MAT 614 Notes on Infinitesimal Deformation Theory

These are some notes accompanying the discussion in lecture on deformation theory and obstruction theory. The canonical sources are Schlessinger's thesis, [Sch68], and Artin's articles on algebraization, [Art69], [Art74]. For the obstruction theory of the Hilbert scheme, the original source is [Art69], and an excellent treatment is also given in [Kol96, I.2].

1 Local Artin Algebras and Complete, Noetherian, Local Algebras.

For every local ring R in what follows, $\mathfrak{m}_R$ denotes the maximal ideal.

Let $\widetilde{\Lambda}$ be a complete, regular, local Noetherian ring, let $E \subset \mathfrak{m}^2_{\widetilde{\Lambda}}$ be an ideal, and denote the quotient by Λ. Thus, Λ is also a complete, local Noetherian ring. Denote the residue field $\Lambda/\mathfrak{m}_\Lambda$ by k.

Denote by $\mathcal{C} = \mathcal{C}_\Lambda$ the category whose objects are Λ-algebras A such that

(i) A is a local, Artin ring with $\mathfrak{m}_\Lambda A \subset \mathfrak{m}_A$, and

(ii) the induced homomorphism $k \to A/\mathfrak{m}_\Lambda A$ is an isomorphism.

The morphisms in $\mathcal{C}_\Lambda$ are homomorphisms of Λ-algebras; these are automatically local homomorphisms. Similarly, denote by $\widehat{\mathcal{C}} = \widehat{\mathcal{C}}_\Lambda$ the category whose objects are Λ-algebras $\widehat{A}$ such that

(i) $\widehat{A}$ is a complete, local, Noetherian ring with $\mathfrak{m}_\Lambda \cdot \widehat{A} \subset \mathfrak{m}_{\widehat{A}}$, and

(ii) the induced homomorphism $k \to \widehat{A}/\mathfrak{m}_{\widehat{A}}$ is an isomorphism.

The morphisms in $\widehat{\mathcal{C}}_\Lambda$ are local homomorphisms of Λ-algebras. Of course $\mathcal{C}_\Lambda$ is a full subcategory of $\widehat{\mathcal{C}}_\Lambda$. Moreover, for every $\widehat{A}$ in $\widehat{\mathcal{C}}_\Lambda$, for every integer $N > 0$, the Λ-algebra $\widehat{A}/\mathfrak{m}^N_{\widehat{A}}$ is an object of $\mathcal{C}_\Lambda$.

For every object $\widehat{R}$ in $\widehat{\mathcal{C}}_\Lambda$, denote by $h_{\widehat{\mathcal{C}},\widehat{R}}$ the covariant functor

$$h_{\widehat{\mathcal{C}},\widehat{R}} : \widehat{\mathcal{C}}_\Lambda \to \text{Sets}, \quad \widehat{S} \mapsto \text{Hom}_{\widehat{\mathcal{C}}}(\widehat{R}, \widehat{S}).$$

Also denote by $h_{\mathcal{C},\widehat{R}}$ the restriction of $h_{\widehat{\mathcal{C}},\widehat{R}}$ to the full subcategory $\mathcal{C}$. Observe that $h_{\mathcal{C},\widehat{R}}(k)$ is a singleton set. A *pointed functor* on $\mathcal{C}_\Lambda$ is a covariant functor

$$F_{\mathcal{C}} : \mathcal{C}_\Lambda \to \text{Sets}$$

such that $F_{\mathcal{C}}(k)$ is a singleton set, and similarly for a pointed functor $F_{\widehat{\mathcal{C}}}$ on $\widehat{\mathcal{C}}_\Lambda$. Each of the categories $\mathcal{C}_\Lambda$ and $\widehat{\mathcal{C}}_\Lambda$ has a small, reflective subcategory, e.g., the full category of all objects that are quotients of $\Lambda[\![t_1, t_2, \dots]\!]$ as complete, local, Λ-algebras. Every natural transformation between pointed functors on $\mathcal{C}_\Lambda$, resp. $\widehat{\mathcal{C}}_\Lambda$, is uniquely determined by its restriction to this small, reflective subcategory. There is a set of natural transformations between these restriction functors. Therefore, there is a set of natural transformations between two pointed functors (the "realization" of this set depends on the small, reflective subcategory only up to unique bijection). Thus there is a category $\text{Fun}(\mathcal{C}_\Lambda)$, resp. $\text{Fun}(\widehat{\mathcal{C}}_\Lambda)$ of pointed functors on $\mathcal{C}_\Lambda$, resp. on $\widehat{\mathcal{C}}_\Lambda$. Finally, by the Yoneda Lemma there is a fully faithful embedding

$$h : \mathcal{C} \to \text{Fun}(\mathcal{C}_\Lambda), \quad A \mapsto h_{\mathcal{C},A},$$

and there is a fully faithful embedding

$$h : \widehat{\mathcal{C}} \to \text{Fun}(\widehat{\mathcal{C}}_\Lambda), \quad \widehat{R} \mapsto h_{\widehat{\mathcal{C}},\widehat{R}}.$$

Moreover, for every object A of $\mathcal{C}_\Lambda$ and for every pointed functor F on $\mathcal{C}_\Lambda$, there is a bifunctorial bijection

$$\text{Hom}_{\text{Fun}(\mathcal{C})}(h_{\mathcal{C},A}, F) \leftrightarrow F(A)$$

sending a natural transformation θ to the image under θ of $\text{Id}_A \in h_{\mathcal{C},A}(A)$. There is a similar bifunctorial bijection for $\widehat{\mathcal{C}}_\Lambda$.

Associated to the fully faithful embedding of $\mathcal{C}_\Lambda$ in $\widehat{\mathcal{C}}_\Lambda$, there is a restriction functor

$$\bullet|_{\mathcal{C}} : \text{Fun}(\widehat{\mathcal{C}}_\Lambda) \to \text{Fun}(\mathcal{C}_\Lambda).$$

For a pointed functor $F_{\widehat{\mathcal{C}}}$ on $\widehat{\mathcal{C}}_\Lambda$, denote by $F_{\widehat{\mathcal{C}}}|_{\mathcal{C}}$ the restriction to $\mathcal{C}_\Lambda$. Similarly, for a natural transformation η of pointed functors on $\widehat{\mathcal{C}})_\Lambda$, denote by $\eta|_{\mathcal{C}}$ the associated natural transformation of restriction functors. As we will see below, there is an important right adjoint to the restriction functor

For the pointed functor $h_{\widehat{\mathcal{C}},\widehat{R}}$, for every object $\widehat{S}$ of $\widehat{\mathcal{C}}_\Lambda$, since $\widehat{S}$ is complete the following natural map is a bijection

$$h_{\widehat{\mathcal{C}},\widehat{R}}(\widehat{S}) \to \varprojlim_N h_{\mathcal{C},\widehat{R}}(\widehat{S}/\mathfrak{m}_{\widehat{S}}^N).$$

In general, a pointed functor $G_{\widehat{\mathcal{C}}}$ on $\widehat{\mathcal{C}}_\Lambda$ is *continuous* if for every object $\widehat{S}$ of $\widehat{\mathcal{C}}_\Lambda$ the following natural map is a bijection

$$G_{\widehat{\mathcal{C}}}(\widehat{S}) \to \varprojlim_N G_{\widehat{\mathcal{C}}}|_{\mathcal{C}}(\widehat{S}/\mathfrak{m}_{\widehat{S}}^N).$$

For every pointed functor $F_{\mathcal{C}}$ on $\mathcal{C}_\Lambda$, there exists a pointed functor $\widehat{F}_{\mathcal{C}}$ on $\widehat{\mathcal{C}}_\Lambda$ defined by

$$\widehat{F}_{\mathcal{C}}(\widehat{S}) := \varprojlim_N F_{\mathcal{C}}(\widehat{S}/\mathfrak{m}_{\widehat{S}}^N).$$

Moreover, for every natural transformation of pointed functors $\theta : F_{\mathcal{C}} \Rightarrow F'_{\mathcal{C}}$, there is an associated natural transformation

$$\widehat{\theta} : \widehat{F}_{\mathcal{C}} \to \widehat{F'}_{\mathcal{C}}, \quad \widehat{\theta}(\widehat{S}) = \varprojlim_N \theta(\widehat{S}/\mathfrak{m}_{\widehat{S}}^N).$$

Together these operations define a functor,

$$\widehat{\bullet} : \mathrm{Fun}(\mathcal{C}_\Lambda) \to \mathrm{Fun}(\widehat{\mathcal{C}_\Lambda}).$$

For every pointed functor $F_{\mathcal{C}}$ on $\mathcal{C}_\Lambda$, there is a natural transformation of pointed functors on $\mathcal{C}_\Lambda$,

$$\alpha'_F : F_{\mathcal{C}} \Rightarrow \widehat{F}_{\mathcal{C}}|_{\mathcal{C}}, \quad F_{\mathcal{C}}(A) \to \varprojlim_N F_{\mathcal{C}}(A/\mathfrak{m}_A^N).$$

In fact this is a natural transformation from the identity functor on $\mathrm{Fun}(\mathcal{C}_\Lambda)$ to the composite functor $(\widehat{\bullet})|_{\mathcal{C}}$,

$$\alpha' : \mathrm{Id}_{\mathrm{Fun}(\mathcal{C})} \Rightarrow (\widehat{\bullet})|_{\mathcal{C}}.$$

Morever, because $A \to A/\mathfrak{m}_A^N$ is an isomorphism for N sufficiently large, each α'_F is a natural equivalence of functors, i.e., α' is a natural equivalence of functors. For this reason we shall usually identify $F_{\mathcal{C}}$ with $\widehat{F}_{\mathcal{C}}|_{\mathcal{C}}$. Moreover, we will denote by α and α_F the inverse natural equivalence,

$$\alpha : (\widehat{\bullet})|_{\mathcal{C}} \Rightarrow \mathrm{Id}_{\mathrm{Fun}(\mathcal{C})}.$$

Similarly, for every pointed functor G on $\widehat{\mathcal{C}}_\Lambda$, there is a natural transformation of pointed functors on $\widehat{\mathcal{C}}_\Lambda$,

$$\beta_G : G \Rightarrow \widehat{G|_{\mathcal{C}}}, \quad G(\widehat{S}) \to \varprojlim_N G|_{\mathcal{C}}(\widehat{S}/\mathfrak{m}_{\widehat{S}}^N).$$

This is also natural in G, hence defines a natural transformation from the identity functor on $\widehat{\mathcal{C}}$ to the composite functor $\widehat{(\bullet)|_{\mathcal{C}}}$,

$$\beta : \mathrm{Id}_{\mathrm{Fun}(\widehat{\mathcal{C}})} \Rightarrow \widehat{(\bullet)|_{\mathcal{C}}}.$$

By definition, G is continuous if and only if β_G is a natural bijection.

For every pointed functor G on $\widehat{\mathcal{C}}$ and for every pointed functor F on $\mathcal{C}$, for every natural transformation of pointed functors on $\mathcal{C}$,

$$\theta : G|_{\mathcal{C}} \Rightarrow F,$$

there is an associated natural transformation of pointed functors on $\widehat{\mathcal{C}}$,

$$\widehat{\theta} \circ \beta_G : F \Rightarrow \widehat{G|_{\mathcal{C}}} \Rightarrow \widehat{F}.$$

Similarly, for every natural transformation of pointed functors on $\mathcal{C}$,

$$\eta : G \Rightarrow \widehat{F}$$

there is an associated natural transformation of pointed functors on $\mathcal{C}$,

$$\alpha_F \circ \eta|_{\mathcal{C}} : G|_{\mathcal{C}} \Rightarrow \widehat{F}|_{\mathcal{C}} \Rightarrow F.$$

Lemma 1.1. *The functors $\bullet|_{\mathcal{C}}$ and $\widehat{\bullet}$ together with the natural transformations α and β form an adjoint pair, i.e., the bifunctorial set maps*

$$Hom_{Fun(\mathcal{C})}(G|_{\mathcal{C}}, F) \to Hom_{Fun(\widehat{\mathcal{C}})}(G, \widehat{F}),\ \theta \mapsto \widehat{\theta} \circ \beta_G,$$

$$Hom_{Fun(\widehat{\mathcal{C}})}(G, \widehat{F}) \to Hom_{Fun(\mathcal{C})}(G|_{\mathcal{C}}, F),\ \eta \mapsto \alpha_F \circ \eta|_{\mathcal{C}},$$

are inverse bijections for every F and G.

