

Problem Set 6

Disclaimer For open-ended problems, part of the problem is to give a precise formulation. Especially for the problems in Part II, you should do as much of the problems as is useful to you. For each problem, it is important you understand how to verify all details. However, if you are pressed for time, you may write-up only the most important steps, instead of every detail.

Late homework policy. Late work will be accepted only with a medical note or for another approved reason.

Cooperation policy. You are strongly encouraged to work with others, but the final write-up must be entirely your own and based on your own understanding.

Part I. These problems are from the textbook. You are expected to read *all* the problems from the sections of the textbook covered that week. You are asked to write-up and turn-in only the problems assigned below.

Part II. These problems are not necessarily from the textbook. Often they will be exercises in commutative algebra, category theory, homological algebra or sheaf theory.

Part I(25 points)

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| (a) | (5 points) | p. 123, | Section II.5, | Problem 5.1 (c) and (d) | See note below |
| (b) | (10 points) | p. 124, | Section II.5, | Problem 5.7 | See note below |
| (c) | (10 points) | p. 125, | Section II.5, | Problem 5.8 | See note below |

Note (a). For 5.1(c), do not assume \mathcal{E} is locally free; this is unnecessary for this part of the problem. For 5.1(d), the more natural statement is that for every \mathcal{O}_Y -module \mathcal{E} and every \mathcal{O}_X -module \mathcal{F} , there is a natural isomorphism

$$\mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{E}, f_*\mathcal{F}) \rightarrow f_*\mathrm{Hom}_{\mathcal{O}_X}(f^*\mathcal{E}, \mathcal{F}).$$

To prove this, use part (c), and functoriality and adjointness of f_* and f^* . To finish the problem, use part (b) for locally free \mathcal{O}_Y -modules (which you may assume) to deduce the projection formula.

Note (b). For 5.7, do not assume X is Noetherian. You do need to assume that \mathcal{F} is locally finitely presented.

Note (c). For 5.8(c), use 5.7 to reduce to the case of a local ring. If M is a finitely presented module over a local ring R , lift a basis of $M/\mathfrak{m}M$. By Nakayama's lemma, the lifts generate M , thus give a surjective map $R^a \rightarrow M$. Because R is an integral domain, every submodule of R^a is

torsion-free. Thus it is (0) if and only if it is zero after localizing at (0) . Now use that the dimension of $M \otimes_R K(R)$ is a to deduce the kernel is (0) , and thus $R^a \rightarrow M$ is an isomorphism.

Part II(25 points)

Problem 1(15 points) Work through Exercise II.5.15 on p. 126 of the textbook.

Problem 2(10 points) Work through Exercise II.5.17 (c), (d) and (e) on p. 128 of the textbook. Also, be sure to read Exercise 5.16 and 5.18. Together with 5.17, this establishes the algebraic geometric approach to studying vector bundles.

Extra credit(5 points) Is there a counterexample to Exercise II.5.7 when X is an affine scheme and \mathcal{F} is a quasi-coherent sheaf that is locally finitely-generated, but not locally finitely-presented?