MAT 614 Problem Set 3

Homework Policy. Please read through all the problems. I will be happy to discuss the solutions during office hours.

Problems.

Problem 1. Let K be a commutative ring with 1. Denote by $K[t]$ the K-algebra of polynomials in one variable t with coefficients in K. Denote by $K[\![t]\!]$ the K-algebra of power series in one variable t with coefficients in K. There is a natural inclusion of $K[t]$ in $K[\![t]\!]$. Let $\lambda(t) = a_0 - a_1 t + \cdots + (-1)^r a_r t^r$ be an element in $K[t]$.

(a) Prove that there exists a (left-right) inverse $\sigma(t)$ in $K[\![t]\!]$ to the image of $\lambda(t)$ if and only if a_0 is invertible in K. In this case, prove that the inverse $\sigma(t)$ is unique. More generally, for every element $a_0 - a_1 t + \dots$ in $K[\![t]\!]$, prove that the element is invertible if and only if a_0 is invertible in K, in which case the inverse is unique.

(b) Assume that $\lambda(t) \in K[t]$ has an inverse $\sigma(t)$ in $K[\![t]\!]$. Prove that the following sequence of $K[t]$-modules is well-defined and exact, where the arrows are scaling by the given element in $K[\![t]\!]$.

$$0 \longrightarrow K[t]/\lambda(t)K[t] \xrightarrow{\sigma(t)} K[\![t]\!]/K[t] \xrightarrow{\lambda(t)} K[\![t]\!]/K[t] \longrightarrow 0.$$

(c) Assume that both a_0 and a_r are invertible, so that $\lambda(t)$ is associate to a monic polynomial. Then show that $K[t]/\lambda(t)K[t]$ is a free K-module with basis $(\overline{1}, \overline{t}, \dots, \overline{t}^{r-1})$.

Problem 2. Let $(X, \mathcal{O}_X)$ be a scheme. Let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of (finite) rank r.

(a) Prove that there exists a pair

$$(\pi_{\mathcal{E}} : \mathbb{P}\mathcal{E} \to X, \phi_{\mathcal{E}} : \pi_{\mathcal{E}}^* \mathcal{E} \to \mathcal{O}(1))$$

representing the contravariant functor,

$$F : \mathbf{Schemes}_X \to \mathbf{Sets},$$

associating to every X-scheme, $p : T \to X$, the set of invertible quotients of $p^*\mathcal{E}$. Moreover, prove that, as an X-scheme, $\mathbb{P}\mathcal{E}$ is isomorphic to $\mathbb{P}_X^{r-1}$ Zariski locally on X.

(b) Denote the "twist" of $\phi_{\mathcal{E}}$ by $\mathcal{O}(-1)$ by

$$\psi_{\mathcal{E}} : \pi_{\mathcal{E}}^* \mathcal{E}(-1) \to \mathcal{O}_{\mathbb{P}\mathcal{E}}.$$

Denote the domain by $K_1(\psi_{\mathcal{E}})$,

$$K_1(\psi_{\mathcal{E}}) := \pi_{\mathcal{E}}^*\mathcal{E}(-1).$$

For every $p = 0, \dots, r$, denote by $K_p(\psi_{\mathcal{E}})$ the locally free $\mathcal{O}_{\mathbb{P}\mathcal{E}}$-module,

$$K_p(\psi_{\mathcal{E}}) := \bigwedge_{\mathcal{O}_{\mathbb{P}\mathcal{E}}}^p K_1(\psi_{\mathcal{E}}) \cong \pi_{\mathcal{E}}^* \bigwedge_{\mathcal{O}_X}^p \mathcal{E}(-p).$$

Thus, the direct sum,

$$K_\bullet(\psi_{\mathcal{E}}) = \bigoplus_{p=0}^r K_p(\psi_{\mathcal{E}}),$$

is naturally the exterior algebra on $K_1(\psi_{\mathcal{E}})$.

Prove that there is a unique collection of $\mathcal{O}_{\mathbb{P}\mathcal{E}}$-module homomorphisms,

$$d_p : K_p(\psi_{\mathcal{E}}) \to K_{p-1}(\psi_{\mathcal{E}}),$$

for $p = 1, \dots, r$ such that d_1 equals $\psi_{\mathcal{E}}$, and so that $(K_\bullet(\psi_{\mathcal{E}}), d_\bullet)$ satisfies the graded Leibniz rule, i.e., for every open $U \subset \mathbb{P}\mathcal{E}$, for every $a \in K_p(\psi_{\mathcal{E}})$, and for every $b \in K_q(\psi_{\mathcal{E}})$,

$$d_{p+q}(a \wedge b) = d_p(a) \wedge b + (-1)^p a \wedge d_q(b).$$

(c) Show that the complex $(K_\bullet(\psi_{\mathcal{E}}), d_\bullet)$ is exact. Use the Euler sequence to prove that $\mathrm{Ker}(d_1)$ is canonically isomorphic to the sheaf of relative differentials Ω_π, and, more generally,

$$\mathrm{Ker}(d_p) = \Omega_\pi^p = \bigwedge_{\mathcal{O}_{\mathbb{P}\mathcal{E}}}^p \Omega_\pi,$$

for every $p = 1, \dots, r$. In particular, for every $p = 1, \dots, r$, this gives a short exact sequence,

$$0 \longrightarrow \Omega_\pi^p \xrightarrow{u_{p,\mathcal{E}}} \pi^* \textstyle\bigwedge^p \mathcal{E}(-p) \xrightarrow{v_{p,\mathcal{E}}} \Omega_\pi^{p-1} \longrightarrow 0,$$

and it also gives a canonical isomorphism,

$$\Omega_\pi^{r-1} \cong \pi_{\mathcal{E}}^* \bigwedge^r \mathcal{E}(-r).$$

(d) Prove that for every integer $m < 0$, the pushforward sheaf,

$$\pi_{\mathcal{E},*}\mathcal{O}(m),$$

is the zero sheaf. Also, the natural $\mathcal{O}_X$-module homomorphism,

$$\pi^\# : \mathcal{O}_X \to \pi_{\mathcal{E},*}\mathcal{O}_{\mathbb{P}\mathcal{E}}$$

is an isomorphism.

Prove that for every integer $q = 1, \dots, r-2$ and for every integer m, the higher direct image sheaf,

$$R^q\pi_{\mathcal{E},*}\mathcal{O}(m),$$

is the zero sheaf on X.

Finally, prove that for every integer $m > -r$, the higher direct image sheaf

$$R^{r-1}\pi_{\mathcal{E},*}\mathcal{O}(m)$$

is the zero sheaf.

(e) Substitute the vanishing results in (d) into the long exact sequences of higher direct image sheaves associated to the short exact sequences in (c). Conclude that $R^q\pi_{\mathcal{E},*}\Omega^p_\pi$ vanishes unless p equals q, in which case it is canonically isomorphic to $\mathcal{O}_X$. In particular, conclude that there is a canonical isomorphism,

$$t_{\mathcal{E}} : R^{r-1}\pi_{\mathcal{E},*}\Omega^{r-1}_\pi \to \mathcal{O}_X.$$

(f) Via adjunction of π_* and π^*, associated to $\phi_{\mathcal{E}}$, there is a natural homomorphism of $\mathcal{O}_X$-modules,

$$\phi_{1,\mathcal{E}} : \mathcal{E} \to \pi_{\mathcal{E},*}\mathcal{O}(1).$$

Prove that $\phi_{1,\mathcal{E}}$ is an isomorphism. More generally, for every integer $d \geq 0$, the induced $\mathcal{O}_X$-module homomorphism,

$$\phi_{d,\mathcal{E}} : \mathrm{Sym}^d_{\mathcal{O}_X}\mathcal{E} \to \pi_{\mathcal{E},*}\mathcal{O}(d),$$

is an isomorphism. Combined with (d) above, this is also valid for $d < 0$ if we adopt the convention that $\mathrm{Sym}^d_{\mathcal{O}_X}\mathcal{E}$ is the zero sheaf for all $d < 0$.