Proof. This will be an exercise on Problem Set 2. □

We are mainly interested in the case where G is a representable functor, $h_{\widehat{\mathcal{C}},\widehat{R}}$. In this case, by Lemma 1.1, every natural transformation $\theta : h_{\mathcal{C},\widehat{R}} \Rightarrow F$ is equivalent to a natural transformation $\widehat{\theta} \circ \beta_{\widehat{R}} : h_{\widehat{\mathcal{C}},\widehat{R}} \Rightarrow \widehat{F}$. By the Yoneda Lemma, this is equivalent to an element of $\widehat{F}(\widehat{R})$, i.e., a datum

$$(\theta_N)_{N>0},\ \theta_N \in F(\widehat{R}/\mathfrak{m}_{\widehat{R}}^N)$$

that is a *compatible family* in the sense that for every $N > 0$, for the set map $F(\widehat{R}/\mathfrak{m}_{\widehat{R}}^{N+1}) \to F(\widehat{R}/\mathfrak{m}_{\widehat{R}}^N)$ associated to the canonical surjection, the element θ_{N+1} maps to θ_N.

Here is the basic definition of this section.

Definition 1.2. A pointed functor F on $\mathcal{C}_\Lambda$ is *prorepresentable* if there exists an object $\widehat{R}$ in $\widehat{\mathcal{C}}_\Lambda$ and a natural equivalence of functors on $\mathcal{C}_\Lambda$, $\theta : h_{\mathcal{C},\widehat{R}} \Rightarrow F$.

By the discussion above, F is prorepresentable if and only if there exists a natural equivalence of functors there exists an object $\widehat{R}$ in $\widehat{\mathcal{C}}_\Lambda$ and a compatible family $(\theta_N)_{N>0}$ of elements $\theta_n \in F(\widehat{R}/\mathfrak{m}_{\widehat{R}}^N)$ such that for every object A of $\mathcal{C}_\Lambda$, for every object $a \in F(A)$, and for one (hence every) integer $N > 0$ such that $\mathfrak{m}_A^N$ equals $\{0\}$, there exists a unique local homomorphism of Λ-algebra, $u_n : \widehat{R}/\mathfrak{m}_R^N \to A$ such that $F(u_n)$ maps θ_n to a.

The thesis of Michael Schlessinger, [Sch68], and also work of Rim, characterizes prorepresentable functors, as well as functors admitting a hull, in terms of an efficient list of axioms that can be verified for many functors of interest.

2 Obstruction Theory.

Let F be a pointed functor on $\mathcal{C}_\Lambda$. There are many different uses of obstruction theories, hence there are many definitions. The following definition is essentially the definition of Artin, [Art74, Definition (2.6) p. 169].

Definition 2.1. An *infinitesimal extension* in $\mathcal{C}_\Lambda$ is a surjective homomorphism $q : A' \to A$ such that $\mathfrak{m}_{A'} \cdot \text{Ker}(q)$ equals $\{0\}$. For infinitesimal extensions $q_A : A' \to A$ and $q_B : B' \to B$, a *morphism* of infinitesimal extensions is a pair (u, u') of morphisms $u : A \to B$ and $u' : A' \to B'$ such that $u \circ q_A$ equals $q_B \circ u'$. The restriction of u' to $\text{Ker}(q_A)$ gives a morphism denoted by $u'_K : \text{Ker}(q_A) \to \text{Ker}(q_B)$.
A *deformation situation* is a pair $(q : A' \to A, a)$ of an infinitesimal extension and an element $a \in F(A)$. For deformation situations $(q_A : A' \to A, a)$ and $(q_B : B' \to B, b)$, a *morphism* of deformation situations is a morphism of infinitesimal extensions, (u, u'), such that $F(u)$ maps a to b.

With these notions, there is a category InfDef_Λ of infinitesimal extensions as well as a category $\text{Def}_{\Lambda,F}$ of deformation situations. There is a forgetful functor

$$\Phi : \text{Def}_{\Lambda,F} \to \text{InfDef}_\Lambda.$$

For a morphism (u, u') of infinitesimal deformations, denote by u'_K the restriction of u' to $\text{Ker}(q_A)$,

$$u'_K : \text{Ker}(q_A) \to \text{Ker}(q_B).$$

This is a k-linear map of finite dimensional k-vector space, and it is functorial in (u, u'). Thus there is a functor from the category of infinitesimal extensions to the category of finite dimensional k-vector space,

$$K : \text{InfDef}_\Lambda \to \text{Vec}_k,\ (q : A' \to A) \mapsto \text{Ker}(q),\ (u, u') \mapsto u'_K.$$

Also there is the composite functor,

$$K \circ \Phi : \text{Def}_{\Lambda,F} \to \text{Vec}_k.$$

Similarly, for every finite dimensional k-vector space $\mathcal{O}$, there is a functor

$$\mathcal{O} \otimes_k K : \text{InfDef}_\Lambda \to \text{Vec}_k,\ (q : A' \to A) \mapsto \mathcal{O} \otimes_k \text{Ker}(q),\ (u, u') \mapsto \text{Id}_\mathcal{O} \otimes_k u'_K,$$

and there is also the functor

$$\mathcal{O} \otimes_k (K \circ \Phi) : \text{Def}_{\Lambda,F} \to \text{Vec}_k.$$

A *section* o of the functor $\mathcal{O} \otimes_k (K \circ \Phi)$ over $\text{Def}_{\Lambda,F}$ is an assignment to every deformation situation $(q : A' \to A, a)$ of an element $o_{q,a} \in \mathcal{O} \otimes_k \text{Ker}(q)$ such that for every morphism of deformation situations, (u, u'), the k-linear map

$$\text{Id}_\mathcal{O} \otimes_k u'_K : \mathcal{O} \otimes_k \text{Ker}(q_A) \to \mathcal{O} \otimes_k \text{Ker}(q_B)$$

maps $o_{q_A,a}$ to $o_{q_B,b}$. Equivalently, introducing the functor

$$\mathbf{1} : \mathrm{Def}_{\Lambda,F} \to \mathrm{Vec}_k, \quad \mathbf{1}(q,a) = k, \quad \mathbf{1}(u,u') = \mathrm{Id}_k,$$

a section o is a natural transformation $o : \mathbf{1} \Rightarrow \mathcal{O} \otimes_k (K \circ \Phi)$.

Definition 2.2. An *obstruction theory* for F is a pair $(\mathcal{O}, o)$ of a finite dimensional k-vector space $\mathcal{O}$ together with a section o of $\mathcal{O} \otimes_k (K \circ \Phi)$ over $\mathrm{Def}_{\Lambda,F}$ such that for every deformation situation $(q : A' \to A, a)$, there exists $a' \in F(A')$ mapping to a under $F(q)$ if and only if $o_{(q,a)}$ equals 0 as elements of $\mathcal{O} \otimes_k \mathrm{Ker}(q)$. A *morphism* of obstruction theories, $(\mathcal{O}, o)$ and $(\mathcal{O}', o')$ is a k-linear map $L : \mathcal{O} \to \mathcal{O}'$ such that for every deformation situation $(q : A' \to A, a)$, $L \otimes_k \mathrm{Id}_{\mathrm{Ker}(f)}$ maps $o_{q,a}$ to $o'_{q,a}$.

Every prorepresentable functor $h_{\mathcal{C},\widehat{R}}$ has a canonical associated obstruction theory that is functorial in $\widehat{R}$. The simplest construction I know of passes through non-Noetherian rings. So the obstruction theory that follows is not quite the canonical one, however it is (non-canonically) isomorphic to the canonical obstruction theory. Before describing the obstruction theory, there is some setup. Denote by $\Lambda_n := \Lambda[\![t_1,\dots,t_n]\!]$ the power series ring over Λ, considered as a complete, local, Noetherian Λ-algebra. For every complete, local, Noetherian Λ-algebra $\widehat{R}$ and for every ordered n-tuple of elements $\underline{r} = (r_1,\dots,r_n)$ in $\mathfrak{m}_{\widehat{R}}$, by the universal property of power series algebras there exists a unique local homomorphism of Λ-algebras,

$$f_{\widehat{R},\underline{r}} : \Lambda[\![t_1,\dots,t_n]\!] \to \widehat{R}, \quad f(t_i) = r_i.$$

In fact, the rule $\underline{r} \mapsto f_{\widehat{R},\underline{r}}$ gives a natural equivalence of functors $\Updownarrow^{\oplus n} \to h_{\widehat{\mathcal{C}},\Lambda_n}$, where $\mathfrak{m}^{\oplus n}$ is the obvious functor

$$\mathfrak{m}^{\oplus n} : \widehat{\mathcal{C}}_\Lambda \to \mathrm{Sets}, \quad \widehat{R} \mapsto \mathfrak{m}_{\widehat{R}}^{\oplus n}, \ (u : \widehat{R} \to \widehat{S}) \mapsto (u^{\oplus n} : \mathfrak{m}_{\widehat{R}}^{\oplus n} \to \mathfrak{m}_{\widehat{S}}^{\oplus n}).$$

Using Nakayama's Lemma and completeness, $f_{\widehat{R},\underline{r}}$ is surjective if and only if the images $\overline{r}_1,\dots,\overline{r}_n \in \mathfrak{m}_{\widehat{R}}/\mathfrak{m}_{\widehat{R}}^2$ generate as a k-vector space. Moreover, the images form a k-basis if and only if the induced map on $\mathfrak{m}/\mathfrak{m}^2$ is an isomorphism of k-vector spaces. In particular, for $r_1,\dots,r_n \in \mathfrak{m}_{\Lambda_n}$, the associated homomorphism

$$f_{\Lambda_n,\underline{r}} : \Lambda[\![t_1,\dots,t_n]\!] \to \Lambda[\![t_1,\dots,t_n]\!], \ f(t_i) = r_i$$

is an isomorphism if and only if $\overline{r}_1,\dots,\overline{r}_n$ forms a basis for $\mathfrak{m}/\mathfrak{m}^2$. Also $f_{\Lambda_n,\underline{r}}$ is an isomorphism inducing the identity map on $\mathfrak{m}/\mathfrak{m}^2$ if and only if each r_i equals $t_i + s_i$ for $s_i \in \mathfrak{m}^2$. For each ordered n-tuple $\underline{s} = (s_1,\dots,s_n)$ of elements $s_i \in \mathfrak{m}^2$, denote by $f_{\Lambda_n,\underline{t}+\underline{s}}$ the associated isomorphism of $\Lambda[\![t_1,\dots,t_n]\!]$ inducing the identity on $\mathfrak{m}/\mathfrak{m}^2$.

For every object $\widehat{R}$ of $\widehat{\mathcal{C}}_\Lambda$, and for every $\underline{r} = (r_1,\dots,r_n)$ in $\mathfrak{m}_{\widehat{R}}$ mapping to a basis $\overline{r}_1,\dots,\overline{r}_n$ of $\mathfrak{m}_{\widehat{R}}/\mathfrak{m}_{\widehat{R}}^2$, there exists a surjection

$$f_{\widehat{R},\underline{r}} : \Lambda[\![t_1,\dots,t_n]\!] \to \widehat{R},$$

mapping the basis $(\overline{t}_i)_i$ of $\mathfrak{m}/\mathfrak{m}^2$ to the basis $(\overline{r}_i)_i$ of $\mathfrak{m}/\mathfrak{m}^2$.

