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MAT 614 Problem Set 1

Homework Policy. Please read through all the problems. I will be happy to discuss the solutions during office hours.

Problems.

Problem 1, Intersection Multiplicity: This problem is essentially (Hartshorne, Exer. I.5.4). Let $F, G \in k[X_0, X_1, X_2]$ be non-constant, irreducible, homogeneous polynomials, and denote $C = \mathbb{V}(F), D = \mathbb{V}(G)$ in $\mathbb{P}^2_k$. Let $p \in C \cap D$ be an element such that $\dim(C \cap D, p) = 0$, i.e., p is an isolated point of $C \cap D$. The *intersection multiplicity of* C *and* D *at* p, $i(C, D; p)$, is defined to be,

$$i(C, D; p) = \dim_k(\mathcal{O}_{\mathbb{P}^2,p}/\langle F_p, G_p\rangle),$$

where $F_p, G_p \in \mathcal{O}_{\mathbb{P}^2,p}$ are germs of dehomogenizations of F and G at p.

Let $P \subset k[X_0, X_1, X_2]$ be the homogeneous ideal corresponding to p. Form the graded $k[X_0, X_1, X_2]$-module, $M = \text{Image}(\phi_p)$, where ϕ_p is the homomorphism of modules,

$$\phi_p : k[X_0, X_1, X_2]/\langle F, G\rangle \to (k[X_0, X_1, X_2]/\langle F, G\rangle)_P.$$

(a) Prove that the Hilbert polynomial of M equals $i(C, D; p)$, i.e., for all $l \gg 0$, $\dim_k M_l = i(C, D; p)$. **Hint:** You may assume existence of a *Jordan-Hölder filtration of* M: a filtration of M by graded submodules, $M = M^0 \supset M^1 \supset \cdots \supset M^r = \{0\}$, such that for every $i = 1, \dots, r$, $M^{i-1}/M^i \cong (k[X_0, X_1, X_2]/P)(d_i)$ for some integer d_i. For every $X \in k[X_0, X_1, X_2]_1 - P$, the dehomogenization of M with respect to X equals $\mathcal{O}_{\mathbb{P}^2,p}/\langle F_p, G_p\rangle$ and has an induced Jordan-Hölder filtration whose associated graded pieces are the dehomogenizations of the graded modules M^{i-1}/M^i. Relate the length of the dehomogenization of M, the Hilbert polynomial of M, and the integer r.

(b) This problem is rather difficult. Attempt it, but you don't have to solve it. Denote by $e(C; p)$, resp. $e(D; p)$, the Hilbert-Samuel multiplicity of C at p, resp. of D at p. Prove that $i(C, D; p)$ is at least $e(C; p)e(D; p)$. **Hint:** Work in affine coordinates for which $p = (0, 0)$. First consider the case that $C = \mathbb{V}(f), D = \mathbb{V}(g)$ where f and g are relatively prime homogeneous polynomials in x, y. Next deduce the case where f and g are not necessarily homogeneous, but the tangent cones of C and D at p have no common irreducible component. The general case can be deduced from this one by a "semicontinuity" argument.

(c) Let X be a plane curve, and let $p \in X$ be an element. Prove that for all but finitely many lines L in $\mathbb{P}^2$ containing p, $i(X, L; p)$ equals $e(X; p)$.

Problem 2, Bézout's Theorem in the Plane: This problem continues the previous problem. Let d denote $\deg(F)$ and let e denote $\deg(G)$. Assume $C \cap D$ is a finite set $\{p_1, \dots, p_m\}$, i.e., $C \cap D$ has no irreducible component of dimension 1. Define M to be the graded module $k[X_0, X_1, X_2]/\langle F, G\rangle$. For every $i = 1, \dots, m$, define M_i to be $\text{Image}(\phi_{P_i})$ where P_i is the homogeneous ideal of p_i and where $\phi_{P_i} : k[X_0, X_1, X_2]/\langle F, G\rangle \to (k[X_0, X_1, X_2]/\langle F, G\rangle)_{P_i}$ is the localization homomorphism.

