Use Zorn's Lemma to prove that there exists a maximal ideal $\mathfrak{m}$ such that $\text{Ker}(L_{R,b})$ is contained in $\mathfrak{m}$. Conclude that the image of b in $R_{\mathfrak{m}}$ is nonzero. Prove that the image of a in $R_{\mathfrak{m}}$ is a zerodivisor (possibly zero). Thus, a is not in S. Conclude that S equals S'.

(c) Let $(X, \mathcal{O}_X)$ be a scheme, let U be an open subset, and let a be an element of $\mathcal{O}_X(U)$ that is a zerodivisor. Thus there exists nonzero b in $\mathcal{O}_X(U)$ such that ab equals 0. Since $\mathcal{O}_X$ is a sheaf, and since U is a scheme, conclude that there exists an open affine V contained in U such that the image of b in V is nonzero. Using (c), prove that the image of a in $\mathcal{O}_X(V)$ is not in $S_X(V)$. Thus, by the previous problem, a is not in $S_X(U)$. Conclude that $S_X(U)$ is contained in the set S' of nonzerodivisors of $\mathcal{O}_X(U)$.

Problem 5. Let $\mathbb{A}^2_k$ be $\text{Spec} k[u, v]$. Let p be the closed point of $\mathbb{A}^2_k$ corresponding to the maximal ideal $\langle u, v\rangle$. Let $\nu : X_0 \to \mathbb{A}^2_k$ denote the blowing up of $\mathbb{A}^2_k$ along the ideal sheaf of $\{p\}$. This is covered by two basic open affines, $D_+(v) = \text{Spec} k[u/v, v]$ and $D_+(u) = \text{Spec} k[u, v/u]$, with the usual glueing data.

Denote by $i : E \hookrightarrow X_0$ the exceptional divisor, i.e., $E = \text{Proj} k[u, v]$. Denote by $\mathcal{O}_X$ the coherent sheaf of graded $\mathcal{O}_{X_0}$-algebras whose degree 0 part is $\mathcal{O}_{X_0}$, whose degree 1 part is $i_*\mathcal{O}_E(-1)$, and whose degree d part is zero for all $d > 1$, i.e.,

$$\mathcal{O}_X = \mathcal{O}_{X_0} \oplus \epsilon i_*\mathcal{O}_E(-1),$$

where, as usual, ϵ^2 equals 0.

(a) Prove that $(X_0, \mathcal{O}_X)$ is a scheme, and the obvious morphism f of locally ringed spaces,

$$(\text{Id}, \phi) : (X_0, \mathcal{O}_X) \to (X_0, \mathcal{O}_{X_0}),$$

is a finite morphism. In particular, the composition

$$\nu \circ f : (X_0, \mathcal{O}_X) \to (\mathbb{A}^2_k, \mathcal{O}_{\mathbb{A}^2})$$

is a projective morphism.

(b) Prove that the natural pullback map,

$$(\nu \circ f)^* : k[u, v] \to \mathcal{O}_X(X_0)$$

is a ring isomorphism. In particular, the set S' of nonzerodivisors in $\mathcal{O}_X(X_0)$ equals the image of $k[u, v] \setminus \{0\}$.

(c) Denoting by η the generic point of E, prove that the stalk $\mathcal{O}_{X_0,\eta}$, considered as a subring of the fraction field $k(u, v)$, equals the DVR

$$\mathcal{O}_{X_0,\eta} = (k(u/v)[v])_{\langle v\rangle} = (k(v/u)[u])_{\langle u\rangle}$$

with prime ideal

$$\mathfrak{p}_\eta = v(k(u/v)[v])_{\langle v\rangle} = u(k(v/u)[u])_{\langle u\rangle}$$

and with residue field $k(u/v) = k(v/u)$. Prove that the stalk $\mathcal{O}_{X,\eta}$ equals the $\mathcal{O}_{X_0,\eta}$-algebra,

$$\mathcal{O}_{X_0,\eta} = (k(u/v)[v])_{\langle v\rangle} \oplus \epsilon(k(u/v)[v])_{\langle v\rangle}/\langle v\rangle = (k(v/u)[u])_{\langle u\rangle} \oplus \epsilon(k(v/u)[u])_{\langle u\rangle}/\langle u\rangle.$$

For every element $f \in \langle u, v\rangle \subset k[u, v]$, prove that the image of f in $\mathcal{O}_{X_0,\eta}$ is in the prime ideal $\mathfrak{p}_\eta$. Conclude that the image in $\mathcal{O}_{X,\eta}$ is in the annihilator of the nonzero element ϵ. Thus, $S_X(X)$ equals the image of $k[u, v] \setminus \langle u, v\rangle$, not the set S'.

