Problem Set 13

Disclaimer For open-ended problems, part of the problem is to give a precise formulation. Especially for the problems in Part II, you should do as much of the problems as is useful to you. For each problem, it is important you understand how to verify all details. However, if you are pressed for time, you may write-up only the most important steps, instead of every detail. As an adult, you are expected to use your judgment in interpreting what this precisely means.
Late homework policy. Late work will be accepted only with a medical note or for another Institute-approved reason.

Cooperation policy. You are strongly encouraged to work with others, but the final write-up must be entirely your own and based on your own understanding.

Part I. These problems are from the textbook. You are expected to read *all* the problems from the sections of the textbook covered that week. You are asked to write-up and turn-in only the problems assigned below.

Part II. These problems are not necessarily from the textbook. Often they will be exercises in commutative algebra, category theory, homological algebra or sheaf theory.

Part I(25 points)

(a)	(10 points)	p. 238,	Section III.6,	Problem 6.3
(b)	(5 points)	p. 238,	Section III.6,	Problem 6.4
(c)	(10 points)	p. 238,	Section III.6,	Problem 6.7

Part II(25 points)

Problem 1(15 points) Do Exercise 6.8 of §III.6 on p. 238. A slightly sharper version of (a) is that for *every* open affine subset U of X, the complement Z is a Cartier divisor. Then for the invertible sheaf $\mathcal{L} = \mathcal{O}_X(Z)$ and the global section s defining the Cartier divisor Z, X_s equals U.

Because X is locally factorial, to prove Z is a Cartier divisor, it suffices to prove Z is a Weil divisor. To prove this, let V be any open affine in X intersecting Z. Because X is separated, $U \cap V$ is affine. What does Proposition II.6.3A say about the closed subset $V - (V \cap U)$ of V?

Problem 2(10 points) Let X be the scheme from Exercise III.6.7 on p. 148. This is projective, equidimensional (of dimension 1) and a local complete intersection (in fact just a hypersurface in $\mathbb{P}^2_k$). By Theorem III.7.11, X has an invertible dualizing sheaf ω_X.

Which invertible sheaf is ω_X? More precisely, what is its degree? Among invertible sheaves of that degree, which one is ω_X? Use this to give a one-line solution to the $\lambda \neq 1$ case of Problem 2 from Problem Set 10, Part II.

Extra credit(5 points) Let X be a reduced, connected, Cohen-Macaulay, projective curve over an algebraically closed field k. By the duality theorem, $H^1(X, \omega_X) \cong k$. Of course $H^1(X, \omega_X) = \text{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_X, \omega_X)$. By Exercise III.6.1, every nonzero element of $\text{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_X, \omega_X)$ gives a short exact sequence,

$$0 \longrightarrow \omega_X \longrightarrow E \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

The extension class depends on the element, but the isomorphism class of the $\mathcal{O}_X$-module E is independent of the element (so long as it is nonzero).

For the curve X from Problem 2 above, give a locally free presentation of E in the form,

$$\oplus_{i=1}^{M} \mathcal{O}_{\mathbb{P}^1}(-d_i)|_X \xrightarrow{\phi} \oplus_{i=1}^{N} \mathcal{O}_{\mathbb{P}^1}(-e_i)|_X \longrightarrow E \longrightarrow 0$$

where the integers M, N, d_i and e_i are explicitly given, and the $\mathcal{O}_X$-module homomorphism ϕ is an explicit matrix whose $(\mathcal{O}_{\mathbb{P}^2}(-d_i)|_X, \mathcal{O}_{\mathbb{P}^2}(-e_j)|_X)$-entry is an explicit polynomial of degree $d_i - e_j$. There is such a presentation with $M = 1$ and $N = 3$.