Similarly, for every integer $d \geq 0$, prove that the natural pairing of $\mathcal{O}_X$-modules,

$$\pi_{\mathcal{E},*}\mathit{Hom}_{\mathcal{O}_{\mathbb{P}E}}(\Omega^{r-1}_\pi(-d), \Omega^{r-1}_\pi) \otimes_{\mathcal{O}_X} R^{r-1}\pi_{\mathcal{E},*}\Omega^{r-1}_\pi(-d) \to R^{r-1}\pi_{\mathcal{E},*}\Omega^{r-1}_\pi \xrightarrow{t_{\mathcal{E}}} \mathcal{O}_X,$$

is a perfect pairing, i.e., the induced $\mathcal{O}_X$-module homomorphism,

$$R^{r-1}\pi_{\mathcal{E},*}\Omega^{r-1}_\pi(-d) \to \mathit{Hom}_{\mathcal{O}_X}(\pi_{\mathcal{E},*}\mathit{Hom}_{\mathcal{O}_{\mathbb{P}E}}(\Omega^{r-1}_\pi(-d), \Omega^{r-1}_\pi), \mathcal{O}_X),$$

is an isomorphism, i.e.,

$$R^{r-1}\pi_{\mathcal{E},*}\Omega^{r-1}_\pi(-d) \cong \mathit{Hom}_{\mathcal{O}_X}(\mathrm{Sym}^d_{\mathcal{O}_X}\mathcal{E}, \mathcal{O}_X).$$

Combined with the projection formula, this gives a canonical isomorphism of $\mathcal{O}_X$-modules,

$$R^{r-1}\pi_{\mathcal{E},*}\mathcal{O}(-r-d) \cong \mathit{Hom}_{\mathcal{O}_X}(\mathrm{Sym}^d_{\mathcal{O}_X}\mathcal{E} \otimes_{\mathcal{O}_X} \bigwedge^r_{\mathcal{O}_X}\mathcal{E}, \mathcal{O}_X).$$

(g) For every integer $m > 0$, consider the exact complex of $\mathcal{O}_{\mathbb{P}\mathcal{E}}$-modules $(K_\bullet(\psi_{\mathcal{E}})(m), d_\bullet)$. Prove that for every $p = 0, \dots, r$ and for every $q > 0$, the higher direct image sheaf,

$$R^q\pi_{\mathcal{E},*}[K_p(\psi_{\mathcal{E}})(m)] = R^q\pi_{\mathcal{E},*}[\pi_{\mathcal{E}}^* \bigwedge_{\mathcal{O}_X}^p \mathcal{E}(m-q)],$$

is the zero sheaf. Moreover, the pushforward sheaf,

$$\pi_{\mathcal{E},*}[K_p(\psi_{\mathcal{E}})(m)] = \bigwedge_{\mathcal{O}_X}^p \mathcal{E} \otimes_{\mathcal{O}_X} \mathrm{Sym}_{\mathcal{O}_X}^{m-q}\mathcal{E},$$

is a locally free $\mathcal{O}_X$-module; the zero sheaf if $m < q$. Conclude that the pushforward complex of $\mathcal{O}_X$-modules,

$$(\pi_{\mathcal{E},*}[K_\bullet(\psi_{\mathcal{E}})(m)], \pi_{\mathcal{E},*}d_\bullet),$$

is an exact complex of locally free $\mathcal{O}_X$-modules.

In particular, for $m = 0$, in $K^0(X)$ we have the identity,

$$\sum_{p=0}^{r}(-1)^p[\mathrm{Sym}_{\mathcal{O}_X}^{-p}\mathcal{E} \otimes_{\mathcal{O}_X} \bigwedge_{\mathcal{O}_X}^p \mathcal{E}] = [\mathcal{O}_X].$$

Also, for every $m > 0$, we have the identity,

$$\sum_{p=0}^{r}(-1)^p[\mathrm{Sym}_{\mathcal{O}_X}^{m-p}\mathcal{E} \otimes_{\mathcal{O}_X} \bigwedge_{\mathcal{O}_X}^p \mathcal{E}] = 0.$$

Problem 3. Assume now that X is quasi-compact and connected. Thus also $\mathbb{P}\mathcal{E}$ is quasi-compact and connected.

(a) For every finite rank, locally free $\mathcal{O}_{\mathbb{P}\mathcal{E}}$-module, $\mathcal{F}$, prove that there exists an integer m_0, depending on $\mathcal{F}$, such that for every integer $m \geq m_0$, $\pi_{\mathcal{E},*}\mathcal{F}(m)$ is a finite rank, locally free $\mathcal{O}_X$-module and for every integer $q > 0$, the higher direct image sheaf,

$$R^q\pi_{\mathcal{E},*}\mathcal{F}(m)$$

is the zero sheaf. In particular, the image in $K^0(X)\llbracket t \rrbracket / K^0(X)[t]$ of the element,

$$\sum_{m \geq m_0} [\pi_{\mathcal{E},*}\mathcal{F}(m)]t^m$$

is independent of the choice of m_0. Denote this image by $P(t, \mathcal{F})$.

(b) For every short exact sequence of finite rank, locally free $\mathcal{O}_{\mathbb{P}\mathcal{E}}$-modules,

$$0 \longrightarrow \mathcal{F}_1 \xrightarrow{e} \mathcal{F}_2 \xrightarrow{f} \mathcal{F}_3 \longrightarrow 0,$$

choosing m_0 to be the maximum of $m_0(\mathcal{F}_i)$, $i = 1, 2, 3$, prove that for every integer $m \geq m_0$, the pushforward

$$0 \longrightarrow \pi_{\mathcal{E},*}\mathcal{F}_1 \xrightarrow{e} \mathcal{F}_2 \xrightarrow{f} \mathcal{F}_3 \longrightarrow 0,$$

is a short exact sequence of finite rank, locally free $\mathcal{O}_X$-modules. Conclude that $P(t, \mathcal{F}_2)$ equals $P(t, \mathcal{F}_1) + P(t, \mathcal{F}_3)$. Thus $P(t, -)$ is a generalized Euler characteristic. Hence there exists a unique group homomorphism,

$$P(t, -) : K^0(\mathbb{P}\mathcal{E}) \to K^0(X) [\![t]\!] / K^0(X)[t],$$

that extends the definition of $P(t, -)$ above. In particular, check that $P(t, -)$ is a homomorphism of $K^0(X)$-modules.

(c) For every a in $K^0(\mathbb{P}\mathcal{E})$, check that $tP(t, a[\mathcal{O}(1)])$ equals $P(t, a)$, and $P(t, a[\mathcal{O}(-1)])$ equals $tP(t, a)$. Thus, if we extend the $K^0(X)$-module structure on $K^0(\mathbb{P}\mathcal{E})$ to a $K^0(X)[t]$-module structure by defining $t * a := a[\mathcal{O}(-1)]$, then $P(t, -)$ is a homomorphism of $K^0(X)[t]$-modules.

(d) For the $K^0(X)[t]$-module structure from the previous part, check that the class in $K^0(\mathbb{P}\mathcal{E})$ corresponding to the complex $(K_\bullet(\psi_\mathcal{E}), d_\bullet)$ is the image of the polynomial in $K^0(X)[t]$,

$$\lambda(t) = \sum_{p=0}^{r} (-1)^p [\bigwedge_{\mathcal{O}_X}^{p} \mathcal{E}] t^p.$$

Use **Problem 1(a)** to prove that $\lambda(t)$ is invertible in $K^0(X) [\![t]\!]$. More precisely, use **Problem 2(g)** to prove that an inverse is,

$$\sigma(t) = \sum_{m=0}^{\infty} [\mathrm{Sym}^m_{\mathcal{O}_X} \mathcal{E}] t^m.$$

Thus, $P(t, -)$ has image contained in the kernel of the homomorphism of $K^0(X)[t]$-modules,

$$K^o(X) [\![t]\!] / K^o(X)[t] \xrightarrow{\lambda(t)} K^o(X) [\![t]\!] / K^o(X)[t].$$

(e) Use **Problem 2(b)** to conclude that there is a unique homomorphism of $K^0(X)[t]$-modules,

$$R(t, -) : K^0(\mathbb{P}\mathcal{E}) \to K^0(X)[t]/\lambda(t)K^0(X)[t],$$

such that $P(t, -)$ equals $\sigma(t)R(t, -)$. Moreover, since $P(t, [\mathcal{O}_{\mathbb{P}\mathcal{E}}])$ equals the image of $\sigma(t)$, conclude that $R(t, \mathcal{O}_{\mathbb{P}\mathcal{E}})$ equals 1 in $K^0(X)[t]/\lambda(t)K^0(X)[t]$. Conclude that $R(t, -)$ is surjective. Moreover, since $\lambda(t)$ annihilates $K^0(\mathbb{P}\mathcal{E})$, conclude that $R(t, -)$ is a homomorphism of $K^0(X)[t]/\lambda(t)K^0(X)[t]$-modules (that sends 1 to 1).

Finally, use the relation $\lambda(t) = 0$ in $K^0(\mathbb{P}\mathcal{E})$ to prove that for every integer m (possibly negative), the class $[\mathcal{O}(-m)] = t^m$ is in the $K^0(X)[t]/\lambda(t)K^0(X)[t]$-module generated by 1.