For the following homomorphism of graded modules, prove both the kernel and cokernel have finite length:

$$\phi : M \to \oplus_{i=1}^m M_i.$$

Hint: This requires more about the Jordan-Hölder filtration and associated primes. For a graded module M, there exists a filtration of M, $M = M^0 \supset \cdots \supset M^r = \{0\}$, such that for every $j = 1, \dots, r$, $M^{j-1}/M^j \cong (k[X_0, X_1, X_2]/Q_j)(d_j)$ where Q_j is an associated prime of M. If Q is a minimal associated prime, then $(M^{j-1}/M^j)_P$ is nonzero if and only if P_j equals P. So the graded pieces in the filtration of M_i are the associated graded pieces in the filtration of M such that Q_j equals P_i.

Remark: It follows that the Hilbert polynomial of M equals the sum over i of the Hilbert polynomial of M_i. On the one hand, there is an exact sequence of graded modules,

$$0 \to k[X_0, X_1, X_2](-d-e) \xrightarrow{(G,-F)^\dagger} k[X_0, X_1, X_2](-d) \oplus k[X_0, X_1, X_2](-e)$$
$$\xrightarrow{(F,G)} k[X_0, X_1, X_2] \xrightarrow{k} [X_0, X_1, X_2]/\langle F, G\rangle \to 0,$$

from which it easily follows the Hilbert polynomial of M is the constant polynomial with value de. On the other hand, by Problem 1, the Hilbert polynomial of each M_i is the intersection multiplicity $i(C, D; p_i)$. This gives *Bézout's theorem in the plane*,

$$\deg(C) \cdot \deg(D) = \sum_{p_i \in C \cap D} i(C, D; p_i).$$

Problem 3: This is essentially (Hartshorne, Exer. I.7.5). Let $C \subset \mathbb{P}^2_k$ be a plane curve of degree $d \geq 1$.

(a) If there exists $p \in C$ such that $e(C; p)$ equals d, prove that C is a union of lines containing p.

(b) If C is irreducible, and $p \in C$ is a point such that $e(C; p)$ equals $d-1$, prove that the projection from p is birational: $\pi_p : (C - \{p\}) \to \mathbb{P}^1_k$.

Problem 4: This is a "multilinear algebra problem" introducing the derivative and Hessian of a polynomial. The next problem relates the Hessian of a homogeneous polynomial on $\mathbb{P}^2$ to the *flex lines* of the associated plane curve.

For every finite-dimensional k-vector space V, denote by $V^\vee$ the dual vector space $\text{Hom}_k(V, k)$. Denote by $k[V^\vee]$ the ring of polynomial functions on V, i.e., the k-subalgebra of $\text{Hom}_{\text{Set}}(V, k)$ generated by $V^\vee$. There is a unique $\mathbb{Z}_{\geq 0}$-grading on $k[V^\vee]$ such that $k[V^\vee]_0$ is the field of constant functions k and such that $k[V^\vee]_1$ is the k-vector space of linear functionals $V^\vee$. For every integer

$r \geq 0$, denote by $S^r(V^\vee)$ the k-vector space $k[V^\vee]_r$, called the r^{th} *symmetric power of* $V^\vee$. Denote by $(\mathbb{A}V, \mathcal{O}_{\mathbb{A}V})$ the unique affine variety whose underlying point-set is V and whose coordinate ring $\mathcal{O}_{\mathbb{A}V}(\mathbb{A}V)$ is $k[V^\vee]$. (Usually this variety is just denoted $(V, \mathcal{O}_V)$, but in this problem this notation distinguishes V as a k-vector space from V as an affine variety.)

(a) Denote by M the (left) $k[V^\vee]$-module $M = k[V^\vee] \otimes_k V^\vee$ where $f \cdot (g \otimes x) := (fg) \otimes x$ for every $f, g \in k[V^\vee]$ and $x \in V^\vee$. Prove that there exists a unique k-derivation $d : k[V^\vee] \to M$ such that $d(x) = 1 \otimes x$ for every $x \in V^\vee = k[V^\vee]_1$. The induced homomorphism of $k[V^\vee]$-modules, $\Omega_{k[V^\vee]/k} \to M$, is an isomorphism (you need not prove this).