Definition 2.3. For every object $\widehat{R}$ in $\widehat{\mathcal{C}}_\Lambda$ and for every lift $\underline{r}$ to $\mathfrak{m}_{\widehat{R}}$ of a k-basis of $\mathfrak{m}_{\widehat{R}}/\mathfrak{m}_{\widehat{R}}^2$, define the *associated obstruction space* to be the finite dimensional "dual" k-vector space

$$\mathcal{O}_{\widehat{R},\underline{r}} := \mathrm{Hom}_k(\mathrm{Ker}(f_{\widehat{R},\underline{r}})/\mathfrak{m}\cdot\mathrm{Ker}(f_{\widehat{R},\underline{r}}), k).$$

If $\underline{r}'$ maps to the same basis, then $f_{\widehat{R},\underline{r}'}$ equals $f_{\widehat{R},\underline{r}} \circ f_{\Lambda_n,\underline{t}+\underline{s}}$ for some choice of $\underline{s}$.

Lemma 2.4. *For every integer $e \geq 0$, the Λ-algebra automorphism $f_{\Lambda_n,\underline{t}+\underline{s}}$ maps $\mathfrak{m}_{\Lambda_n}\cdot Ker(f_{\widehat{R},\underline{r}'})$ isomorphically onto $\mathfrak{m}_{\Lambda_n}\cdot Ker(f_{\widehat{R},\underline{r}})$. In particular, $f_{\Lambda_n,\underline{t}+\underline{s}}$ induces a k-linear isomorphism*

$$f_{\Lambda_n,\underline{t}+\underline{s},e} : Ker(f_{\widehat{R},\underline{r}'})/\left(\mathfrak{m}_{\Lambda_n}\cdot Ker(f_{\widehat{R},\underline{r}'}) + \mathfrak{m}_{\Lambda_n}^e \cap Ker(f_{\widehat{R},\underline{r}'})\right) \to Ker(f_{\widehat{R},\underline{r}'})/\left(\mathfrak{m}_{\Lambda_n}\cdot Ker(f_{\widehat{R},\underline{r}'}) + \mathfrak{m}_{\Lambda_n}^e \cap Ker(f_{\widehat{R},\underline{r}'}\right.$$

If $Ker(f_{\widehat{R},\underline{r}'})$ is contained in $Ker(f_{\widehat{R},\underline{r}}) + \mathfrak{m}_{\Lambda_n}^e$, then the following ideals in $\Lambda[\![t_1,\dots,t_n]\!]$ are equal,

$$Ker(f_{\widehat{R},\underline{r}'}) + \mathfrak{m}_{\Lambda_n}^e = Ker(f_{\widehat{R},\underline{r}}) + \mathfrak{m}_{\Lambda_n}^e,$$

$$\mathfrak{m}_{\Lambda_n}\cdot Ker(f_{\widehat{R},\underline{r}'}) + \mathfrak{m}_{\Lambda_n}^e = \mathfrak{m}_{\Lambda_n}\cdot Ker(f_{\widehat{R},\underline{r}}) + \mathfrak{m}_{\Lambda_n}^e,$$

and the k-linear map $f_{\Lambda_n,\underline{t}+\underline{s},e}$ above is an automorphism.

Proof. This will be an exercise on Problem Set 2. □

When $e = 0$, the morphism $f_{\Lambda_n,\underline{t}+\underline{s}}$ induces an isomorphism of obstruction spaces, $\mathcal{O}_{\widehat{R},\underline{r}} \to \mathcal{O}_{\widehat{R},\underline{r}'}$ (note that the variance is reversed).

Let $q : A' \to A$ be an infinitesimal extension in $\mathcal{C}_\Lambda$. Let $a : \widehat{R} \to A$ be a morphism in $\widehat{\mathcal{C}}_\Lambda$. Then there is an induced morphism

$$\widetilde{a} : \Lambda[\![t_1,\dots,t_n]\!] \to A, \quad \widetilde{a} = a \circ f_{\widehat{R},\underline{r}}.$$

Denote by $(\widetilde{a}_1,\dots,\widetilde{a}_n)$ the images of $(t_1,\dots,t_n)$ under $\widetilde{a}$. Since q is surjective, there exists an ordered n-tuple of elements $(\widetilde{a}'_1,\dots,\widetilde{a}'_n)$ mapping to $(\widetilde{a}_1,\dots,\widetilde{a}_n)$ under q. By the universal property of power series algebras, there exists a unique local homomorphism of Λ-algebras,

$$\widetilde{a}' : \Lambda[\![t_1,\dots,t_n]\!] \to A'$$

mapping $(t_1,\dots,t_n)$ to $(\widetilde{a}'_1,\dots,\widetilde{a}'_n)$. By construction $q\circ\widetilde{a}'$ equals $\widetilde{a}$. Thus $\widetilde{a}'$ maps $\mathrm{Ker}(\widetilde{a})$ to $\mathrm{Ker}(q)$. In particular $\widetilde{a}'$ maps $\mathrm{Ker}(f_{\widehat{R},\underline{r}})$ to $\mathrm{Ker}(q)$. Denote this induced $\Lambda[\![t_1,\dots,t_n]\!]$-module homomorphism by

$$o^{\mathrm{pre}}_{q,a,\underline{r},\widetilde{a}'} : \mathrm{Ker}(f_{\widehat{R},\underline{r}}) \to \mathrm{Ker}(q).$$

Because $q : A' \to A$ is an infinitesimal extension, $\mathfrak{m}_{A'}\cdot\mathrm{Ker}(q)$ equals $\{0\}$. Because $\widetilde{a}'$ is a local homomorphism, the image of $\mathfrak{m}\cdot\mathrm{Ker}(f_{\widehat{R},\underline{r}})$ is contained in $\mathfrak{m}_{A'}\cdot\mathrm{Ker}(q)$, which is $\{0\}$. Thus the map above factors uniquely through a k-linear map

$$o_{q,a,\underline{r},\widetilde{a}'} : \mathrm{Ker}(f_{\widehat{R},\underline{r}})/\mathfrak{m}_{\Lambda_n}\cdot\mathrm{Ker}(f_{\widehat{R},\underline{r}}) \to \mathrm{Ker}(q).$$

Because the k-vector spaces involved are finite dimensional, this is equivalent to an element in $\mathcal{O}_{\widehat{R},\underline{r}} \otimes_k \mathrm{Ker}(q)$.

Proposition 2.5. *Each element $o_{q,a,\underline{r},\widetilde{a}'}$ in $\mathcal{O}_{\widehat{R},\underline{r}} \otimes_k Ker(q)$ is independent of the choice of lift $\widetilde{a}'$ of $\widetilde{a}$. Moreover, the pair $(\mathcal{O}_{\widehat{R},\underline{r}}, o)$ is an obstruction theory for the Yoneda functor $h_{\mathcal{C},\widehat{R}}$.*

Proof. Every other lift is $\widehat{a}''_i = \widehat{a}'_i + \kappa_i$ for elements $\kappa_i \in \mathrm{Ker}(q)$. In particular, every quadratic monomial is

$$\widehat{a}''_i \cdot \widehat{a}''_j = \widehat{a}'_i \cdot \widehat{a}'_j + (\widehat{a}'_i \cdot \kappa_j + \widehat{a}'_j \cdot \kappa_i + \kappa_i \cdot \kappa_j).$$

Since $\mathfrak{m}_{A'} \cdot \mathrm{Ker}(q)$ equals $\{0\}$, each element in the parentheses equals 0 in A'. Therefore the restricted map

$$\widetilde{a}'_2 : \mathfrak{m}^2_{\Lambda_n} \to A'$$

is independent of the choice of the lift $\widetilde{a}'$. Denote this map by $\widetilde{a}_2$. Since $\mathrm{Ker}(f_{\widehat{R},\underline{r}})$ is contained in $\mathfrak{m}^2_{\Lambda_n}$, it follows that $o_{q,a,\underline{r},\widetilde{a}'}$ is independent of the choice of $\widehat{a}'$.

If there exists a lift $a' : \widehat{R} \to A'$ of a, then we can define $\widetilde{a}' = a' \circ f_{\widehat{R},\underline{r}}$. This maps $\mathrm{Ker}(f_{\widehat{R},\underline{r}})$ to $\{0\}$, so that $o_{q,a,ulr}$ equals 0. Conversely, if $o_{q,a,\underline{r}}$ equals 0, then $\widetilde{a}'$ maps $\mathrm{Ker}(f_{\widehat{R},\underline{r}})$ to $\{0\}$, hence factors through a morphism $a' : \widehat{R} \to A'$ that lifts a. Therefore $o_{q,a,\underline{r}}$ equals 0 if and only if there exists a lift a' of a.

For a morphism (u, u') of deformation situations $(q_A : A' \to A, a) \to (q_B : B' \to B, b)$, for every lift $\widetilde{a}'$ of $\widetilde{a}$, the composition $\widetilde{b}' := u' \circ \widetilde{a}'$ is a lift of $\widetilde{b}$. Using these two lifts, it follows directly that $\mathrm{Id}_{\mathcal{O}} \otimes_k u'_K$ maps $o_{q_A,a,\underline{r}}$ to $o_{q_B,b,\underline{r}}$. Therefore o is a section of $\mathcal{O}_{\widehat{R},\underline{r}} \otimes (K \circ \Phi)$ over the category of deformation situations. Therefore $(\mathcal{O}_{\widehat{R},\underline{r}}, o_{\underline{r}})$ is an obstruction theory for $h_{\mathcal{C},\widehat{R}}$. □

One can eliminate the dependence of this obstruction theory on $\underline{r}$, and also make the obstruction theory functorial in $\widehat{R}$, but this requires a detour through non-Noetherian rings. For every $\widehat{R}$, denote by $M = M_{\widehat{R}}$ the free Λ-module on the underlying set of $\mathfrak{m}_{\widehat{R}}$. One can begin with a canonical choice of a free Λ-module M together with a surjection $M \to \mathfrak{m}_{\widehat{R}}$, e.g., the free module on the underlying set of $\mathfrak{m}_{\widehat{R}}$. Then one can form the (infinitely generated) symmetric algebra $\Lambda[M]$. One can complete this with respect to the maximal ideal $\mathfrak{m}_\Lambda \cdot \Lambda[M] + M \cdot \Lambda[M]$ to obtain a complete, local Λ-algebra $\Lambda\llbracket M \rrbracket$ (that is very non-Noetherian). This comes with a *canonical* surjection $f : \Lambda\llbracket M \rrbracket \to \widehat{R}$. One can define $\mathcal{O}_{\widehat{R}}$ to be the kernel of the map $\mathrm{Ker}(f)/\mathfrak{m} \cdot \mathrm{Ker}(f) \to \mathfrak{m}/\mathfrak{m}^2$. For every choice of $f_{\widehat{R},\underline{r}} : \Lambda\llbracket t_1, \dots, t_n \rrbracket \to \widehat{R}$, there is a canonical associated local homomorphisms of Λ-algebras, $\Lambda\llbracket t_1, \dots, t_n \rrbracket \to \Lambda\llbracket M \rrbracket$ that induces an isomorphism of k-vector spaces

$$i_{\underline{r}} : \mathrm{Ker}(f_{\widehat{R},\underline{r}})/\mathfrak{m} \cdot \mathrm{Ker}(f_{\widehat{R},\underline{r}}) \to \mathcal{O}_{\widehat{R}}.$$

Then for every deformation situation $(q : A' \to A, a)$, this isomorphism maps the element $o_{q,a,\underline{r}}$ above to an element $o_{q,a} \in \mathcal{O}_{\widehat{R}} \otimes_k \mathrm{Ker}(q)$. It is not hard to check that $o_{q,a}$ is independent of the choice of $\underline{r}$, essentially because the dependence of the isomorphism $i_{\underline{r}}$ on the choice of $\underline{r}$ is "inverse" to the dependence of the element $o_{q,a,\underline{r}}$ on $\underline{r}$.