Problem 4. Denote by $w_\mathcal{E}$ the twist of $u_{1,\mathcal{E}}$ by $\mathcal{O}(1)$,

$$w_\mathcal{E} : \Omega^1_\pi(1) \to \pi^*\mathcal{E}.$$

Form the Cartesian diagram,

$$\begin{array}{ccc} \mathbb{P}\mathcal{E} \times_X \mathbb{P}\mathcal{E} & \xrightarrow{\mathrm{pr}_1} & \mathbb{P}\mathcal{E} \\ {\scriptstyle \mathrm{pr}_2}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathbb{P}\mathcal{E} & \xrightarrow[\pi]{} & X \end{array}.$$

Using the natural isomorphism of coherent sheaves,

$$\mathrm{pr}_1^*\pi^*\mathcal{E} \cong (\pi \circ \mathrm{pr}_1)^*\mathcal{E} = (\pi \circ \mathrm{pr}_2)^*\mathcal{E} \cong \mathrm{pr}_2^*\pi^*\mathcal{E},$$

form the composition homomorphism,

$$\mathrm{pr}_1^*[\Omega_\pi^1(1)] \xrightarrow{\mathrm{pr}_1^* w_{\mathcal{E}}} \mathrm{pr}_1^*\pi^*\mathcal{E} \cong \mathrm{pr}_2^*\pi^*\mathcal{E} \xrightarrow{\mathrm{pr}_2^*\phi_{\mathcal{E}}} \mathrm{pr}_2^* O(1).$$

Twisting by $\mathrm{pr}_2^* O(-1)$ gives a morphism of coherent sheaves on $\mathbb{P}\mathcal{E} \times_X \mathbb{P}\mathcal{E}$,

$$\alpha_{\mathcal{E}} : \mathrm{pr}_1^*[\Omega_\pi^1(1)] \otimes_{\mathcal{O}} \mathrm{pr}_2^*[\mathcal{O}(-1)] \to \mathcal{O}_{\mathbb{P}\mathcal{E} \times_X \mathbb{P}\mathcal{E}}.$$

Denote by $K_1(\alpha_{\mathcal{E}})$ the domain,

$$K_1(\alpha_{\mathcal{E}}) := \mathrm{pr}_1^*[\Omega_\pi^1(1)] \otimes_{\mathcal{O}} \mathrm{pr}_2^*[\mathcal{O}(-1)]$$

(a) Prove that the image of $\alpha_{\mathcal{E}}$ equals the ideal sheaf $\mathcal{I}_\Delta$ of the diagonal closed immersion,

$$\Delta : \mathbb{P}\mathcal{E} \to \mathbb{P}\mathcal{E} \times_X \mathbb{P}\mathcal{E}.$$

Thus, there is an exact sequence,

$$K_1(\alpha_{\mathcal{E}}) \xrightarrow{\alpha_{\mathcal{E}}} \mathcal{O}_{\mathbb{P}\mathcal{E} \times_X \mathbb{P}\mathcal{E}} \xrightarrow{\Delta^\#} \Delta_*\mathcal{O}_{\mathbb{P}\mathcal{E}} \longrightarrow 0.$$

(b) As in **Problem 2(b)**, for $p = 0, \dots, r-1$, form the locally free sheaves on $\mathbb{P}\mathcal{E} \times_X \mathbb{P}\mathcal{E}$,

$$K_p(\alpha_{\mathcal{E}}) = \bigwedge_{\mathcal{O}}^p K_1(\alpha_{\mathcal{E}}) \cong \mathrm{pr}_1^*[\Omega_\pi^p(p)] \otimes_{\mathcal{O}} \mathrm{pr}_2^*[O(-p)].$$

Prove that there is a unique collection of $\mathcal{O}_{\mathbb{P}\mathcal{E}}$-module homomorphisms,

$$d_p : K_p(\alpha_{\mathcal{E}}) \to K_{p-1}(\alpha_{\mathcal{E}}),$$

for $p = 1, \dots, r-1$ such that d_1 equals $\alpha_{\mathcal{E}}$, and so that $(K_\bullet(\alpha_{\mathcal{E}}), d_\bullet)$ satisfies the graded Leibniz rule.

(c) Read about Koszul complexes in a textbook on commutative algebra or homological algebra. Check that with respect to local trivializations of $K_1(\alpha_{\mathcal{E}})$, the coordinates of $\alpha_{\mathcal{E}}$ are a regular sequence. Conclude that the complex $(K_\bullet(\alpha_{\mathcal{E}}), d_\bullet)$ is *acyclic*, i.e., it is exact except in degree $q = 0$, where the homology is $\Delta_*\mathcal{O}_{\mathbb{P}\mathcal{E}}$. Thus the complex is a locally free resolution of $\Delta_*\mathcal{O}_{\mathbb{P}\mathcal{E}}$. This is the **Beilinson resolution** of the diagonal (this also works for all the standard projective homogeneous varieties).

(c) For every locally free sheaf $\mathcal{F}$ on $\mathbb{P}\mathcal{E}$, prove that the tensor product complex,

$$(\mathrm{pr}_1^*\mathcal{F} \otimes_{\mathcal{O}} K_\bullet(\alpha_{\mathcal{E}}), d_\bullet),$$

is a locally free resolution of $\Delta_*\mathcal{F}$. For every $p = 0, \dots, r-1$, check that there is a natural isomorphism of locally free sheaves,

$$\mathrm{pr}_1^*\mathcal{F} \otimes_{\mathcal{O}} K_p(\alpha_{\mathcal{E}}) \cong \mathrm{pr}_1^*[\mathcal{F} \otimes_{\mathcal{O}_{\mathbb{P}\mathcal{E}}} \Omega_\pi^p(p)] \otimes_{\mathcal{O}} \mathrm{pr}_2^*\mathcal{O}(-p).$$

By **Problem 3(a)**, there exists an integer m_0 such that for all integers $m \geq m_0$, for every $p = 0, \dots, r-1$,

$$\pi_{\mathcal{E},*}[\mathcal{F} \otimes_{\mathcal{O}_{\mathbb{P}\mathcal{E}}} \Omega_\pi^p(p+m)]$$

is a locally free $\mathcal{O}_X$-module and, for every integer $q > 0$, the higher direct image sheaf,

$$R^q\pi_{\mathcal{E},*}[\mathcal{F} \otimes_{\mathcal{O}_{\mathbb{P}\mathcal{E}}} \Omega_\pi^p(p+m)],$$

is the zero sheaf. Conclude that the complex $(\mathrm{pr}_1^*\mathcal{F}(m) \otimes_{\mathcal{O}} K_\bullet(\alpha_{\mathcal{E}}), d_\bullet)$ is acyclic for $\mathrm{pr}_{2,*}$, i.e., the complex on $\mathbb{P}\mathcal{E}$

$$(\mathrm{pr}_{2,*}[\mathrm{pr}_1^*\mathcal{F}(m) \otimes_{\mathcal{O}} K_\bullet(\alpha_{\mathcal{E}})], \mathrm{pr}_{2,*}d_\bullet)$$

is an acyclic complex of locally free sheaves that is a resolution of

$$\mathrm{pr}_{2,*}\Delta_*\mathcal{F}(m) \cong \mathcal{F}(m).$$

Finally, twisting by $\mathcal{O}(-m)$, this gives a locally free resolution of $\mathcal{F}$ by terms

$$\mathrm{pr}_{2,*}[\mathrm{pr}_1^*\mathcal{F}(m) \otimes_{\mathcal{O}} K_p(\alpha_{\mathcal{E}})](-m) \cong$$

$$\pi^*\pi_*[\mathcal{F} \otimes_{\mathcal{O}_{\mathbb{P}\mathcal{E}}} \Omega_\pi^p(p+m)](-m).$$

(d) Conclude that the class in $K^0(\mathbb{P}\mathcal{E})$ associated to

$$\mathrm{pr}_1^*\mathcal{F}(m) \otimes_{\mathcal{O}} K_p(\alpha_{\mathcal{E}})$$

is in the $K^0(X)$-submodule generated by all of the classes $[\mathcal{O}(-m)]$. Now use **(e)** to conclude that this is in the $K^0(X)[t]/\lambda(t)K^0(X)[t]$-submodule generated by 1. Since the complex is an exact resolution of $\mathcal{F}$, conclude that also $[\mathcal{F}]$ is in this submodule. Since this holds for every locally free sheaf $\mathcal{F}$, conclude that $K^0(\mathbb{P}\mathcal{E})$ equals the $K^0(X)[t]/\lambda(t)K^0(X)[t]$-submodule generated by 1. Finally, conclude that the homomorphism,

$$R(t,-) : K^0(\mathbb{P}\mathcal{E}) \to K^0(X)[t]/\lambda(t)K^0(X)[t],$$

is an isomorphism of $K^0(X)[t]/\lambda(t)K^0(X)[t]$-algebras.