(b) For every integer $r \geq 0$, denote by $d_r : S^r(V^\vee) \to S^{r-1}(V^\vee) \otimes V^\vee$ the restriction of d, and denote by $\widetilde{d_r} : S^r(V^\vee) \to \mathrm{Hom}_k(V, S^{r-1}(V^\vee))$ the composition of d_r with the canonical isomorphism $S^{r-1}(V^\vee) \otimes_k V^\vee \cong \mathrm{Hom}_k(V, S^{r-1}(V^\vee))$. Given $F \in S^r(V^\vee)$, denote the image under d_r by $d_r F$, and denote the induced linear map by $\widetilde{d_r}F : V \to S^{r-1}(V^\vee)$. Let $(\mathbf{e}_1, \dots, \mathbf{e}_n)$ be an ordered basis for V and let $(x_1, \dots, x_n)$ be the dual ordered basis for $V^\vee$. Prove for every $F \in S^r(V^\vee)$ and every $i = 1, \dots, n$,

$$\widetilde{d_r}F(\mathbf{e}_i) = \frac{\partial F}{\partial x_i}.$$

(c) For every integer $r \geq 0$, denote by $\mathrm{Hess}_r : S^r(V^\vee) \to \mathrm{Hom}_k(V, S^{r-2}(V^\vee) \otimes_k V^\vee)$ the unique linear map $F \mapsto \mathrm{Hess}_r(F)$ such that for every $v \in V$, $\mathrm{Hess}_r(F)(v) = d_{r-1}((\widetilde{d_r}F)(v))$. This is the *Hessian of* F. Let $(\mathbf{e}_1, \dots, \mathbf{e}_n)$ be an ordered basis for V, and let $(x_1, \dots, x_n)$ be the dual ordered basis for $V^\vee$. Prove that for every $F \in S^r(V^\vee)$ and every $1 \leq j \leq n$,

$$\mathrm{Hess}_r(F)(\mathbf{e}_j) = \sum_{i=1}^{n} \frac{\partial^2 F}{\partial x_i \partial x_j} \otimes x_i.$$

Considering the terms $\partial^2 F/\partial x_i \partial x_j$ to be "coefficients", $Hess_r(F)$ is an $n \times n$ matrix whose (i,j)-entry is the degree $r-2$ homogeneous polynomial $\partial^2 F/\partial x_i \partial x_j$. For every point $p \in \mathbb{A}V$, denote by $\mathrm{Hess}_r(F)(p) : V \to V^\vee$ the k-linear map obtained by evaluating these degree $r-2$ homogeneous polynomials at p.

Problem 5: This problem continues the previous problem. Let $\dim_k V = 3$ so that $\mathbb{A}V \cong \mathbb{A}^3_k$. Denote by $(\mathbb{P}V, \mathcal{O}_{\mathbb{P}V})$ the projective variety $(\mathbb{A}V - \{0\})/(v \sim \lambda v) \cong \mathbb{P}^2_k$. Let $r \geq 1$, let $F \in S^r(V^\vee)$ be an irreducible polynomial, and let $C = \mathbb{V}(F) \subset \mathbb{P}V$ be the associated plane curve.

(a) Let $p \in C$ be an element, and let $v \in V$ be a vector. Prove that $(\widetilde{d_r}F(v))(p) = 0$ if and only if there exists a line $L \subset \mathbb{P}V$ tangent to C at p and such that the associated affine cone $\mathbb{A}L \subset \mathbb{A}V$ contains v. **Hint:** If v is in $\mathbb{A}\{p\}$ this is trivial, and if v is not in $\mathbb{A}\{p\}$, choose an ordered basis $(\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2)$ for V such that $p = [1, 0, 0]$ and $v = (0, 1, 0)$.