The canonical obstruction theory on a representable functor is minimal in the following sense.

Proposition 2.6. *For every obstruction theory $(\mathcal{O}, o)$ of $h_{\mathcal{C},\widehat{R}}$, there exists an injective, k-linear map $L : \mathcal{O}_{\widehat{R},\underline{r}} \to \mathcal{O}$ that is a morphism of obstruction theories.*

Proof. First of all, notice that $\mathcal{O}_{\widehat{R},\underline{r}}$ equals $\{0\}$ if and only if If $f_{\widehat{R},\underline{r}}$ is an isomorphism. In this case the zero map, $L = 0$, is the unique k-linear map, and it is trivially a map of obstruction theories. Thus, without loss of generality, assume that $\mathcal{O}_{\widehat{R},\underline{r}}$ is nonzero.

By the Artin-Rees Lemma, there exists an integer E such that $\mathfrak{m}_{\Lambda_n}^E \cap \text{Ker}(f_{\widehat{R},\underline{r}})$ is contained in $\mathfrak{m}_{\Lambda_n} \cdot \text{Ker}(f_{\widehat{R},\underline{r}})$ as ideals in $\Lambda\,[\![t_1,\dots,t_n]\!]$. Consider the following natural surjection of local, Artin Λ-algebras,

$$Q : \Lambda\,[\![t_1,\dots,t_n]\!] \,/\left(\mathfrak{m}_{\Lambda_n}\cdot\text{Ker}(f_{\widehat{R},\underline{r}}) + \mathfrak{m}_{\Lambda_n}^E\right) \to \Lambda\,[\![t_1,\dots,t_n]\!] \,/\left(\text{Ker}(f_{\widehat{R},\underline{r}}) + \mathfrak{m}_{\Lambda_n}^E\right).$$

Denote by $\widetilde{\alpha}$, resp. $\widetilde{\alpha}'$, the natural surjection from $\Lambda\,[\![t_1,\dots,t_n]\!]$ to the target of Q, resp. the source of Q. Since $f_{\widehat{R},\underline{r}}$ is a surjection, there exists a unique surjection

$$\alpha : \widehat{R} \to \Lambda\,[\![t_1,\dots,t_n]\!] \,/\left(\text{Ker}(f_{\widehat{R},\underline{r}}) + \mathfrak{m}_{\Lambda_n}^E\right)$$

such that $\alpha \circ f_{\widehat{R},\underline{r}}$ equals $\widetilde{\alpha}$. The kernel of Q equals

$$\left(\text{Ker}(f_{\widehat{R},\underline{r}}) + \mathfrak{m}_{\Lambda_n}^E\right) / \left(\mathfrak{m}_{\Lambda_n}\cdot\text{Ker}(f_{\widehat{R},\underline{r}}) + \mathfrak{m}_{\Lambda_n}^E\right) \cong \left(\text{Ker}(f_{\widehat{R},\underline{r}})\right) / \left(\mathfrak{m}_{\Lambda_n}\cdot\text{Ker}(f_{\widehat{R},\underline{r}}) + \mathfrak{m}_{\Lambda_n}^E \cap \text{Ker}(f_{\widehat{R},\underline{r}})\right).$$

By construction of E, this is canonically isomorphic to $\text{Ker}(f_{\widehat{R},\underline{r}})/\mathfrak{m}_{\Lambda_n}\cdot\text{Ker}(f_{\widehat{R},\underline{r}})$. Thus the obstruction element $o_{Q,\alpha}$ is an element in

$$\mathcal{O} \otimes_k \left(\text{Ker}(f_{\widehat{R},\underline{r}})/\mathfrak{m}_{\Lambda_n}\cdot\text{Ker}(f_{\widehat{R},\underline{r}})\right) = \text{Hom}_k(\mathcal{O}_{\widehat{R},\underline{r}}, \mathcal{O}).$$

For every nonzero element

$$\phi : \text{Ker}(f_{\widehat{R},\underline{r}})/\mathfrak{m}_{\Lambda_n}\cdot\text{Ker}(f_{\widehat{R},\underline{r}}) \to k$$

in $\mathcal{O}_{\widehat{R},\underline{r}}$, denote by I_ϕ the unique ideal in $\Lambda\,[\![t_1,\dots,t_n]\!]$ containing $\mathfrak{m}_{\Lambda_n}\cdot\text{Ker}(f_{\widehat{R},\underline{r}}) + \mathfrak{m}_{\Lambda_n}^E$, contained in $\text{Ker}(f_{\widehat{R},\underline{r}}) + \mathfrak{m}_{\Lambda_n}^E$, and whose image in $\text{Ker}(f_{\widehat{R},\underline{r}})/\mathfrak{m}_{\Lambda_n}\cdot\text{Ker}(f_{\widehat{R},\underline{r}})$ equals $\text{Ker}(\phi)$. Denote by $\widetilde{\alpha}'_\phi$ the following natural surjection,

$$\widetilde{\alpha}'_\phi : \Lambda\,[\![t_1,\dots,t_n]\!] \to \Lambda\,[\![t_1,\dots,t_n]\!]\,/I_\phi.$$

Since I_ϕ is contained in $\text{Ker}(f_{\widehat{R},\underline{r}}) + \mathfrak{m}_{\Lambda_n}^E$, there exists a unique surjection

$$q_\phi : \Lambda\,[\![t_1,\dots,t_n]\!]\,/I_\phi \to \Lambda\,[\![t_1,\dots,t_n]\!] \,/\left(\mathfrak{m}_{\Lambda_n}\cdot\text{Ker}(f_{\widehat{R},\underline{r}}) + \mathfrak{m}_{\Lambda_n}^E\right)$$

such that $q_\phi \circ \widetilde{\alpha}'_\phi$ equals $\widetilde{\alpha}$. Similarly, since I_ϕ contains $\mathfrak{m}_{\Lambda_n}\cdot\text{Ker}(f_{\widehat{R},\underline{r}}) + \mathfrak{m}_{\Lambda_n}^E$, there exists a unique surjection

$$u'_\phi : \Lambda\,[\![t_1,\dots,t_n]\!] \,/\left(\mathfrak{m}_{\Lambda_n}\cdot\text{Ker}(f_{\widehat{R},\underline{r}}) + \mathfrak{m}_{\Lambda_n}^E\right) \to \Lambda\,[\![t_1,\dots,t_n]\!]\,/I_\phi,$$

such that $u'_\phi \circ \widetilde{\alpha}'$ equals $\widetilde{\alpha}'_\phi$. Finally, the morphism ϕ determines an isomorphism of $\text{Ker}(q_\phi)$ with k such that the following diagram commutes,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \text{Ker}(f)/\mathfrak{m}\cdot\text{Ker}(f) & \longrightarrow & \Lambda[\![t_1,\dots,t_n]\!]/\left(\mathfrak{m}\cdot\text{Ker}(f)+\mathfrak{m}^E\right) & \xrightarrow{Q} & \Lambda[\![t_1,\dots,t_n]\!]/\left(\text{Ker}(f)+\mathfrak{m}^E\right) \\ & & \phi\downarrow & & u'_\phi\downarrow & & \downarrow\text{Id} \\ 0 & \longrightarrow & k & \longrightarrow & \Lambda[\![t_1,\dots,t_n]\!]/I_\phi & \xrightarrow{q_\phi} & \Lambda[\![t_1,\dots,t_n]\!]/\left(\text{Ker}(f)+\mathfrak{m}^E\right). \end{array}$$

In particular, (u'_ϕ, Id) is a morphism of deformation situations. Since o is a section of $\mathcal{O}\otimes_k K$, the associated k-linear map

$$\text{Id}_{\mathcal{O}}\otimes\phi : \mathcal{O}\otimes_k\left(\text{Ker}(f_{\widehat{R},\underline{r}})/\mathfrak{m}_{\Lambda_n}\cdot\text{Ker}(f_{\widehat{R},\underline{r}})\right)\to\mathcal{O}\otimes_k$$

maps $o_{Q,\alpha}$ to $o_{q_\phi,\alpha}$. Finally, $o_{\widehat{R},\underline{r},q_\phi,\alpha}$ is the element ϕ as an element in $\mathcal{O}_{\widehat{R},\underline{r}}\otimes_k k = \mathcal{O}_{\widehat{R},\underline{r}}$. By hypothesis this is nonzero. Thus there exists no lift α'_ϕ such that $q_\phi\circ\alpha'_\phi$ equals α. Therefore, since $(\mathcal{O}, o)$ is an obstruction theory, also $o_{q_\phi,\alpha}$ is nonzero. On the other hand, $o_{q_\phi,\alpha}$ equals $L(\phi)$ in $\mathcal{O}$. Therefore $L(\phi)$ is nonzero for every nonzero ϕ in $\mathcal{O}_{\widehat{R},\underline{r}}$, i.e., L is an injective k-linear map.

For every deformation situation $(q : A'\to A, a)$, there exists an integer $e\geq E$ such that $\mathfrak{m}^e_{A'}$ equals $\{0\}$. Then there exists a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \text{Ker}(f)/\mathfrak{m}\cdot\text{Ker}(f) & \longrightarrow & \Lambda[\![t_1,\dots,t_n]\!]/\left(\mathfrak{m}\cdot\text{Ker}(f)+\mathfrak{m}^E\right) & \xrightarrow{Q} & \Lambda[\![t_1,\dots,t_n]\!]/\left(\text{Ker}(f)+\mathfrak{m}^E\right) \\ & & u'_K\downarrow & & u'\downarrow & & \downarrow u \\ 0 & \longrightarrow & \text{Ker}(a) & \longrightarrow & A' & \xrightarrow{q} & A, \end{array}$$

where u is the unique homomorphism such that $u\circ\alpha$ equals a, and where u' exists for the same reason that $\widetilde{a}'$ exists in the proof of Proposition 2.5. Because o is a section of $\mathcal{O}\otimes_k K$, the k-linear map

$$\text{Id}\otimes_k u'_K : \mathcal{O}\otimes_k\left(\text{Ker}(f_{\widehat{R},\underline{r}})/\mathfrak{m}_{\Lambda_n}\cdot\text{Ker}(f_{\widehat{R},\underline{r}})\right)\to\mathcal{O}\otimes_k\text{Ker}(q)$$

maps $o_{Q,\alpha}$ to $o_{q,a}$. Also, by the definition of L, $o_{Q,\alpha}$ equals the image under $L\otimes\text{Id}$ of the identity map

$$\text{Id}\in\text{Hom}_k\left(\text{Ker}(f_{\widehat{R},\underline{r}})/\mathfrak{m}_{\Lambda_n}\cdot\text{Ker}(f_{\widehat{R},\underline{r}}),\text{Ker}(f_{\widehat{R},\underline{r}})/\mathfrak{m}_{\Lambda_n}\cdot\text{Ker}(f_{\widehat{R},\underline{r}})\right)=\mathcal{O}\otimes_k\left(\text{Ker}(f_{\widehat{R},\underline{r}})/\mathfrak{m}_{\Lambda_n}\cdot\text{Ker}(f_{\widehat{R},\underline{r}})\right).$$

Thus $o_{q,a}$ equals the image of Id under the k-linear map

$$(\text{Id}\otimes_k u'_K)\circ(L\otimes_k\text{Id}) = L\otimes_k u'_K.$$

But of course this also equals $(L\otimes_k\text{Id})\circ(\text{Id}\otimes_k u'_K)$. The image of Id under $\text{Id}\otimes_k u'_K$ is the definition of $o_{\widehat{R},\underline{r},q,a}$. Therefore, $L\otimes\text{Id}$ maps $o_{\widehat{R},\underline{r},q,a}$ to $o_{q,a}$ for every deformation situation (q,a). In other words, L is a morphism of obstruction theories. $\square$

The main application is the following.