(e) The isomorphism above is equivalent to another isomorphism that is more conventional. First of all, note that t is invertible modulo $\lambda(t)$. Thus, it makes sense to consider the rational expression,

$$t^{-r}\lambda(t) = \sum_{p=0}^{r}(-1)^p[\bigwedge_{\mathcal{O}_X}^{p}\mathcal{E}](t^{-1})^{r-p}.$$

Define s to be

$$s = t^{-1} - 1 = [\mathcal{O}(1)] - [\mathcal{O}_{\mathbb{P}\mathcal{E}}].$$

Recall from lecture that this is the first Chern class of $\mathcal{O}(1)$ in K-theory (with respect to one of the two standard conventions). Using the substitution $t^{-1} = 1+s$, rewrite,

$$(1+s)^r\lambda(1/(1+s)) = \sum_{p=0}^{r}(-1)^p[\bigwedge_{\mathcal{O}_X}^{p}\mathcal{E}](1+s)^{r-p} = \sum_{m=0}^{r}\left(\sum_{l=0}^{r-m}(-1)^{l+m-r}\binom{r-l}{m}[\bigwedge_{\mathcal{O}_X}^{l}\mathcal{E}]\right)(-1)^{r-m}s^m.$$

Defining the *Chern classes* in $K^0(X)$,

$$c_m(\mathcal{E}) := \sum_{j=0}^{m}(-1)^j\binom{r-m+j}{r-m}[\bigwedge_{\mathcal{O}_X}^{m-j}\mathcal{E}],$$

the polynomial is,

$$\mu_{\mathcal{E}}(s) = (1+s)^r\lambda(1/(1+s)) = \sum_{m=0}^{r}(-1)^m c_m(\mathcal{E})s^{r-m}.$$

Altogether, this gives an isomorphism of $K^0(X)$-algebras,

$$K^0(\mathbb{P}\mathcal{E}) \cong K^0(X)[s]/\langle s^r - c_1(\mathcal{E})s^{r-1} + \cdots + (-1)^m c_m(\mathcal{E})s^{r-m} + \cdots + (-1)^r c_r(\mathcal{E})\rangle,$$

where s corresponds to the first Chern class of $\mathcal{O}(1)$, i.e., $s = [\mathcal{O}(1)] - [\mathcal{O}_{\mathbb{P}\mathcal{E}}]$.

(f) In particular, observe that the ring homomorphism,

$$K^0(\pi_{\mathcal{E}}) : K^0(X) \to K^0(\mathbb{P}\mathcal{E}),$$

is injective. Thus, for every locally free $\mathcal{O}_X$-module $\mathcal{E}$ of finite rank r, there exists a projective morphism that is Zariski locally a product (admitting Zariski local sections)

$$\pi : P \to X,$$

such that $K^0(\pi)$ is injective and such that there exists an invertible quotient of $\pi^*\mathcal{E}$ on P. By using induction on r, and applying the previous step to the kernel of the invertible quotient on P, prove that there exists a projective morphism that is Zariski locally a product (admitting Zariski local sections)

$$\rho : Q \to X,$$

such that $K^0(\rho)$ is injective and such that $\rho^*\mathcal{E}$ admits a filtration by $\mathcal{O}_Q$-submodules,

$$\rho^*\mathcal{E} = F^0\rho^*\mathcal{E} \subset F^1\rho^*\mathcal{E} \subset \cdots \subset F^r\rho^*\mathcal{E} = \underline{0},$$

such that for $q = 0, \dots, r-1$, the associated graded $\mathcal{O}_Q$-module,

$$F^q\rho^*\mathcal{E}/F^{q+1}\rho^*\mathcal{E}$$

is an invertible $\mathcal{O}_Q$-module.

This leads to the *splitting principle*. Let there be given a class of schemes and morphisms between these schemes such that for each scheme X as above, and for every locally free $\mathcal{O}_X$-module $\mathcal{E}$ of finite rank r, the morphism ρ constructed above is in the class. Suppose given a rule that associates to certain collections of locally free sheaves $\mathcal{E}_1, \dots, \mathcal{E}_m$ on a scheme X in our class an element $a(X, \mathcal{E}_1, \dots, \mathcal{E}_m)$ in $K^0(X)$. Assume moreover that the rule is functorial, i.e., for every morphism $f : Y \to X$ in our class, $f^*a(X, \mathcal{E}_1, \dots, \mathcal{E}_m)$ equals $a(Y, f^*\mathcal{E}_1, \dots, f^*\mathcal{E}_m)$. Finally, assume that $a(X, \mathcal{E}_1, \dots, \mathcal{E}_m)$ equals 0 whenever the sheaves admit filtrations whose associated graded pieces are invertible quotients, i.e., whenever the locally free sheaves "split" as a sum of invertible sheaves in K-theory. Then prove that for all $(X, \mathcal{E}_1, \dots, \mathcal{E}_r)$, the class $a(X, \mathcal{E}_1, \dots, \mathcal{E}_r)$ equals 0. The idea is to introduce a morphism $\rho : Q \to X$ as above such that each $\rho^*\mathcal{E}_i$ has such a filtration, and then use that $K^0(\rho)$ is injective.

Problem 5. By **(a)**, the subset $1 + tK^0(X)[\![t]\!] \subset K^0(X)[\![t]\!]$ is an Abelian group under multiplication. For every locally free $\mathcal{O}_X$-module $\mathcal{E}$ of finite rank r, define the following element in $1 + tK^0(X)[\![t]\!]$,

$$\lambda(\mathcal{E}, t) = \sum_{p=0}^{r}[\bigwedge_{\mathcal{O}_X}^{p}\mathcal{E}]t^p = \sum_{p=0}^{\infty}[\bigwedge_{\mathcal{O}_X}^{p}\mathcal{E}]t^p.$$

Thus the polynomial $\lambda(t)$ from the previous problems equals $\lambda(\mathcal{E}, -t)$.

For every short exact sequence of locally free $\mathcal{O}_X$-modules of finite ranks r', resp. r, r'',

$$\Sigma : 0 \longrightarrow \mathcal{E}' \xrightarrow{u} \mathcal{E} \xrightarrow{v} \mathcal{E}'' \longrightarrow 0,$$

for every integer $p = 0, \dots, r$, for every integer $q = 0, \dots, p$, define the subsheaf $F^q_\Sigma \bigwedge^p_{\mathcal{O}_X} \mathcal{E}$ of $\bigwedge^p_{\mathcal{O}_X} \mathcal{E}$ to be the image of the composition,

$$\bigwedge_{\mathcal{O}_X}^{q}\mathcal{E}' \otimes_{\mathcal{O}_X} \bigwedge_{\mathcal{O}_X}^{p-q}\mathcal{E} \xrightarrow{\bigwedge^p u \otimes \mathrm{Id}} \bigwedge_{\mathcal{O}_X}^{q}\mathcal{E} \otimes_{\mathcal{O}_X} \bigwedge_{\mathcal{O}_X}^{p-q}\mathcal{E} \xrightarrow{-\wedge-} \bigwedge_{\mathcal{O}_X}^{p}\mathcal{E}.$$

(a) Prove that for every pair of integers p_1, p_2 with $0 \le p_1, p_2 \le r$ and for every pair of integers q_1, q_2 with $0 \le q_i \le p_i$, the multiplication homomorphism

$$F^{q_1}_\Sigma \bigwedge_{\mathcal{O}_X}^{p_1}\mathcal{E} \otimes_{\mathcal{O}_X} F^{q_2}_\Sigma \bigwedge_{\mathcal{O}_X}^{p_2}\mathcal{E} \xrightarrow{-\wedge-} \bigwedge_{\mathcal{O}_X}^{p_1+p_2}\mathcal{E}$$

surjects onto the subsheaf $F_\Sigma^{q_1+q_2}\bigwedge_{\mathcal{O}_X}^{p_1+p_2}\mathcal{E}$.

(b) Prove that $F_\Sigma^1\bigwedge_{\mathcal{O}_X}^p\mathcal{E}$ equals the kernel of the natural surjection,

$$\bigwedge^p(v):\bigwedge^p_{\mathcal{O}_X}\mathcal{E}\to\bigwedge^p_{\mathcal{O}_X}\mathcal{E}'',$$

so that associated graded sheaf F_Σ^0/F_Σ^1 equals $\bigwedge_{\mathcal{O}_X}^p\mathcal{E}''$.

(c) Combine (a) and (b) with associativity of the multiplication in the exterior algebra, to prove that there is a well-defined multiplication homomorphism to the associated graded sheaf of this filtration,

$$\bigwedge^q_{\mathcal{O}_X}\mathcal{E}'\otimes_{\mathcal{O}_X}\bigwedge^{p-q}_{\mathcal{O}_X}\mathcal{E}''\to F_\Sigma^q/F_\Sigma^{q+1}.$$

Prove that this homomorphism is an isomorphism.