(b) Assume $\mathrm{char}(k)$ does not divide $2(r-1)$. For every point $p \in C$, a tangent line L to C at p, $L \subset \mathbb{P}V$, is defined to be a *flex line to* C *at* p if the germ at p of the restriction to L of the dehomogenization of F is contained in $\mathfrak{m}_p^3 \mathcal{O}_{L,p}$, i.e., the restriction of F to L vanishes to order ≥ 3 at p. Prove that there is a flex line to C at p if and only if the 3×3 Hessian $\mathrm{Hess}_r(F)(p)$ is not an isomorphism, i.e., if and only if, with respect to some (and hence any) basis, the determinant of the 3×3 Hessian matrix equals 0. **Hint:** There are two cases depending on whether p is a smooth or a singular point of C. In both cases, choose an ordered basis $(\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2)$ for V such that $p = [1, 0, 0]$ and such that tangent line under consideration is $\{[a, b, 0] | a, b \in k\}$.

(c) Assume $\text{char}(k)$ does not divide 6. Compute all the flex lines to the smooth cubic plane curve $\mathbb{V}(x_0^3 + x_1^3 + x_2^3) \subset \mathbb{P}^2_k$. **Hint:** There are 9 of them.

Problem 6: Assume $\text{char}(k)$ does not divide $d(d-1)$. Combine Problem 2 with Problem 5 to deduce that every smooth plane curve C of degree $d \geq 3$ has at most $3d(d-2)$ flex lines.

Problem 7: If $\text{char}(k) = p$, give an example of a smooth plane curve C of degree $d = p + 1$ having infinitely many flex lines.

Problem 8: Use the same technique from Lecture 1 to prove that for every integer $r \geq 1$, for a general quadruple $(\Pi_1, \Pi_2, \Pi_3, \Pi_4)$ of linear subvarieties $\Pi_i \subset \mathbb{P}^{2r-1}$ of dimension $r - 1$, there are precisely r lines $L \subset \mathbb{P}^{2r-1}$ such that for every $i = 1, \dots, 4$, L intersects Π_i.

Problem 9: With the same notation as in the previous problem, for a general triple (Π_1, Π_2, Π_3), describe the union Σ of all lines L that intersect each of Π_1, Π_2 and Π_3. Show that Σ is irreducible of dimension r. For a general codimension r linear space $\Pi_4 \subset \mathbb{P}^r$, what can you say about $\Sigma \cap \Pi_4$? What can you conclude about the degree of Σ? Do you know another way to compute this degree? (If so, double-check your answer.)

Problem 10: Recall the heuristic "parameter count" from lecture: for every integer $n \geq 2$, for $d = 2n - 3$, for a general hypersurface $X \subset \mathbb{P}^n$ of degree d, we expect a finite number c_n of lines $L \subset \mathbb{P}^n$ to be contained in X. Recall also that the list $c_2 = 1$, $c_3 = 27$, $c_4 = 2875$. Now, inside the ring $\mathbb{Z}[s, t]$, graded in the usual way so that s and t have degree 1, consider the homogeneous ideal

$$I = \langle s^{n+1}, s^n + s^{n-1}t + \cdots + s^{n-r}t^r + \cdots + st^{n-1} + t^n, t^{n+1} \rangle,$$

which is invariant under the action of $\mathbb{Z}/2\mathbb{Z}$ on the graded ring permuting s and t. Consider the graded quotient ring $\mathbb{Z}[s, t]/I$ with its induced $\mathbb{Z}/2\mathbb{Z}$-action, and denote by A^* the invariant graded subring $(\mathbb{Z}[s, t]/I)^{\mathbb{Z}/2\mathbb{Z}}$.

(a) Check that the top non-zero graded piece of A^* has degree $2(n-1)$ generated by the image $\overline{s^{n-1}t^{n-1}}$ of the invariant monomial $s^{n-1}t^{n-1}$. Recall that we computed that $2(n-1)$ equals the dimension of the Grassmannian of lines in $\mathbb{P}^n$.

(b) Now, for $d = 2n - 3$, compute the image in A^* of the invariant, homogeneous polynomial of degree $d + 1 = 2(n-1)$,

$$f(s, t) = (ds + 0t) \cdot ((d-1)s + 1t) \cdots ((d-r)s + rt) \cdots (1s + (d-1)t) \cdot (0s + dt).$$

Write your answer as $b_n \overline{s^{n-1}t^{n-1}}$ for some integer b_n. How do the integers b_n compare to the integers c_n for $n = 2, 3, 4$? Based on this, what is your guess for c_5, the number of lines contained in a septic fourfold?