Corollary 2.7. *Let $(\mathcal{O}, o)$ be an obstruction theory for $F = h_{\mathcal{C},\widehat{R}}$. Then there exists an isomorphism*

$$f_{\widehat{R},\underline{r}} : \Lambda [\![t_1, \dots, t_n]\!] / \langle p_1, \dots, p_m \rangle \to \widehat{R}, \quad g_i \in \mathfrak{m}_{\Lambda_n}^2$$

where n equals $dim_k F(k[\epsilon]/\langle \epsilon^2 \rangle)$ and where m equals $dim_k \mathcal{O}$. In particular, Krull-dim$(\widehat{R}/\mathfrak{m}_\Lambda \widehat{R})$ is $\geq n - m$. If Krull-dim$(\widehat{R}/\mathfrak{m}_\Lambda \widehat{R})$ equals $n - m$, then $\widehat{R}$ is Λ-flat.

Proof. Let $f_{\widehat{R},\underline{r}}$ be as above. Since there exists a k-linear injection of $\mathcal{O}_{\widehat{R},\underline{r}}$ into $\mathcal{O}$, it follows that m is greater than or equal to the dimension of $\text{Ker}(f_{\widehat{R},\underline{r}})/\mathfrak{m}_{\Lambda_n} \cdot \text{Ker}(f_{\widehat{R},\underline{r}})$. Therefore there exist elements $g_1, \dots, g_m \in \text{Ker}(f_{\widehat{R},\underline{r}})$ whose images are a k-spanning set modulo $\mathfrak{m} \cdot \text{Ker}(f_{\widehat{R},\underline{r}})$. By Nakayama's Lemma, the elements $g_1, \dots, g_m$ generate the ideal $\text{Ker}(f_{\widehat{R},\underline{r}})$. Thus $f_{\widehat{R},\underline{r}}$ is the isomorphism as above.

By the above, the image ideal $\overline{\text{Ker}(f)}$ in

$$\Lambda [\![t_1, \dots, t_n]\!] / \mathfrak{m}_\Lambda \cdot \Lambda [\![t_1, \dots, t_n]\!] = k [\![t_1, \dots, t_n]\!]$$

is generated by the elements $\overline{g}_1, \dots, \overline{g}_m$. By the Krull Hauptidealsatz, for every minimal prime $\mathfrak{p} \subset k [\![t_1, \dots, t_n]\!]$ over $\overline{\text{Ker}(f)}$, $\mathfrak{p}$ has height $\leq m$, i.e., the quotient domain has Krull dimension $\geq n - m$. $\text{Ker}(f_{\widehat{R},\underline{r}}) + \mathfrak{m}_\Lambda$. Finally, if $\widehat{R}/\mathfrak{m}_\Lambda \widehat{R}$ has dimension equals to $n - m$, then $\overline{g}_1, \dots, \overline{g}_m$ is a regular sequence in $k [\![t_1, \dots, t_n]\!]$ by [Mat89, Theorem 17.4]. Then, by [Mat89, Corollary 22.5', p. 177], also $g_1, \dots, g_m$ is a regular sequence in $\Lambda [\![t_1, \dots, t_n]\!]$, and the quotient ring is Λ-flat. Since this quotient ring is Λ-isomorphic to $\widehat{R}$, also $\widehat{R}$ is Λ-flat. □

3 An Obstruction Theory for the Hilbert Scheme.

Let X_Λ be a separated, flat, finitely presented scheme over Spec Λ. For every Λ-algebra $\widehat{R}$, denote by $X_{\widehat{R}}$ the base change Spec $\widehat{R} \times_{\text{Spec } \Lambda} X_\Lambda$. Let Z_k be a closed subscheme of X_k. Denote by

$$e_{Z_k/X_k} : \mathcal{I}_{Z_k/X_k} \to \mathcal{O}_{X_k}$$

the ideal sheaf of Z_k inside $\mathcal{O}_{X_k}$.

Definition 3.1. The *Hilbert functor*, $\text{Hilb}_{X_\Lambda/\Lambda, Z_k}$, is the functor $\mathcal{C}_\Lambda \to \text{Sets}$ sending each object A of $\mathcal{C}_\Lambda$ to the set of A-flat, closed subschemes $Z_A \subset X_A$ such that Spec $k \times_{\text{Spec } A} Z_A$ equals Z_k as a closed subscheme of Spec $k \times_{\text{Spec } A} X_A = X_k$. For every morphism $q : A' \to A$ in $\mathcal{C}_\Lambda$, the induced map

$$\text{Hilb}_{X_\Lambda/\Lambda, Z_k}(A') \to \text{Hilb}_{X_\Lambda/\Lambda, Z_k}(A)$$

sends each A'-flat closed subscheme $Z_{A'}$ to the base change Spec $A \times_{\text{Spec } A'} Z_{A'}$ as a closed subscheme of Spec $A \times_{\text{Spec } A'} X_{A'} = X_A$.

The *standard obstruction group* of $\text{Hilb}_{X_\Lambda/\Lambda,Z_k}$ is

$$\text{Ext}^1_{\mathcal{O}_{X_k}}(\mathcal{I}_{Z_k/X_k}, \mathcal{O}_{X_k}/\mathcal{I}_{Z_k/X_k}).$$

The *normal sheaf* $\mathcal{Z}_{Z_k/X_k}$ is the $\mathcal{O}_{Z_k}$-module,

$$\mathcal{N}_{Z_k/X_k} = \mathcal{H}om_{\mathcal{O}_{X_k}}(\mathcal{I}_{Z_k/X_k}, \mathcal{O}_{X_k}/\mathcal{I}_{Z_k/X_k}).$$

The *global subgroup* is

$$H^1(X_k, \mathcal{H}om_{\mathcal{O}_{X_k}}(\mathcal{I}_{Z_k/X_k}, \mathcal{O}_{X_k}/\mathcal{I}_{Z_k/X_k})) = H^1(Z_k, \mathcal{N}_{Z_k/X_k}).$$

By construction $\text{Hilb}_{X_\Lambda/\Lambda,Z_k}$ is a pointed functor. For every closed subscheme Z_A of X_A (not necessarily flat), denote the ideal sheaf of Z_A by

$$e_{Z_A/X_A} : \mathcal{I}_{Z_A/X_A} \hookrightarrow \mathcal{O}_{X_A}.$$

Lemma 3.2. *The closed subscheme Z_A is A-flat if and only if both $\mathcal{I}_{Z_A/X_A}$ is flat and the following $\mathcal{O}_{X_k}$-module homomorphism is injective,*

$$e_{Z_A/X_A} \otimes_A Id_k : \mathcal{I}_{Z_A/X_A} \otimes_A k \to \mathcal{O}_{X_A} \otimes_A k,$$

i.e., the natural surjection $\mathcal{I}_{Z_A/X_A} \otimes_A k \to \mathcal{I}_{Z_k/X_k}$ is an isomorphism.

Proof. One direction is straightforward. Assume that Z_A is flat, i.e., $\mathcal{O}_{X_A}/\mathcal{I}_{Z_A/X_A}$ is A-flat. Then from the short exact sequence,

$$0 \longrightarrow \mathcal{I}_{Z_A/X_A} \longrightarrow \mathcal{O}_{X_A} \longrightarrow \mathcal{O}_{X_A}/\mathcal{I}_{Z_A/X_A} \longrightarrow 0,$$

for every A-module M there is a long exact sequence of Tor sheaves,

$$\mathcal{T}or_2^A(\mathcal{O}_{X_A}/\mathcal{I}_{Z_A/X_A}, M) \longrightarrow \mathcal{T}or_1^A(\mathcal{I}_{Z_A/X_A}, M) \longrightarrow \mathcal{T}or_1^A(\mathcal{O}_{X_A}, M).$$

Since $\mathcal{O}_{X_A}/\mathcal{I}_{Z_A/X_A}$ and $\mathcal{O}_{X_A}$ are A-flat, the outer terms are zero, hence also the middle term is zero. Therefore $\mathcal{I}_{Z_A}$ is A-flat. Similarly, since $\mathcal{T}or_1^A(\mathcal{O}_{X_A}/\mathcal{I}_{Z_A/X_A}, k)$ is zero, the short exact sequence above gives rise to a short exact sequence,

$$0 \longrightarrow \mathcal{I}_{Z_A/X_A} \otimes_A k \longrightarrow \mathcal{O}_{X_A} \otimes_A k \longrightarrow \mathcal{O}_{X_A}/\mathcal{I}_{Z_A/X_A} \otimes_A k \longrightarrow 0.$$

Thus $e_{Z_A/X_A} \otimes_A \text{Id}_k$ is injective.

The opposite direction follows from the Local Flatness Criterion, [Mat89, Theorem 22.3]. The details are left to the reader. □

Let $q : A' \to A$ be an infinitesimal extension and let $Z_A \subset X_A$ be an element of $\text{Hilb}_{X_\Lambda/\Lambda, Z_k}$. Denote by

$$q_X : \mathcal{O}_{X_{A'}} \to \mathcal{O}_{X_A}$$

the surjective map of structure sheaves associated to q. There is a short exact sequence of $\mathcal{O}_{X_{A'}}$-modules,

$$0 \longrightarrow \text{Ker}(q) \otimes_k \mathcal{O}_{X_k} \longrightarrow q_X^{-1}(\mathcal{I}_{Z_A/X_A}) \longrightarrow \mathcal{I}_{Z_A/X_A} \longrightarrow 0.$$

Multiplying by $\mathfrak{m}_{A'}$ induces a commutative diagram,

$$\begin{array}{ccccccccc} & & \mathfrak{m}_{A'} \otimes_{A'} \text{Ker}(q) \otimes_k \mathcal{O}_{X_k} & \longrightarrow & \mathfrak{m}_{A'} \otimes_{A'} q_X^{-1}(\mathcal{I}_{Z_A/X_A}) & \longrightarrow & \mathfrak{m}_{A'} \otimes_A \mathcal{I}_{Z_A/X_A} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \text{Ker}(q) \otimes_k \mathcal{O}_{X_k} & \longrightarrow & q_X^{-1}(\mathcal{I}_{Z_A/X_A}) & \longrightarrow & \mathcal{I}_{Z_A/X_A} & \longrightarrow & 0. \end{array}$$

Since $\mathfrak{m}_{A'} \cdot \text{Ker}(q)$ equals $\{0\}$, the first vertical map is zero, hence the second vertical map factors through the surjection to $\mathfrak{m}_{A'} \otimes_{A'} \mathcal{I}_{Z_A/X_A}$. This has two consequences. Using the commutative diagram,

$$\begin{array}{ccccccccc} & & \text{Ker}(q) \otimes_{A'} \mathcal{I}_{Z_A/X_A} & \longrightarrow & \mathfrak{m}_{A'} \otimes_{A'} \mathcal{I}_{Z_A/X_A} & \longrightarrow & \mathfrak{m}_A \otimes_A \mathcal{I}_{Z_A/X_A} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \text{Ker}(q) \otimes_k \mathcal{O}_{X_k} & \longrightarrow & q_X^{-1}(\mathcal{I}_{Z_A/X_A}) & \longrightarrow & \mathcal{I}_{Z_A/X_A} & \longrightarrow & 0, \end{array}$$

and the injectivity of the last vertical map, the intersection of $\text{Ker}(q) \otimes_k \mathcal{O}_{X_k}$ and $\mathfrak{m}_{A'} \cdot q_X^{-1}(\mathcal{I}_{Z_A/X_A})$ equals the image of the first vertical map, i.e., $\text{Ker}(q) \otimes_k \mathcal{I}_{Z_k/X_k}$. Thus the quotient of $\text{Ker}(q) \otimes_k \mathcal{O}_{X_k}$ by this intersection is $\text{Ker}(q) \otimes_k (\mathcal{O}_{X_k}/\mathcal{I}_{Z_k/X_k})$. Also, by Lemma 3.2, the quotient of $q_X^{-1}(\mathcal{I}_{Z_A/X_A})$ by both $\text{Ker}(q) \otimes_k \mathcal{O}_{X_k}$ and $\mathfrak{m}_{A'} \cdot q_X^{-1}(\mathcal{I}_{Z_A/X_A})$, i.e., the quotient $\mathcal{I}_{Z_A/X_A}/\mathfrak{m}_{A'} \cdot \mathcal{I}_{Z_A/X_A}$, equals $\mathcal{I}_{Z_k/X_k}$. Thus we have a short exact sequence of $\mathcal{O}_{X_k}$-modules,

$$0 \longrightarrow \text{Ker}(q) \otimes_k (\mathcal{O}_{X_k}/\mathcal{I}_{Z_k/X_k}) \longrightarrow q_X^{-1}(\mathcal{I}_{Z_A/X_A})/\mathfrak{m}_{A'} \cdot q_X^{-1}(\mathcal{I}_{Z_A/X_A}) \xrightarrow{q_\mathcal{I}} \mathcal{I}_{Z_k/X_k} \longrightarrow 0.$$