(d) Conclude the identity in $K^0(X)$,

$$[\bigwedge^p_{\mathcal{O}_X}\mathcal{E}]=\sum_{q=0}^p[\bigwedge^q_{\mathcal{O}_X}\mathcal{E}'][\bigwedge^{p-q}_{\mathcal{O}_X}\mathcal{E}''].$$

Finally, use this to prove the identity in $1+tK^0(X)\,[\![t]\!]$,

$$\lambda(\mathcal{E},t)=\lambda(\mathcal{E}',t)\lambda(\mathcal{E}'',t).$$

Conclude that $\lambda(-,t)$ is a generalized Euler characteristic. Thus, there exists a unique extension to a group homomorphism,

$$\lambda(-,t):K^0(X)\to 1+tK^0(X)\,[\![t]\!]\,.$$

For every integer $p\geq 0$, define

$$\lambda^p:K^0(X)\to K^0(X)$$

to be the unique set map such that for every $a\in K^0(X)$,

$$\lambda(a,t)=\sum_{p=0}^\infty\lambda^p(a).$$

(e) Prove that $\lambda^0(a)$ equals 1, prove that $\lambda^1(a)$ equals a, and prove that for every $q>1$, for every invertible sheaf $\mathcal{L}$, prove that $\lambda^q([\mathcal{L}])$ equals 0. In particular, prove that for every $q>1$, $\lambda^q(1)$ equals 0. Also, use the fact that $\lambda(-,t)$ is a group homomorphism to prove that for every integer $p\geq 0$,

$$\lambda^p(a+b)=\sum_{q=0}^p\lambda^q(a)\lambda^{p-q}(b).$$

(f) Let $\mathcal{A}_1, \dots, \mathcal{A}_r$ be invertible $\mathcal{O}_X$-modules, and denote $a_i = [\mathcal{A}_i]$ in $K^0(X)$. Denote $a_1 + \cdots + a_r$ by a. Prove that

$$\lambda(a,t) = \prod_{i=1}^{r}(1 + a_i t).$$

Similarly, for invertible sheaves $\mathcal{B}_1, \dots, \mathcal{B}_s$ with classes $b_j = [\mathcal{B}_j]$, denoting $b_1 + \cdots + b_s$ by b, prove that also

$$\lambda(b,t) = \prod_{j=1}^{s}(1 + b_j t).$$

Finally, since $ab = \sum_{i,j} a_i b_j$, prove that

$$\lambda(ab,t) = \prod_{i=1}^{r}\prod_{j=1}^{s}(1 + a_i b_j t).$$

(g) Now let the symmetric group $\mathfrak{S}_r$, resp. $\mathfrak{S}_s$ act on the polynomial ring $\mathbb{Z}[x_1, \dots, x_r]$, resp. $\mathbb{Z}[y_1, \dots, y_s]$, in the obvious manner. This induces an action of the product $\mathfrak{S}_r \times \mathfrak{S}_s$ on the tensor product $\mathbb{Z}[x_1, \dots, x_r, y_1, \dots, y_s]$. The main theorem of invariant theory guarantees that the invariant ring $\mathbb{Z}[x_1, \dots, x_r]^{\mathfrak{S}_r}$ is the polynomial ring generated by the algebraically independent elements

$$\chi_l = \sum_{I \subset \{1,\dots,r\}, \#I = l} \prod_{i \in I} x_i,$$

for $l = 1, \dots, r$. Similarly, $\mathbb{Z}[y_1, \dots, y_s]^{\mathfrak{S}_s}$ is the polynomial ring generated by the algebraically independent elements

$$\upsilon_m = \sum_{J \subset \{1,\dots,s\}, \#J = m} \prod_{j in J} y_j.$$

Finally, the subring of $\mathbb{Z}[x_1, \dots, x_r, y_1, \dots, y_s]$ of $\mathfrak{S}_r \times \mathfrak{S}_s$-invariants is the subring generated by $\chi_1, \dots, \chi_r$ and $\upsilon_1, \dots, \upsilon_s$. Thus, for every integer $n \geq 0$, there exists a unique polynomial

$$P_{(r,s),m}(\chi_1, \dots, \chi_r; \upsilon_1, \dots, \upsilon_s) \in \mathbb{Z}[\chi_1, \dots, \chi_r, \upsilon_1, \dots, \upsilon_s],$$

such that the $\mathfrak{S}_r \times \mathfrak{S}_s$-invariant polynomial,

$$\prod_{i=1}^{r}\prod_{j=1}^{s}(1 + x_i y_j t)$$

equals

$$\sum_{n=0}^{\infty} P_{(r,s),n}(\chi_1, \dots, \chi_r; \upsilon_1, \dots, \upsilon_s) t^n.$$

If we make the invariant ring a bigraded ring with $\deg(\chi_l) = (l, 0)$ and $\deg(\upsilon_m) = (0, m)$, then prove that $P_{(r,s),n}$ is homogeneous of bidegree (n, n). In particular, it cannot involve any χ_l or υ_m with $l > n$, resp. $m > n$.

For $r \leq r'$ and $s \leq s'$, prove that

$$P_{(r',s'),n}(\chi_1,\dots,\chi_r,0,\dots,0;\upsilon_1,\dots,\upsilon_s,0,\dots,0) = P_{(r,s),n}(\chi_1,\dots,\chi_r;\upsilon_1,\dots,\upsilon_s).$$

Thus, there exists a unique homogeneous, degree n polynomial,

$$P_n(\chi_1,\dots,\chi_n;\upsilon_1,\dots,\upsilon_n) = P_{(n,n),n}(\chi_1,\dots,\chi_n;\upsilon_1,\dots,\upsilon_n)$$

in the graded ring $\mathbb{Z}[\chi_1,\dots,\chi_n,\upsilon_1,\dots,\upsilon_n]$ such that for every (r,s), in $\mathbb{Z}[\chi_1,\dots,\chi_r,\upsilon_1,\dots,\upsilon_s]$,

$$P_{(r,s),n}(\chi_1,\dots,\chi_r;\upsilon_1,\dots,\upsilon_s) = P_n(\chi_1,\dots,\chi_n;\upsilon_1,\dots,\upsilon_n),$$

with the convention that χ_l, resp. υ_m, is zero if $l > r$, resp. if $m > s$.

In particular, compute the low degree polynomials,

$$P_1 = \chi_1\upsilon_1, \quad P_2 = \chi_1^2\upsilon_2 + \chi_2\upsilon_1^2 - 2\chi_2\upsilon_2,$$

$$P_3 = \chi_1^3\upsilon_3 + \chi_3\upsilon_1^3 + 2\chi_1\chi_2\upsilon_1\upsilon_2 - 3\chi_1\chi_2\upsilon_3 - 3\chi_3\upsilon_1\upsilon_2 + 18\chi_3\upsilon_3.$$

(h) Combine (f) and (g) to prove that for $a = a_1 + \cdots + a_r$ and $b = b_1 + \cdots + b_s$ in $K^0(X)$,

$$\lambda^n(ab) = P_n(\lambda^1(a),\dots,\lambda^n(a);\lambda^1(b),\dots,\lambda^n(b)).$$

Combine this with the splitting principle to prove that for all finite rank, locally free sheaves $\mathcal{A}$ and $\mathcal{B}$ on X and for every integer $n \geq 0$,

$$\lambda^n([\mathcal{A}][\mathcal{B}]) = P_n(\lambda^1([\mathcal{A}]),\dots,\lambda^n([\mathcal{A}]);\lambda^1([\mathcal{B}]),\dots,\lambda^n([\mathcal{B}])).$$

Combined with the additivity relations, conclude that for every $a, b \in K^0(X)$,

$$\lambda^n(ab) = P_n(\lambda^1(a),\dots,\lambda^n(a);\lambda^1(b),\dots,\lambda^n(b)).$$

(i) Via the same strategy as in (f), (g) and (h), for every pair of integers m and n, prove that there are canonically defined homogeneous polynomials of degree mn,

$$Q_{m,n}(\chi_1,\dots,\chi_{mn}) \in \mathbb{Z}[\chi_1,\dots,\chi_{mn}],$$

such that for every quasi-compact scheme X, and for every $a, b \in K^0(X)$,

$$\lambda^n(\lambda^m(a)) = Q_{n,m}(\lambda^1(a),\dots,\lambda^{mn}(a)).$$

Let the polynomials P_n and $Q_{m,n}$ be as above, A *lambda ring* is a commutative, unital ring K with set maps, $(\lambda^n : K \to K)_{n=0,1,\dots}$ satisfying the identities above, i.e., for every a and b in K and for every pair m, n of nonnegative integers,

(i) $\lambda^0(a) = 1$ and $\lambda^1(a) = a$,

(ii) $\lambda^m(1) = 0$ for all $m > 1$,

(iii) $\lambda^n(a+b) = \lambda^n(a)\lambda^0(b) + \cdots + \lambda^{n-m}(a)\lambda^m(b) + \cdots + \lambda^0(a)\lambda^n(b)$,

(iv) $\lambda^n(ab) = P_n(\lambda^1(a), \dots, \lambda^n(a); \lambda^1(b), \dots, \lambda^n(b))$, and

(v) $\lambda^n(\lambda^m(a)) = Q_{n,m}(\lambda^1(a), \dots, \lambda^{mn}(a))$.