(d) Prove that $D_+(v)$ is an affine open subset of X_0 with

$$\mathcal{O}_{X_0}(D_+(v)) = k[u/v, v], \quad \mathcal{O}_X(D_+(v)) = k[u/v, v] \oplus \epsilon"1/v"k[u/v, v]/vk[u, v],$$

where "$1/v$" is just a placeholder. Show that $\mathcal{K}_X(D_+(v))$ is the ring of fractions,

$$\mathcal{K}_X(D_+(v)) = (k[u/v, v])_{\langle v\rangle} \oplus \epsilon"1/v"(k[u/v, v])_{\langle v\rangle}/\langle v\rangle = (k(u/v)[v])_{\langle v\rangle} \oplus \epsilon"1/v"k(u/v).$$

In particular, $\mathcal{K}_X(f^{-1}D(v))$ equals $k(u, v) \oplus \epsilon\{0\}$. For every $a \in k[u, v] \setminus \langle u, v\rangle$, observe that a is already an invertible element in $k(u/v)[v]_{\langle v\rangle}$, so that $\mathcal{K}_X(D_+(v) \cap f^{-1}D(a))$ equals $\mathcal{K}_X(D_+(v))$.

Next prove the analogous result for $D_+(u)$. Using the basic open affine cover $(D_+(v), D_+(u))$ of X_0, conclude that the ring homomorphism

$$i_X(X) : \mathcal{O}_X(X) \to \mathcal{K}_X(X)$$

equals

$$k[u, v] \mapsto \mathcal{O}_{X,\eta} \oplus \mathcal{O}_{X,\eta} \oplus \eta\mathcal{O}_{X,\eta}/\mathfrak{p}_\eta.$$

Thus $\theta_X(X)$ is not surjective. So the presheaf $\mathcal{K}_X^{\text{pre}}$ is not a sheaf.

Problem 6. This is similar to the last example. Let Y_0 be $\mathbb{A}^2_k = \text{Spec}k[u, v]$. Let $\mathcal{O}_Y$ be the sheaf

$$\mathcal{O}_Y = \mathcal{O}_{Y_0} \oplus \epsilon\kappa(p),$$

where $\kappa(p)$ is the skyscraper sheaf supported at p whose stalk equals the residue field $\kappa(p)$. For every $a \in k[u, v] \setminus \langle u, v\rangle$, we have

$$\mathcal{K}_Y^{\text{pre}}(D(a)) = k[u, v]_{\langle u,v\rangle} \oplus \epsilon k[u, v]/\langle u, v\rangle = \mathcal{O}_{Y_0,p} \oplus \epsilon\kappa(p).$$

On the other hand, for every nonzero $b \in \langle u, v\rangle$, we have

$$\mathcal{K}_Y^{\text{pre}}(D(b)) = k(u, v) \oplus \epsilon\{0\} = \text{Frac}(Y_0) \oplus \epsilon\{0\}.$$

Thus, $\mathcal{K}_Y$ is the sheaf on Y_0 that associates $R = \mathcal{O}_{Y_0,p} \oplus \epsilon\kappa(p)$ to every open subset containing p and that associates $\text{Frac}(Y_0) \oplus \epsilon\{0\}$ to every nonempty open subset that does not contain p. By adjunction, there is a natural map,

$$\psi : \mathcal{K}_Y(Y) \otimes_{\mathcal{O}_Y(Y)} \mathcal{O}_Y\mathcal{K}_Y.$$

Since Y is affine, $\mathcal{K}_Y$ is quasi-coherent if and only if ψ is an isomorphism. But now, on the open subset $D(u)$, the associated ring homomorphism is,

$$\psi(D(u)) : (k[u,v])_{\langle u,v\rangle} \otimes_{k[u,v]} k[u,v][1/u] \to k(u,v),$$

which is not surjective: the element $1/v$ is not in the image. Thus, $\mathcal{K}_X$ is not a quasi-coherent $\mathcal{O}_Y$-module.

A bit more precisely, prove that ψ is injective, and prove that $\text{Coker}(\psi)$ equals the non-quasi-coherent $\mathcal{O}_Y$-module $i_!(\mathcal{K}_W/\mathcal{O}_W)$ where $i : W \hookrightarrow Y_0$ is the open immersion of the complement of $\{p\}$.