Definition 3.3. For a deformation situation $(q : A' \to A, Z_A)$ for $\text{Hilb}_{X_\Lambda/\Lambda, Z_k}$, the *standard obstruction class* is the class o_{q,Z_A} of the short exact sequence

$$o_{q,Z_A} : 0 \longrightarrow \text{Ker}(q) \otimes_k (\mathcal{O}_{X_k}/\mathcal{I}_{Z_k/X_k}) \longrightarrow q_X^{-1}(\mathcal{I}_{Z_A/X_A})/\mathfrak{m}_{A'} \cdot q_X^{-1}(\mathcal{I}_{Z_A/X_A}) \xrightarrow{q_\mathcal{I}} \mathcal{I}_{Z_k/X_k} \longrightarrow 0.$$

in the Yoneda Ext group $\text{Ext}^1_{\mathcal{O}_{X_k}}(\mathcal{I}_{Z_k/X_k}, \mathcal{O}_{X_k}/\mathcal{I}_{Z_k/X_k}) \otimes_k \text{Ker}(q)$.

This definition is justified by the following.

Proposition 3.4. *The element o_{q,Z_A} equals 0 if and only if there exists an A'-flat closed subscheme $Z_{A'}$ of $X_{A'}$ such that $Z_{A'} \times_{\text{Spec } A'} \text{Spec } A$ equals Z_A as closed subschemes of $X_{A'} \times_{\text{Spec } A'} \text{Spec } A = X_A$.*

Proof. For every $\mathcal{O}_{X_{A'}}$-submodule $\mathcal{I}_{Z_{A'}/X_{A'}}$ of $q_X^{-1}(\mathcal{I}_{Z_A/X_A})$ that surjects onto $\mathcal{I}_{Z_A/X_A}$, the subsheaf $\mathfrak{m}_{A'} \cdot \mathcal{I}_{Z_{A'}/X_{A'}}$ of $q_X^{-1}(\mathcal{I}_{Z_A/X_A})$ equals $\mathfrak{m}_{A'} \cdot q_X^{-1}(\mathcal{I}_{Z_A/X_A})$. This implies that $\mathcal{I}_{Z_{A'}/X_{A'}}$ contains $\mathfrak{m}_{A'} \cdot q_X^{-1}(\mathcal{I}_{Z_A/X_A})$. This also implies that the natural map

$$\mathfrak{I}_{Z_{A'}/X_{A'}}/\mathfrak{m}_{A'} \cdot q_X^{-1}(\mathcal{I}_{Z_A/X_A}) \to \mathcal{I}_{Z_{A'}/X_{A'}} \otimes_{A'} k$$

is an isomorphism. If $Z_{A'}$ is A'-flat, then by Lemma 3.2, the induced map

$$\mathcal{I}_{Z_{A'}/X_{A'}} \otimes_{A'} k \to \mathcal{I}_{Z_k/X_k}$$

is an isomorphism. Thus the $\mathcal{O}_{X_k}$-submodule $\mathcal{I}_{Z_{A'}/X_{A'}}/\mathfrak{m}_{A'} \cdot q_X^{-1}(\mathcal{I}_{Z_A/X_A})$ of $q_X^{-1}(\mathcal{I}_{Z_A/X_A})/\mathfrak{m}_{A'} \cdot q_X^{-1}(\mathcal{I}_{Z_A/X_A})$ is a splitting of o_{q,Z_A}, i.e., o_{q,Z_A} equals 0 in the Yoneda Ext group.

Conversely, given a submodule of $q_X^{-1}(\mathcal{I}_{Z_A/X_A})/\mathfrak{m}_{A'} \cdot q_X^{-1}(\mathcal{I}_{Z_A/X_A})$ splitting o_{q,Z_A}, define $\mathcal{I}_{Z_{A'}/X_{A'}}$ to be the inverse image of the submodule in $q_X^{-1}(\mathcal{I}_{Z_A/X_A})$. Then $\mathcal{I}_{Z_{A'}/X_{A'}}$ surjects onto I_{Z_A/X_A}. Since $\mathcal{I}_{Z_{A'}/X_{A'}}/\mathfrak{m}_{A'} \cdot q_X^{-1}(\mathcal{I}_{Z_A/X_A})$ has trivial intersection with $\text{Ker}(q_{\mathcal{I}})$, it follows that the intersection of $\mathcal{I}_{Z_{A'}/X_{A'}}$ and $\text{Ker}(q_X)$ equals $\text{Ker}(q) \otimes_k \mathcal{I}_{Z_k/X_k}$, which also equals $\text{Ker}(q_X) \cdot \mathcal{I}_{Z_{A'}/X_{A'}}$. Thus we have a short exact sequence

$$0 \longrightarrow \text{Ker}(q) \otimes_k \mathcal{I}_{Z_k/X_k} \longrightarrow \mathcal{I}_{Z_{A'}/X_{A'}} \longrightarrow \mathcal{I}_{Z_A/X_A} \longrightarrow 0.$$

By the previous paragraph, the natural map $\mathcal{I}_{Z_{A'}/X_{A'}} \otimes_{A'} k \to \mathcal{I}_{Z_k/X_k}$ is an isomorphism. Thus the short exact sequence above proves that the map

$$\text{Ker}(q) \otimes_{A'} \mathcal{I}_{Z_{A'}/X_{A'}} \to \mathcal{I}_{Z_{A'}/X_{A'}}$$

is injective. Since $\mathcal{I}_{Z_A/X_A}$ is A-flat, by the Local Flatness Criterion, [Mat89, Theorem 22.3], also

$$\mathfrak{m}_A \otimes_A \mathcal{I}_{Z_A/X_A} \to \mathcal{I}_{Z_A/X_A}$$

is injective. Putting these together with the short exact sequence above,

$$\mathfrak{m}_{A'} \otimes_{A'} \mathcal{I}_{Z_{A'}/X_{A'}} \to \mathcal{I}_{Z_{A'}/X_{A'}}$$

is injective. Thus, by the Local Flatness Criterion once more, $\mathcal{I}_{Z_{A'}/X_{A'}}$ is A'-flat. Finally, since the natural map

$$\mathcal{I}_{Z_{A'}/X_{A'}} \otimes_{A'} k \to \mathcal{I}_{Z_k/X_k}$$

is an isomorphism, Lemma 3.2 implies that $Z_{A'}$ is A'-flat. Therefore o_{q,Z_A} equals 0 if and only if there exists an A'-flat closed subscheme $Z_{A'}$ of $X_{A'}$ with $Z_{A'} \times_{\text{Spec } A'} \text{Spec } A$ equal to Z_A as closed subschemes of $X_{A'} \times_{\text{Spec } A'} \text{Spec } A = X_A$. $\square$

Finally, it is left to the reader to verify that the elements o_{q,Z_A} are functorial for morphisms of deformation situations.

The "standard" obstruction group above is often larger than strictly necessary. Because of Corollary 2.7, it is crucial to identify the smallest possible obstruction group. One of the basic reductions has to do with the case that $\mathcal{I}_{Z_k/X_k}$ is everywhere locally generated by a regular sequence. Thus, for now let X_Λ be an affine, flat, finitely presented scheme over $\text{Spec } \Lambda$. Let $\underline{b} = (b_1, \dots, b_r)$ be a regular sequence in $B_k = H^0(X_\Lambda, \mathcal{O}_{X_\Lambda})$. For every A in $\mathcal{C}_\Lambda$, denote by B_A the A-algebra $H^0(X_A, \mathcal{O}_{X_A})$.

Proposition 3.5. *Every ideal in* $Hilb_{X_\Lambda/\Lambda,Z_k}$ *is generated by a regular sequence* $\underline{b}_A = (b_{A,1}, \dots, b_{A,r})$ *in* B_A *that maps to* $\underline{b}$. *Conversely, every sequence* $\underline{b}_A$ *in* B_A *that maps to* $\underline{b}$ *is regular and generates an ideal in* $Hilb_{X_\Lambda/\Lambda,Z_k}$.

Proof. For a regular sequence $\underline{b}_A$ that maps to $\underline{b}$, the corresponding ideal is in $\text{Hilb}_{X_\Lambda/\Lambda,Z_k}$ by the Local Flatness Criterion, cf. [Mat89, Corollary, Theorem 22.6, p. 177]. Conversely, for every ideal $\mathcal{I}_{Z_A/X_A}$ in $\text{Hilb}_{X_\Lambda/\Lambda,Z_k}$, since the map to $\mathcal{I}_{Z_k/X_k}$ is surjective on global sections, there exists a sequence of global sections $\underline{b}_A$ of $\mathcal{I}_{Z_A/X_A}$ that maps to $\underline{b}$. Denote by $\mathcal{I}'_{Z_A/X_A}$ the sub-ideal sheaf of $\mathcal{I}_{Z_A/X_A}$ generated by $\underline{b}_A$. Consider the commutative diagram,

$$\begin{array}{ccc} \mathcal{I}'_{Z_A/X_A} \otimes_A k & \longrightarrow & \mathcal{I}_{Z_A/X_A} \otimes_A k \\ \downarrow & & \downarrow \\ \mathcal{I}_{Z_k/X_k} & \xrightarrow{=} & \mathcal{I}_{Z_k/X_k} \end{array}.$$

By Lemma 3.2, both vertical arrows are isomorphisms. Hence the top horizontal arrow is surjective. Thus, by Nakayama's Lemma, the map $\mathcal{I}'_{Z_A/X_A} \to \mathcal{I}_{Z_A/X_A}$ is surjective, i.e., $\mathcal{I}_{Z_A/X_A}$ is generated by $\underline{b}_A$. ☐

Corollary 3.6. *With hypotheses as above, for every deformation situation* $(q : A' \to A, Z_A)$ *there exists a lift of* Z_A *to* $Z_{A'}$ *in* $Hilb_{X_\Lambda/\Lambda,Z_k}(A')$. *In particular, every standard obstruction class* o_{q,Z_A} *equals* 0.