Equivalently, the data $(\lambda^n)_n$ can be encoded as the power series map,

$$\lambda(-,t) : K \to 1 + tK\,[\![t]\!]\,,\ \ \lambda(a,t) = \sum_{n=0}^{\infty} \lambda^n(a)t^n.$$

All of the above proves that $K = K^0(X)$ with the lambda operations constructed above is a lambda ring. Moreover, for every $f : Y \to X$, the pullback homomorphisms $K^0(f)$ are homomorphisms of lambda rings.

(j) For the initial commutative, unital ring $\mathbb{Z}$, prove that there exists a unique structure of lambda ring, namely,

$$\lambda_{\mathbb{Z}}(a,t) := (1+t)^a \in 1 + t\mathbb{Z}\,[\![t]\!]\,.$$

With this structure, prove that $\mathbb{Z}$ is the initial lambda ring, i.e., for every lambda ring (K, λ_K), the unique homomorphism of commutative, unital rings,

$$\mathbb{Z} \to K,$$

is a homomorphism of lambda rings.

An *augmented lambda ring* is a lambda ring (K, λ_K) together with a homomorphism of lambda rings,

$$\epsilon : K \to \mathbb{Z}.$$

The homomorphism ϵ is called an *augmentation.* Assume now that X is quasi-compact and connected. Recall that the *rank* is a generalized Euler characteristic on locally free $\mathcal{O}_X$-modules, hence extends to a unique group homomorphism,

$$\epsilon : K^0(X) \to \mathbb{Z},\ \ \epsilon([\mathcal{E}]) = \text{rank}(\mathcal{E}).$$

Prove that ϵ is a homomorphism of lambda rings. Thus, $(K^0(X), \lambda, \epsilon)$ is an augmented lambda ring.

Problem 6. Let (K, λ_K) be a lambda ring. Define the set map $\gamma(-,t)$ as follows,

$$\gamma(-,t) : K \to 1 + tK\,[\![t]\!]\,,\ \ \ \gamma(a,t) = \lambda(a, t/(1-t)) = \sum_{n=0}^{\infty} \lambda^n(a)(t + t^2 + t^3 + ..)^n.$$

For every integer $n \geq 0$, define

$$\gamma^n : K \to K, \ \gamma^0(a) = 1, \ \gamma^{n>0}(a) = \sum_{m=1}^{n} \binom{n-1}{m-1} \lambda^m(a),$$

to be the unique set map such that

$$\gamma(a,t) = \sum_{n=0}^{\infty} \gamma^n(a) t^n.$$

Since $\lambda(a,t)$ equals $\gamma(a, t/(1+t))$ is uniquely recovered from γ, every axiom for the lambda operations is equivalent to a corresponding axiom for the gamma operations. Thus lambda rings could be alternatively axiomatized in terms of the gamma operations.

(a) Use the identities of a lambda ring to prove the following identities,

$$\gamma(a+b,t) = \gamma(a,t)\gamma(b,t),$$

$$\gamma(1,t) = \sum_{n=0}^{\infty} 1t^n = 1/(1-t) = 1 + t/(1-t),$$

and

$$\gamma(-1,t) = 1-t.$$

(b) For every element a of K such that $\lambda^n(a)$ vanishes for all $n > 1$, prove that

$$\gamma(a,t) = 1 + at/(1-t) = 1 + \sum_{n=1}^{\infty} at^n.$$

Conclude that,

$$\gamma(a-1,t) = \gamma(a,t)\gamma(-1,t) = 1 + (a-1)t.$$

More generally, for every integer $r \geq 1$, for every element a such that $\lambda^n(a)$ vanishes for all $n > 1$, check that

$$\gamma(a-r,t) = \gamma(a,t)(1-t)^r = 1 + \sum_{n=1}^{r} \left(\sum_{m=0}^{n} (-1)^{n-m} \binom{r-m}{r-n} \lambda^m(a) \right).$$

You may use the combinatorial identity,

$$\binom{r-m}{r-n} = \sum_{l=m}^{n} (-1)^{n-l} \binom{r}{n-l} \binom{l-1}{m-1}.$$

(c) Let (K, λ, ϵ) be an augmented lambda ring. First, check that ϵ is compatible with the gamma operations. For every a in K, define the *total Chern class* as follows,

$$c(-,t) : K \to 1 + tK\,[\![t]\!], \quad c(a,t) = \gamma(a - \epsilon(a), t) = \gamma(a,t)(1-t)^{\epsilon(a)}.$$

Prove that the lambda operations are recovered from the total Chern class and the augmentation by the identity,
$$\lambda(a,t) = c(a, t/(1+t))(1+t)^{\epsilon(a)}.$$
Hence every axiom for the augmented lambda ring leads to an axiom for the total Chern class and augmentation. Thus, augmented lambda rings could be alternatively axiomatized in terms of the total Chern class and the augmentation. Prove the *Whitney sum formula*,
$$c(a+b,t) = c(a,t)c(b,t).$$

(d) For every integer $n \geq 0$, define the *degree n Chern class*,
$$c_n : K \to K,$$
to be the unique set map such that
$$c(a,t) = \sum_{n=0}^{\infty} c_n(a)t^n.$$
Let a be an element of K such that $\epsilon(a)$ is a nonnegative integer r and such that $\lambda^n(t)$ vanishes for all $n > r$. Use (b) to prove that $c_n(a)$ vanishes for all $n > r$, and
$$c_n(a) = \sum_{m=0}^{n} (-1)^{n-m} \binom{r-m}{r-n} \lambda^m(a)$$
for all integers $n = 0, \dots, r$. Also in this case, use the identity $\lambda(a,t) = c(a,t/(1+t))(1+t)^r$ to prove the identity
$$\lambda^n(a) = \sum_{m=0}^{n} \binom{r-m}{r-n} c_n(a)$$
for all integers $n = 0, \dots, r$.

Define $\Gamma^1 K$ to be the ideal that is the kernel of ϵ. Since $c_1(a)$ equals $a - \epsilon(a)$, prove that $\Gamma^1 K$ equals the ideal generated by all first Chern classes $c_1(a)$. More generally, since ϵ is compatible with the gamma operations, prove that the image in $1 + t\mathbb{Z}[\![t]\!]$,
$$\epsilon(c(a,t)) := \sum_{n=0}^{\infty} \epsilon(c_n(a))t^n$$
equals
$$c(\epsilon(a),t) = \gamma(\epsilon(a) - \epsilon(a), t) = \gamma(0,t) = 1.$$
Hence, for every $n > 0$, prove that $\epsilon(c_n(a))$ equals 0, i.e., $c_n(a)$ is in $\Gamma^1 K$. Thus $\Gamma^1 K$ equals the ideal generated by all expressions $c_{d_1}(a_1) \cdots c_{d_m}(a_m)$ for all positive integers m, for all m-tuples

$(a_1, \dots, a_m)$ of elements in K, and for every m-tuple of nonnegative integers $(d_1, \dots, d_m)$ such that $d_1 + \dots + d_m \geq 1$.

More generally, define the *gamma filtration* to be the descending filtration of K by ideals $\Gamma^d K$, for $d = 0, 1, 2, \dots$, where $\Gamma^d K$ equals the additive subgroup of K generated by all expressions $c_{d_1}(a_1) \cdots c_{d_m}(a_m)$ for all nonnegative integers m, for all elements $a_1, \dots, a_m$ in K, and for all m-tuples of nonnegative integers $d_1, \dots, d_m$ such that $d_1 + \dots + d_m \geq d$. For every $b \in K$, since $c_1(b)$ equals $b - \epsilon(b)$, prove that

$$bc_{d_1}(a_1) \cdots c_{d_m}(a_m) = \epsilon(b)c_{d_1}(a_1) \cdots c_{d_m}(a_m) + c_1(b)c_{d_1}(a_1) \cdots c_{d_m}(a_m)$$

is again in $\Gamma^d K$. Hence $\Gamma^d K$ is an ideal in K. Moreover, for all nonnegative integers d and e, $\Gamma^d K \cdot \Gamma^e K$ is contained in $\Gamma^{d+e} K$.

(e) Formalize the equivalence between the lambda operations as follows. For every nonnegative integer r, let Φ_r be the commutative, unital ring $\mathbb{Z}[\chi_1, \dots, \chi_r]$ and let F_r be the commutative, unital ring $\mathbb{Z}[u_1, \dots, u_r]$. Define inverse ring isomorphisms,

$$C_r : \Phi_r \to F_r, \quad C_r(\chi_i) := \sum_{p=0}^{i} \binom{r-p}{r-i} u_p.$$

and

$$L_r : F_r \to \Phi_r, L_r(u_i) := \sum_{p=0}^{i} (-1)^{i-p} \binom{r-p}{r-i} u_p.$$

For every element a in K such that $\epsilon(a)$ equals r and such that $\lambda^n(a)$ vanishes for all $n > r$, there is a corresponding ring homomorphism,

$$\phi_r(a) : \Phi_r \to K, \quad \chi_i \mapsto \lambda^i(a),$$

and a ring homomorphism,

$$f_r(a) : F_r \to K, \quad u_i \mapsto c_i(a).$$

Prove that $f_r(a) \circ C_r$ equals $\phi_r(a)$, and prove that $\phi_r(a) \circ L_r$ equals f_r. Moreover, if we make F_r into a graded ring with $\deg(u_m) = m$, and if we define the corresponding ideal $\Gamma^d F_r$ to be the ideal generated by all homogeneous elements of degree $\geq d$, then prove that $f_r(\Gamma^d F_r)$ is contained in $\Gamma^d K$.