Problem 7. Find an example of a finite type, separated k-scheme $(X, \mathcal{O}_X)$ and an invertible $\mathcal{O}_X$-module $\mathcal{L}$ that does not admit any injective $\mathcal{O}_X$-module homomorphism into $\mathcal{K}_X$. These examples are rather rare, cf. Kleiman's example in *Ample Subvarieties of Algebraic Varieties* by Robin Hartshorne.

Problem 8. Let $(X, \mathcal{O}_X)$ be a nodal plane cubic, e.g., $\text{Proj}k[u,v,w]/\langle v^2w - u^2(w-u)\rangle$. Let $p \in X$ be the node. Find an example of a Cartier divisor $D = (\tau : \mathcal{L}^\vee \hookrightarrow \mathcal{K}_X)$ such that the support of D equals $\{p\}$, yet the associated cycle $[D]$ in $Z_1(X)$ is zero.

Problem 9. Assume that k does not have characteristic 2. Starting from

$$\mathbb{G}_m^2 = \text{Spec } A, \quad A = k[s, s^{-1}, t, t^{-1}],$$

let p be $(1,1)$, and let q be $(-1,1)$. Let

$$f : \mathbb{G}_m^2 \to X$$

be the universal finite morphism to a seminormal scheme such that $f(p)$ equals $f(q)$. Explicitly, X is the affine scheme

$$X = \text{Spec } B, \quad B = k[s^2, s^{-2}, s^{-1}-s, t, t^{-1}, s^{-1}(1-t)] = k[\sigma, \sigma^{-1}, \rho, t, t^{-1}, r]/\langle \sigma\rho^2 - (\sigma-1)^2, \sigma r^2 - (t-1)^2, (\sigma-1)r$$

where $\sigma = s^2$, $\rho = s^{-1}(1-s^2)$, and $r = s^{-1}(1-t)$. Since X is integral, by **Problem 3(d)**, $\mathcal{K}_X$ is the constant sheaf with value $k(s,t)$. Let D be the principal Cartier divisor on X associated to $s \in k(s,t)^\times$. Prove that the support of the Cartier divisor is the singleton set $\{f(p)\} = \{f(q)\}$. In particular, the irreducible components of a Cartier divisor in an integral scheme need not have codimension 1. On the other hand, the irreducible components of a Cartier divisor do have codimension 1 if the integral scheme is normal, or even just $S2$ (Serre's Condition 2).

Problem 10. Let X be an integral, normal scheme of dimension n, and let $D = (\tau : \mathcal{L}^\vee \to \mathcal{K}_X)$ be a Cartier divisor. Let the associated cycle $[D]$ be $a_+ - a_-$, where both a_+ and a_- in $Z_{n-1}(X)$ have nonnegative coefficients.

(a) If X is regular, prove that there exist unique Cartier divisors

$$D_+ = (\tau_+ : \mathcal{L}_+^\vee \hookrightarrow \mathcal{K}_X), \quad D_- = (\tau_- : \mathcal{L}_-^\vee \hookrightarrow \mathcal{K}_X),$$

on X such that $[D_+]$ equals a_+, resp. $[D_-]$ equals a_-. Moreover, prove that D_+ and D_- are effective Cartier divisors, i.e., the images of τ_+ and τ_- are contained in $\mathcal{O}_X$ with its standard embedding i_X in $\mathcal{K}_X$. (Just to keep that convention straight, this means that the transposes s_+, resp. s_-, of τ_+, resp. τ_-, induce nonzero global sections $\mathcal{O}_X \hookrightarrow \mathcal{L}_+$, resp. $\mathcal{O}_X \hookrightarrow \mathcal{L}_-$.) Finally, prove that D equals $D_+ - D_-$ as Cartier divisors.

(b) Let $\mathbb{A}^4_k$ be $\text{Spec} k[s,t,u,v]$, and let $X \hookrightarrow \mathbb{A}^4_k$ be the closed subscheme $\text{Spec} k[s,t,u,v]/\langle sv - tu \rangle$. Prove that X is integral and normal. Thus, by **Problem 3(d)**, $\mathcal{K}_X$ is the constant sheaf with value $k(X)$. Let D be the principal Cartier divisor on X associated to the element $\overline{t}/\overline{s} = \overline{v}/\overline{u}$ inside $k(X)$. Prove that $[D]$ equals $1[A_+] - 1[A_-]$, where A_+ denotes $Z(s,u)$ and A_- denotes $Z(t,v)$. Prove that there is no Cartier divisor D_+, resp. D_-, in X such that $[D_+]$ equals $[A_+]$, resp. $[D_-]$ equals $[A_-]$.