Proof. Since $X_{A'}$ is affine, the surjective homomorphism of sheaves of algebras $q_X : \mathcal{O}_{X_{A'}} \to \mathcal{O}_{X_A}$ induces a surjection $B_{A'} \to B_A$. Thus every sequence $\underline{b}_A$ in B_A lifts to a sequence $\underline{b}_{A'}$ in $B_{A'}$. Therefore, by Proposition 3.5, every element $\mathcal{I}_{Z_A/X_A}$ in $\text{Hilb}_{X_\Lambda/\Lambda,Z_k}(A)$ lifts to an element $\mathcal{I}_{Z_{A'}/X_{A'}}$ in $\text{Hilb}_{X_\Lambda/\Lambda,Z_k}(A')$. ☐

Of course, typically we are interested in *proper* Λ-schemes X_Λ, not affine Λ-schemes. So now assume that X_Λ is a separated, flat, finitely presented scheme over Spec Λ that is not necessarily affine. For every pair of coherent sheaves $\mathcal{E}, \mathcal{F}$ on X_k, the local-to-global spectral sequence for Ext gives an exact sequence

$$0 \to H^1(X_k, Hom_{\mathcal{O}_{X_k}}(\mathcal{E},\mathcal{F})) \to \text{Ext}^1_{\mathcal{O}_{X_k}}(\mathcal{E},\mathcal{F}) \to H^0(X_k, Ext^1_{\mathcal{O}_{X_k}}(\mathcal{E},\mathcal{F})) \to H^2(X_k, Hom_{\mathcal{O}_{X_k}}(\mathcal{E},\mathcal{F})).$$

In particular, this gives an exact sequence,

$$0 \to H^1(Z_k, \mathcal{N}_{Z_k/X_k}) \to \text{Ext}^1_{\mathcal{O}_{X_k}}(\mathcal{I}_{Z_k/X_k}, \mathcal{O}_{X_k}/\mathcal{I}_{Z_k/X_k}) \to H^0(X_k, Ext^1_{\mathcal{O}_{X_k}}(\mathcal{I}_{Z_k/X_k}, \mathcal{O}_{X_k}/\mathcal{I}_{Z_k/X_k}))$$

The first term is the *global group* and the third term is the *local group.*

Corollary 3.7. *Let* Z_k *be a closed subscheme of* X_k *whose ideal sheaf* $\mathcal{I}_{Z_k/X_k}$ *is generated by regular sequences on the opens in some open affine covering. Then the normal sheaf* $\mathcal{N}_{Z_k/X_k}$ *is a locally free* $\mathcal{O}_{Z_k}$*-module of finite rank, and every obstruction class* o_{q,Z_A} *is contained in the global subgroup,*

$$H^1(Z_k, \mathcal{N}_{Z_k/X_k}) \otimes_k Ker(q).$$

Thus the global subgroup is the obstruction group of an obstruction theory for $Hilb_{X_\Lambda/\Lambda,Z_k}$.

Proof. Since $\mathcal{I}_{Z_k/X_k}$ is locally generated by a regular sequence, say $(b_1, \dots, b_r)$, then the $\mathcal{O}_{X_k}/\mathcal{I}_{Z_k/X_k}$-module $\mathcal{I}_{Z_k/X_k}/\mathcal{I}^2_{Z_k/X_k}$ is locally freely generated by the images of $b_1, \dots, b_r$. Thus the dual sheaf is also locally free of finite rank.

The restriction of o_{q,Z_A} to each of the opens in this open covering equals 0 by Corollary 3.6. Thus the image of o_{q,Z_A} in the local group $H^0(X_k, \mathcal{E}xt^1_{\mathcal{O}_{X_k}}(\mathcal{E}, \mathcal{F})) \otimes_k \mathrm{Ker}(q)$ is a global section that is zero when restricted to the opens of an open covering. Therefore this is the zero global section. So o_{q,Z_A} is contained in the global subgroup. □

If X_k is smooth over k, then by [Mat89, Theorem 21.2], the ideal sheaf of Z_k is everywhere locally generated by a regular sequence if and only if Z_k is locally a complete intersection scheme. In particular, if Z_k is also smooth over k, then the ideal sheaf is everywhere locally generated by a regular sequence.

When the ideal sheaf is everywhere locally generated by a regular sequence, then the Zariski tangent space of the fiber of $\mathrm{Hilb}_{X_\Lambda/\Lambda, Z_k}$ over Spec k is $H^0(Z_k, \mathcal{N}_{Z_k/X_k})$ and the global subgroup is $H^1(Z_k, \mathcal{N}_{Z_k/X_k})$. Combined with Corollary 2.7, this gives a lower bound on the dimension of any pro-representing object. The main pro-representability result is the following.

Theorem 3.8. *[Art69, Corollary 6.2] For X_Λ a separated, flat, finitely presented scheme over Spec Λ, and for Z_k a closed subscheme of X_k that is proper over Spec k, the functor $\mathrm{Hilb}_{X_\Lambda/\Lambda, Z_k}$ is pro-representable by an object $\widehat{R}$ in $\widehat{\mathcal{C}}_\Lambda$.*

In fact [Art69, Corollary 6.2] proves much more. Although pro-representability is a straightforward application of Schlessinger's thesis, this application is not contained in [Sch68].

Corollary 3.9. *Assume that Z_k is proper, and assume that everwhere locally $\mathcal{I}_{Z_k/X_k}$ is generated by a regular sequences. Then every irreducible component of $\widehat{R}/\mathfrak{m}_\Lambda \widehat{R}$ has Krull dimension $\geq h^0(Z_k, \mathcal{N}_{Z_k/X_k}) - h^1(Z_k, \mathcal{N}_{Z_k/X_k})$. When this is equality, then $\widehat{R}$ is Λ-flat. In particular, if Z_k is a curve (or whenever $h^q(Z_k, \mathcal{N}_{Z_k/X_k})$ equals 0 for all $q > 1$), then the Krull dimension is $\geq \chi(Z_k, \mathcal{N}_{Z_k/X_k})$, and equality implies that $\widehat{R}$ is Λ-flat.*

Proof. This follows immediately from Corollaries 2.7 and 3.7. □

A special case is when X_k is smooth and Z_k is a geometrically reduced curve that is locally a complete intersection. Then the adjunction formula gives,

$$\det(\mathcal{N}_{Z_k/X_k}) = \det(T_{X_k/k})|_{Z_k} \otimes_{\mathcal{O}_{Z_k}} \omega_{Z_k/k},$$

where $T_{X_k/k}$ is the tangent sheaf $\mathcal{H}om_{\mathcal{O}_{X_k}}(\Omega_{X_k/k}, \mathcal{O}_{X_k})$ and where $\omega_{Z_k/k}$ is the dualizing invertible sheaf on Z_k. Denote by $p_a(Z_k)$ the arithmetic genus $1 - \chi(Z_k, \mathcal{O}_{Z_k})$ of Z_k. Then we can apply Riemann-Roch to compute the Euler characteristic $\chi(Z_k, \mathcal{N}_{Z_k/X_k})$ above.

Corollary 3.10. *Assume that X_k is smooth of pure dimension $dim(X_k)$ over k, and assume that Z_k is a proper, reduced curve that is locally a complete intersection. Then every irreducible component of $whR/\mathfrak{m}_\Lambda \widehat{R}$ has Krull dimension*

$$\geq deg_{Z_k} det(T_{X_k/k}|_{Z_k}) + (1 - p_a(Z_k))(dim(X_k) - 3).$$

If this is equality, then $\widehat{R}$ is Λ-flat.

Proof. This follows by computing $\chi(Z_k, \mathcal{N}_{Z_k/X_k})$ using Riemann-Roch and the adjunction isomorphism above. □

4 Variants of the Hilbert Scheme.

There are two variants of this obstruction theory for the Hilbert scheme. First, let $W_\Lambda \subset X_\Lambda$ be a closed subscheme that is Λ-flat. Let Z_k be a closed subscheme of X_k that contains W_k, i.e., such that $\mathcal{I}_{Z_k/X_k}$ is contained in $\mathcal{I}_{W_k/X_k}$.

Definition 4.1. The *Hilbert functor relative to* W_Λ, $\text{Hilb}_{X_\Lambda/\Lambda,W_\Lambda,Z_k}$, is the subfunctor of $\text{Hilb}_{X_\Lambda/\Lambda,Z_k}$ whose A-points parameterize closed subschemes Z_A in $\text{Hilb}_{X_\Lambda/\Lambda,Z_k}(A)$ that contain W_A, i.e., such that $\mathcal{I}_{Z_A/X_A}$ is contained in $\mathcal{I}_{W_A/X_A}$. The *standard obstruction group* of $\text{Hilb}_{X_\Lambda/\Lambda,W_\Lambda,Z_k}$ is

$$\text{Ext}^1_{\mathcal{O}_{X_k}}(\mathcal{I}_{Z_k/X_k}, \mathcal{I}_{W_k/X_k}/\mathcal{I}_{Z_k/X_k}).$$

The *relative normal sheaf* is the $\mathcal{O}_{Z_k}$-module,

$$\mathcal{N}_{Z_k/X_k,W_k} = \mathcal{H}om_{\mathcal{O}_{X_k}}(\mathcal{I}_{Z_k/X_k}, \mathcal{I}_{W_k/X_k}/\mathcal{I}_{Z_k/X_k}).$$

The *global subgroup* is

$$H^1(Z_k, \mathcal{N}_{Z_k/X_k,W_k}).$$

Given a deformation situation $(q : A' \to A, Z_A)$ for $\text{Hilb}_{X_\Lambda/\Lambda,W_\Lambda,Z_k}$, using flatness as in the absolute case, there is a commutative diagram of short exact sequences,

$$\begin{array}{ccccccc}
0 \longrightarrow & \text{Ker}(q) \otimes_k (\mathcal{I}_{W_k/X_k}/\mathcal{I}_{Z_k/X_k}) & \longrightarrow & [q_X^{-1}(\mathcal{I}_{Z_A/X_A}) \cap \mathcal{I}_{W_{A'}/X_{A'}}]/\mathfrak{m}_{A'} \cdot q_X^{-1}(\mathcal{I}_{Z_A/X_A}) & \xrightarrow{q_\mathcal{I}} & \mathcal{I}_{Z_k/X_k} \\
& \downarrow & & \downarrow & & \downarrow = \\
0 \longrightarrow & \text{Ker}(q) \otimes_k (\mathcal{O}_{X_k}/\mathcal{I}_{Z_k/X_k}) & \longrightarrow & q_X^{-1}(\mathcal{I}_{Z_A/X_A})/\mathfrak{m}_{A'} \cdot q_X^{-1}(\mathcal{I}_{Z_A/X_A}) & \xrightarrow{q_\mathcal{I}} & \mathcal{I}_{Z_k/X_k}.
\end{array}$$

Definition 4.2. For a deformation situation $(q : A' \to A, Z_A)$ for $\text{Hilb}_{X_\Lambda/\Lambda,W_\Lambda,Z_k}$, the *standard obstruction class* is the class o_{W_Λ,q,Z_A} of the short exact sequence

$$0 \to \text{Ker}(q) \otimes_k (\mathcal{I}_{W_k/X_k}/\mathcal{I}_{Z_k/X_k}) \to [q_X^{-1}(\mathcal{I}_{Z_A/X_A}) \cap \mathcal{I}_{W_{A'}/X_{A'}}]/\mathfrak{m}_{A'} \cdot q_X^{-1}(\mathcal{I}_{Z_A/X_A}) \to \mathcal{I}_{Z_k/X_k} \to 0$$

in the Yoneda Ext group $\text{Ext}^1_{\mathcal{O}_{X_k}}(\mathcal{I}_{Z_k/X_k}, \mathcal{I}_{W_k/X_k}/\mathcal{I}_{Z_k/X_k}) \otimes_k \text{Ker}(q)$.

Proposition 4.3. *The standard obstruction element o_{W_Λ,q,Z_A} equals 0 if and only if there exists $Z_{A'}$ in $Hilb_{X_\Lambda/\Lambda,W_\Lambda,Z_k}(A')$ mapping to Z_A in $Hilb_{X_\Lambda/\Lambda,W_\Lambda,Z_k}(A)$.*

Proof. The proof is very similar to the proof of Proposition 3.4. □

The local-to-global spectral sequence for Ext gives an exact sequence

$$0 \to H^1(Z_k, \mathcal{N}_{Z_k/X_k,W_k}) \to \mathrm{Ext}^1_{\mathcal{O}_{X_k}}(\mathcal{I}_{Z_k/X_k}, \mathcal{I}_{W_k/X_k}/\mathcal{I}_{Z_k/X_k}) \to H^0(X_k, \mathcal{E}xt^1_{\mathcal{O}_{X_k}}(\mathcal{I}_{Z_k/X_k}, \mathcal{I}_{W_k/X_k}/\mathcal{I}_{Z_k/X_k}))$$

The first term is the *global group* and the third term is the *local group*.