Let T_r be the commutative, unital ring $\mathbb{Z}[x_1, \dots, x_r]$. Define ring homomorphisms,

$$\tau_r : \Phi_r \to T_r, \quad \chi_q \mapsto \sigma_q(x_1, x_2, \dots, x_r) := \sum_{I \subset \{1, \dots, r\}, \#I = q} \left(\prod_{i \in I} x_i \right).$$

Thus, σ_q is the degree q *elementary symmetric polynomial.* Similarly, define the ring homomorphism,

$$t_r : F_r \to R_r, \quad u_q \mapsto s_q(x_1 - 1, x_2 - 1, \dots, x_r - 1).$$

Prove that $t_r \circ C_r$ equals τ_r, and prove that $\tau_r \circ L_r$ equals t_r. Moreover, prove that τ_r and t_r are injective, and each image equals the subring of invariants $\mathbb{Z}[x_1, \dots, x_r]^{\mathfrak{S}_r}$. Of course defining elements,

$$\tilde{x}_i = x_i - 1, \quad x_i = \tilde{x}_i + 1,$$

there is an $\mathfrak{S}_r$-equivariant ring automorphism,

$$\iota_r : R_r \to R_r, \quad \iota_r(x_i) = \tilde{x}_i$$

sending every χ_m to u_m.

(f) Let r and s be nonnegative integers. As above, denote $\Phi_r = \mathbb{Z}[\chi_1, \dots, \chi_r]$, denote $F_r = \mathbb{Z}[u_1, \dots, u_r]$, and denote $T_r = \mathbb{Z}[x_1, \dots, x_r]$, with the corresponding ring homomorphisms C_r, L_r, τ_r and t_r. For the integer s, for notation's sake, denote $\Phi_s = \mathbb{Z}[\upsilon_1, \dots, \upsilon_s]$, denote $F_s = \mathbb{Z}[v_1, \dots, v_s]$, and denote $T_s = \mathbb{Z}[y_1, \dots, y_s]$, with the corresponding ring homomorphisms C_s, L_s, τ_s and t_s. In particular, we have tensor product rings,

$$\Phi_{r,s} := \Phi_r \otimes \Phi_s = \mathbb{Z}[\chi_1, \dots, \chi_r, \upsilon_1, \dots, \upsilon_s],$$

$$F_{r,s} := F_r \otimes F_s = \mathbb{Z}[u_1, \dots, u_r, v_1, \dots, v_s],$$

and

$$T_{r,s} := T_r \otimes T_s = \mathbb{Z}[x_1, \dots, x_r, y_1, \dots, y_s].$$

We also have the tensor product homomorphisms,

$$C_{r,s} : \Phi_{r,s} \to F_{r,s}, \quad C_{r,s}(\chi_i) = C_r(\chi_i),\ C_{r,s}(\upsilon_j) = C_s(\upsilon_j),$$

$$\tau_{r,s} : \Phi_{r,s} \to T_{r,s}, \quad \tau_{r,s}(\chi_i) = \tau_r(\chi_i),\ \tau_{r,s}(\upsilon_j) = \tau_s(\upsilon_j),$$

$$L_{r,s} : F_{r,s} \to \Phi_{r,s}, \quad L_{r,s}(u_i) = L_r(u_i),\ L_{r,s}(v_j) = L_s(v_j),$$

and

$$t_{r,s} : F_{r,s} \to T_{r,s}, \quad t_{r,s}(u_i) = t_r(u_i),\ t_{r,s}(v_j) = t_s(v_j).$$

As above, $t_{r,s} \circ C_{r,s}$ equals $\tau_{r,s}$, and $\tau_{r,s} \circ L_{r,s}$ equals $t_{r,s}$. Finally, for every pair of elements (a, b) in K such that $\epsilon(a) = r$, resp. $\epsilon(b) = s$, and such that $\lambda^n(a)$ vanishes for $n > r$, resp. such that $\lambda^n(b)$ vanishes for $n > s$, there exists a unique ring homomorphism,

$$\phi_{r,s}(a, b) : \Phi_{r,s} \to K, \quad \chi_i \mapsto \lambda^i(a),\ \upsilon_j \mapsto \lambda^j(b),$$

and there exists a unique ring homomorphism,

$$f_{r,s}(a, b) : F_{r,s} \to K, \quad u_i \mapsto c_i(a),\ v_j \mapsto c_j(b).$$

As above, $f_{r,s}(a, b) \circ C_{r,s}$ equals $\phi_{r,s}(a, b)$, and $\phi_{r,s}(a, b) \circ L_{r,s}$ equals $f_{r,s}(a, b)$. If we make $F_{r,s}$ into a graded ring with $\deg(u_m) = \deg(v_m) = m$, and if we define $\Gamma^d F_{r,s}$ to be the ideal generated by all homogeneous elements of degree $\geq d$, then also prove that $f_{r,s}(\Gamma^d F_{r,s})$ is contained in $\Gamma^d K$.

For every integer $n = 1, \dots, rs$, define the element

$$\tilde{P}_{(r,s),n}(u_1, \dots, u_r; v_1, \dots, v_s) := C_{r,s}(\sum_{m=0}^{n} (-1)^{n-m} \binom{rs-m}{rs-n} P_{(r,s),m}),$$

where $P_{(r,s),m}$ is as in **Problem 5(g)**. For elements a, b of K as above, prove that $\epsilon(ab) = rs$, prove that $\lambda^n(ab)$ vanishes for all $n > rs$, and prove the identity,

$$c_n(ab) = \tilde{P}_{(r,s),n}(c_1(a), \dots, c_r(a); c_1(b), \dots, c_s(b))$$

for every $n = 0, \dots, rs$. Also, inside the polynomial ring $T_{r,s}[t]$, prove that

$$\sum_{n=0}^{rs} t_{r,s}(\tilde{P}_{(r,s),n}(u_1, \dots, u_r; v_1, \dots, v_r))t^n = \prod_{i=1}^{r}\prod_{j=1}^{s}(1 + (x_i y_j - 1)t) =$$

$$\prod_{j=1}^{s}(1 + ((\tilde{x}_i + \tilde{y}_j + \tilde{x}_i\tilde{y}_j)t) = \sum_{n=0}^{rs} \sigma_n(\tilde{x}_i + \tilde{y}_j + \tilde{x}_i\tilde{y}_j)t^n,$$

where σ_n is, as above, the degree n elementary symmetric polynomial. This gives another way to compute $\tilde{P}_{(r,s),n}$. In particular, conclude that

$$\tilde{P}_{(r,s),n}(u_1, \dots, u_r; v_1, dots, v_r) \cong \sigma_n(\tilde{x}_i + \tilde{y}_j) \pmod{\Gamma^{n+1}F_{r,s}}.$$

One special case is when s equal to 1,

$$\tilde{P}_{(r,1),n}(u_1, \dots, u_r; v_1) = \sigma_n(\tilde{x}_1(1+v_1) + v_1, \dots, \tilde{x}_n(1+v_1) + v_1) =$$

$$\sum_{m=0}^{n} \binom{r-m}{n-m} u_m v_1^{n-m}(1+v_1)^m \equiv \sum_{m=0}^{n} \binom{r-m}{n-m} u_m v_1^{n-m} \pmod{\Gamma^{n+1}F_{r,1}}.$$

(g) Now let $(K^0(X), \lambda, \epsilon)$ be the augmented lambda ring from **Problem 5**. For every locally free sheaf $\mathcal{E}$ of rank r, conclude that the "lambda-theoretic" Chern classes defined above agree with the earlier definition, i.e.,

$$c_n([\mathcal{E}]) = \sum_{m=0}^{n} (-1)^{n-m} \binom{r-m}{r-n} [\bigwedge_{\mathcal{O}_X}^{m} \mathcal{E}].$$

In particular, conclude the Whitney sum formula for this new definition.

(h) Now assume that X is separated and finite type over a field k, or at least Noetherian of finite dimension. Recall the codimension filtation $F_l K_0(X)$ on the Grothendieck group of coherent sheaves on X: it is the image of the cycle map,

$$\text{cycle}_l(X) : \oplus_{m \le l} Z_m(X) \to K_0(X), \quad [V] \mapsto [\mathcal{O}_V].$$

By devissage, $F_lK_0(X)$ is the same as the subgroup generated by all coherent sheaves that have support of dimension $\leq l$. In particular, for every projective morphism $f : Y \to X$, the pushforward homomorphism,

$$K_0(f) : K_0(Y) \to K_0(X)$$

preserves the filtation, i.e., $K_0(f)(F_lK_0(Y))$ is contained in $F_lK_0(X)$.