Corollary 4.4. *Let Z_k be a closed subscheme of X_k and containing W_k whose ideal sheaf $\mathcal{I}_{Z_k/X_k}$ is generated by regular sequences on the opens in some open affine covering. Then the normal sheaf $\mathcal{N}_{Z_k/X_k,W_k}$ is the tensor product of $\mathcal{I}_{W_k/X_k}/\mathcal{I}_{Z_k/X_k}$ with a locally free $\mathcal{O}_{Z_k}$-module of finite rank, and every obstruction class o_{q,Z_A} is contained in the global subgroup,*

$$H^1(Z_k, \mathcal{N}_{Z_k/X_k,W_k}) \otimes_k Ker(q).$$

Thus the global subgroup is the obstruction group of an obstruction theory for $Hilb_{X_\Lambda/\Lambda,W_\Lambda,Z_k}$.

Proof. The proof is very similar to the proof of Corollary 3.7. □

A special case is when W_k is an effective Cartier divisor in Z_k.

Lemma 4.5. *If W_k is an effective Cartier divisor in Z_k, then for every $Z_A \in Hilb_{X_\Lambda/\Lambda,W_\Lambda,Z_k}(A)$, the ideal sheaf $\mathcal{I}_{W_A/X_A}/\mathcal{I}_{Z_A/X_A}$ is an invertible $\mathcal{O}_{Z_A}$-module, denoted $\mathcal{O}_{Z_A}(-W_A)$. In particular, $\mathcal{N}_{Z_k/X_k,W_k}$ is canonically isomorphic to $\mathcal{N}_{Z_k/X_k}(-W_k)$.*

Proof. By definition, W_k is an effective Cartier divisor in Z_k precisely when the ideal sheaf $\mathcal{I}_{W_k/X_k}/\mathcal{I}_{Z_k/X_k}$ is an invertible sheaf. In this case, the cup product map

$$Hom_{\mathcal{O}_{Z_k}}(\mathcal{I}_{Z_k/X_k}/\mathcal{I}^2_{Z_k/X_k}, \mathcal{O}_{Z_k}) \otimes_{\mathcal{O}_{Z_k}} (\mathcal{I}_{W_k/X_k}/\mathcal{I}_{Z_k/X_k}) \to Hom_{\mathcal{O}_{Z_k}}(\mathcal{I}_{Z_k/X_k}/\mathcal{I}^2_{Z_k/X_k}, \mathcal{I}_{W_k/X_k}/\mathcal{I}_{Z_k/X_k})$$

is an isomorphism. Thus the cup product map gives an isomorphism $\mathcal{N}_{Z_k/X_k}(-W_k) \to \mathcal{N}_{Z_k/X_k,W_k}$.

For any Z_A, to prove that the ideal sheaf $\mathcal{I}_{W_A/X_A}/\mathcal{I}_{Z_A/X_A}$ is an invertible $\mathcal{O}_{Z_A}$-module, it is equivalent to prove that it is a flat $\mathcal{O}_{Z_A}$-module. Since flatness is a local property, it suffices to prove that the stalk at each point of Z_A is flat over the stalk of the structure sheaf. Applying the Local Flatness Criterion, [Mat89, Theorem 22.3], to this stalk, where the nilpotent ideal is the ideal generated by $\mathfrak{m}_{A'}$, it suffices to prove that $\mathcal{I}_{W_k/X_k}/\mathcal{I}_{Z_k/X_k}$ is an invertible $\mathcal{O}_{Z_k}$-module and that the map

$$\mathfrak{m}_A \otimes_A \mathcal{I}_{W_A/X_A}/\mathcal{I}_{Z_A/X_A} \to \mathcal{I}_{W_A/X_A}/\mathcal{I}_{Z_A/X_A}$$

is injective. The first condition is the hypothesis that W_k is an effective Cartier divisor in Z_k. The second condition is A-flatness of the ideal sheaf $\mathcal{I}_{W_A/X_A}/\mathcal{I}_{Z_A/X_A}$ of W_A in Z_A. Since both Z_A and W_A are flat, this ideal sheaf is flat, cf. Lemma 3.2. □

Corollary 4.6. *Assume that X_k is smooth of pure dimension $dim(X_k)$ over k, assume that Z_k is a proper, geometrically reduced curve that is locally a complete intersection, and assume that W_k is an effective Cartier divisor in Z_k. Then $Hilb_{X_\Lambda/\Lambda, W_\Lambda, Z_k}$ is pro-representable by an object $\widehat{R}$ in $\widehat{\mathcal{C}}_\Lambda$. Every irreducible component of $whR/\mathfrak{m}_\Lambda \widehat{R}$ has Krull dimension*

$$\geq deg_{Z_k} det(T_{X_k/k}|_{Z_k}) + (1 - p_a(Z_k))(dim(X_k) - 3) - length(W_k)(dim(X) - 2).$$

If this is equality, then $\widehat{R}$ is Λ-flat.

Proof. The proof is very similar to the proof of Corollary 3.10. □

There is one more variation. Let C_Λ and Y_Λ be separated, flat, finitely presented schemes over Spec Λ. Denote by X_Λ the fiber product $C_\Lambda \times_{\text{Spec }\Lambda} Y_\Lambda$. Let $Z_A \subset X_A$ be an A-flat closed subscheme. Denote by

$$\text{pr}_{Z_A, C_A} : Z_A \to C_A$$

the restriction to Z_A of the projection morphism $\text{pr}_A : X_A \to C_A$.

Proposition 4.7. *If pr_{Z_k,C_k} is an isomorphism of k-schemes, then also pr_{Z_A,C_A} is an isomorphism of A-schemes. In this case, there exists a unique A-morphism $u_A : C_A \to Y_A$ such that Z_A equals the graph of u_A. Conversely, for every A-morphism $u_A : C_A \to Y_A$, the graph Z_A of u_A is an A-flat closed subscheme of X_A such that pr_{Z_A,C_A} is an isomorphism of A-schemes.*

Proof. Assume that the k-morphism pr_{Z_k,C_k} is an isomorphism. Then for every open affine subset of C_A, the inverse image in Z_A is an open subset whose intersection with Z_k is affine. By Chevalley's theorem, cf. [Har77, Exercise III.3.1], the open in Z_A is an affine scheme. Thus, without loss of generality, assume that both C_A and Z_A are affine schemes. Then pr_{Z_A,C_A} is an isomorphism if and only if the associated ring homomorphism $\text{pr}^{\#}_{Z_A,C_A}$ is an isomorphism.

We prove that $\text{pr}^{\#}_{Z_A,C_A}$ is an isomorphism by induction on the smallest integer $e \geq 1$ such that $\mathfrak{m}_A^e$ equals $\{0\}$. If e equals 1, then A equals k and the result is tautological. Thus, by way of induction, assume that $e > 1$, and assume that the result is proved for smaller e. For the infinitesimal extension $q : A \to B = A/\mathfrak{m}^{e-1}$, by the induction hypothesis the morphism $\text{pr}^{\#}_{Z_B,C_B}$ is an isomorphism. Since $\mathcal{O}_{Z_A}$ and $\mathcal{O}_{C_A}$ are A-flat, there is a commutative diagram of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \text{Ker}(q) \otimes_k \mathcal{O}_{Z_k} & \longrightarrow & \mathcal{O}_{Z_A} & \longrightarrow & \mathcal{O}_{Z_B} & \longrightarrow & 0 \\
 & & \downarrow {\scriptstyle \text{Id}_K \otimes \text{pr}^{\#}_{Z_k,C_k}} & & \downarrow {\scriptstyle \text{pr}^{\#}_{Z_A,C_A}} & & \downarrow {\scriptstyle \text{pr}^{\#}_{Z_B,C_B}} & & \\
0 & \longrightarrow & \text{Ker}(q) \otimes_k \mathcal{O}_{C_k} & \longrightarrow & \mathcal{O}_{C_{A'}} & \longrightarrow & \mathcal{O}_{C_A} & \longrightarrow & 0
\end{array}.$$

By hypothesis, the first and third vertical arrows are isomorphisms. Therefore, by the Snake Lemma, also the middle vertical arrow, $\text{pr}^{\#}_{Z_A,C_A}$ is an isomorphism. This proves that pr_{Z_A,C_A} is an isomorphism by induction on e.

Since pr_{Z_A,C_A} is an isomorphism, there exists a unique A-morphism $u_A : C_A \to Y_A$ such that $u_A \circ \text{pr}_{Z_A,C_A}$ equals the projection $\text{pr}_{Z_A,Y_A} : Z_A \to Y_A$. Then Z_A equals the graph of u_A for the unique A-morphism u_A. Finally, for every A-morphism $u_A : C_A \to Y_A$, the graph morphism $\Gamma_{u_A} : C_A \to C_A \times_{\text{Spec } A} Y_A$ is a closed immersion since Y_A is separated over Spec A. Thus, denoting by Z_A the closed image, the morphism $\Gamma_{u_A} : C_A \to Z_A$ is an isomorphism. Since the composition $\text{pr}_{Z_A,C_A} \circ \Gamma_{u_A} : C_A \to C_A$ is an A-isomorphism – in fact the identity morphism – it follows that pr_{Z_A,C_A} is also an A-isomorphism. In particular, Z_A is A-flat. □

In the same way, let W_Λ be a Λ-flat closed subscheme of X_Λ such that $\text{pr}_{W_k,C_k} : W_k \to C_k$ is a closed immersion. Then $\text{pr}_{W_\Lambda,C_\Lambda} : W_\Lambda \to C_\Lambda$ is a closed immersion of Λ-schemes. Denoting by D_Λ the closed image, then there exists a unique Λ-morphism $u_{D_\Lambda} : D_\Lambda \to Y_\Lambda$ such that W_Λ equals the graph of u_{D_Λ}.

Because of Propositon 4.7, for a k-morphism u_k with $u_k|_{D_k}$ equal to u_{D_k}, for the graph Z_k of u_k, the pointed functor $\text{Hilb}_{X_\Lambda/\Lambda,W_\Lambda,Z_k}$ is equivalent to the following Hom functor.

Definition 4.8. The *Hom functor*, $\text{Hom}_\Lambda(C_\Lambda, Y_\Lambda; u_{D_\Lambda}, u_k)$ is the pointed functor $\mathcal{C}_\Lambda \to \text{Sets}$ sending A to the set of A-morphisms $u_A : C_A \to Y_A$ such that $u_A|_{D_A}$ equals u_{D_A} and such that the base change of u_A to k equals u_k.

The obstruction theory for $\text{Hilb}_{X_\Lambda \Lambda,W_\Lambda,Z_k}$, and every result about this obstruction theory immediately gives an analogue for the Hom functor. Here is the application we will use most often.

Corollary 4.9. *Assume that Y_k is smooth of pure dimension $dim(Y_k)$ over k, assume that C_k is a proper, geometrically reduced curve that is locally a complete intersection, and assume assume that D_k is an effective Cartier divisor in C_k. Then $Hom_\Lambda(C_\Lambda, Y_\Lambda; u_{D_\Lambda}, u_k)$ is pro-representable by an object $\widehat{R}$ in $\widehat{\mathcal{C}}_\Lambda$. Every irreducible component of whR$/\mathfrak{m}_\Lambda \widehat{R}$ has Krull dimension*

$$\geq deg_{C_k} u_k^* det(T_{Y_k/k}) + (1 - p_a(Z_k) - length(D_k))(dim(X_k) - 1).$$

If this is equality, then $\widehat{R}$ is Λ-flat.

Proof. This follows immediately from Corollary 4.6 under the identification of $\mathcal{N}_{Z_k/X_k}$ with the pullback of $T_{Y_k/k}$. □

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