A bit more generally, let $\rho : Q \to X$ be any projective morphism such that for every integral closed subscheme $V \subset X$, there exists an integral closed subscheme $\widetilde{V} \subset Q$ whose image equals V and such that

$$\rho|_{\widetilde{V}} : \widetilde{V} \to V$$

is birational; i.e., it admits *rational sections universally*. Use devissage to prove that $F_lK_0(X)$ is the subgroup generated by all proper pushforward classes $K_0(\rho)[\mathcal{O}_W]$ as W varies over integral, closed subschemes of W of dimension $\leq l$, or even just those integral closed subschemes of W of dimension $\leq l$ that map birationally to $\rho(W)$.

For every locally free sheaf $\mathcal{E}$ on X of rank r, for every integral, closed subscheme $V \subset X$ of dimension l, the tensor product $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_V$ has support contained in V (equal to V unless $\mathcal{E}$ is the zero sheaf). Conclude that each subgroup $F_lK_0(X)$ is a $K^0(X)$-submodule. Moreover, since there exists a dense open subscheme $U \subset V$ such that $\mathcal{E}|_U$ is isomorphic to $\mathcal{O}_X^{\oplus r}|_U$, conclude that

$$([mathcalE] - r[\mathcal{O}_X]) \cdot [\mathcal{O}_V]$$

is represented by a coherent sheaf that has support contained in $V \setminus U$. In particular, it is contained in $F_{l-1}K_0(X)$. Conclude that for every $a \in K^0(X)$, $c_1(a) \cdot F_lK_0(X)$ is contained in $F_{l-1}K_0(X)$.

For every locally free sheaf $\mathcal{E}$ on X of rank r, let

$$\rho : Q \to X,$$

be a projective morphism as in **Problem 4(f)**. Such a morphism admits sections Zariski locally, hence it admits rational sections universally. Thus, $F_lK_0(X)$ is generated by the images of all structure sheaves $[\mathcal{O}_W]$ as W varies among integral closed subschemes of Q of dimension $\leq l$. By the projection formula,

$$K_0(\rho)(K^0(\rho)(a) \cdot b) = a \cdot K_0(\rho)(b).$$

In particular, to prove that $a \cdot F_lK_0(X)$ is contained in $F_{l-n}K_0(X)$, it suffices to prove that $K^0(\rho)a \cdot F_lK_0(Q)$ is contained in $F_{l-n}K_0(Q)$. Now apply the Whitney sum formula to express $K^0(\rho)c_n(\mathcal{E})$ as a linear combination of products of d first Chern classes $c_1(F^q\rho^*\mathcal{E}/F^{q+1}\rho^*\mathcal{E})$. Thus, by the result for first Chern classes, prove that $K^0(\rho)c_n(\mathcal{E})$ maps $F_lK_0(Q)$ into $F_{l-n}K_0(Q)$. Thus $c_n(\mathcal{E}) \cdot -$ maps $F_lK_0(Q)$ into $F_{l-n}K_0(Q)$. Conclude that $\Gamma^nK^0(X) \cdot F_lK_0(X)$ is contained in $F_{l-n}K_0(X)$.

Define the associated graded ring of the gamma filtration of $K^0(X)$ by ideals,

$$\text{gr}_\Gamma K^0(X) := \oplus_{n=0}^\infty \text{gr}_\Gamma^n K^0(X), \quad \text{gr}_\Gamma^n K^0(X) := \Gamma^n K^0(X)/\Gamma^{n+1}K^0(X).$$

Similarly, define the associated graded group for the dimension filtration of $K_0(X)$ by

$$\mathrm{gr}^F K_0(X) := \oplus_{l=0}^{\infty} \mathrm{gr}_l^F K_0(X), \quad \mathrm{gr}_l^F K_0(X) := F_l K_0(X)/F_{l-1}K_0(X).$$

Use the previous paragraph to prove that $\mathrm{gr}^F K_0(X)$ admits a well-defined structure of graded module for the graded ring $\mathrm{gr}_\Gamma K^0(X)$. Moreover, check that these graded rings and graded modules are compatible with pushforward and pullback (arbitrary pullback for K^0, flat pullback for K_0, proper pushforward for K_0, etc.).

(i) Assume now that X is a connected, separated, finite type k-scheme. Recall that the cycle class maps factor uniquely through rational equivalence,

$$\mathrm{cycle}_l : A_l(X) \to \mathrm{gr}_l^F K_0(X),$$

and these morphisms are surjective by devissage. For every locally free sheaf $\mathcal{E}$ of rank r on X, use the splitting principle as in the previous section to prove compatible with the Chow theory Chern classes and the K-theory Chern classes, i.e.,

$$\mathrm{cycle}_{l-n}(c_d^A(\mathcal{E}) \cap \beta) = c_d(\mathcal{E}) \cdot \mathrm{cycle}_l(\beta)$$

for every $\beta \in A_l(X)$, where $c_d^A(\mathcal{E}) \cap -$ denotes the Chern class operation $A_l(X) \to A_{l-d}(X)$ defined in the textbook.

Denote by $\mathrm{End}(A_*(X))$ the associative ring of group homomorphisms from $A_*(X)$ to itself. For every integer d, define $\mathrm{End}^d(A_*(X))$ to be the additive subgroup of group homomorphisms such that for every l, the homomorphism maps $A_l(X)$ to $A_{l-d}(X)$. Denote by $\mathrm{End}^*(A_*(X))$ the subring of $\mathrm{End}^*(A_*(X))$ that is the direct sum of all subgroups $\mathrm{End}^d(A_*(X))$. This is a graded (associative) ring. Define $A^*_{\mathrm{pre}}(X)$ to be the graded subring of $\mathrm{End}^*(A_*(X))$ generated by all Chern classes $c_d^A(\mathcal{E}) \in \mathrm{End}^d(A_*(X))$ for all integers $d \geq 0$ and all locally free sheaves $\mathcal{E}$. This is *not* yet the Chow (cohomology) ring, as defined in Fulton. For instance, there is a surjective group homomorphism,

$$\mathrm{Pic}(X) \to A^1_{\mathrm{pre}}(X), \quad [\mathcal{L}] \mapsto c_1(\mathcal{L}) \cap - \,.$$

However, there exists nonreduced, projective schemes X such that Fulton's Chow cohomology group $A^1(X)$ is not generated by the image of $\mathrm{Pic}(X)$. Use the splitting principle to prove that $A^*_{\mathrm{pre}}(X)$ is a commutative, unital, graded ring.

For every locally free sheaf $\mathcal{E}$ on X, define the total Chern class,

$$c^A(\mathcal{E}, t) \in 1 + tA^*_{\mathrm{pre}}(X) \llbracket t \rrbracket$$

by

$$c^A(\mathcal{E}, t) = 1 + \sum_{n=1}^{\mathrm{rank}(\mathcal{E})} t^d c_d^A(\mathcal{E}) \cap - \,.$$

Use the Whitney sum formula for c_d^A to prove that $c^A(-,t)$ is a generalized Euler characteristic. Conclude that there exists a well-defined group homomorphism,

$$c^A(-,t) : K^0(X) \to 1 + tA^*_{\text{pre}}(X)\llbracket t \rrbracket$$

extending the definition above. For every integer $n \geq 0$, define

$$c_n^A : K^0(X) \to A^n_{\text{pre}}(X)$$

to be the unique set map such that

$$c^A(a,t) = \sum_{n=0}^{\infty} c_n^A(a) t^n.$$

In general, there is no reason to expect that there exists a homomorphism of commutative, unital, graded rings,

$$(-)^A : \text{gr}^*_\Gamma K^0(X) \to A^*_{\text{pre}}(X)$$

such that

$$(c_{d_1}(a_1) \cdots c_{d_m}(a_m))^A = c^A_{d_1}(a_1) \cap \cdots \cap c^A_{d_m}(a_m),$$

for every integer $m > 0$, for every m-tuple $(a_1, \dots, a_m)$ of elements in $K^0(X)$, and for every m-tuple of nonnegative integers $(d_1, \dots, d_m)$. The basic difficulty is that a polynomial expression in Chern classes that evaluates to 0 in $\text{gr}^*_\Gamma K^0(X)$ need not evaluate to 0 in $A^*_{\text{pre}}(X)$ (at least not obviously). However, at least when the gamma filtration terminates, after tensoring with $\mathbb{Q}$ there is a unique homomorphism of commutative, unital, graded rings

$$(-)^A_{\mathbb{Q}} : \text{gr}^*_\Gamma K^0(X)_{\mathbb{Q}} \to A^*_{\text{pre}}(X)_{\mathbb{Q}}$$

that is compatible with monomials in Chern classes. The proof of this uses the Adams operations.