MAT 589 Introductory Notes

References.

Algebraic Geometry textbooks.

Hartshorne, Robin. *Algebraic geometry.* Grad. Texts in Math., No. 52, Springer-Verlag, New York-Heidelberg, 1977.

Mumford, David. *The red book of varieties and schemes.* Lecture Notes in Math., 1358, Springer-Verlag, Berlin, 1988.

Mumford, David and Oda, Tadao. *Algebraic geometry II.* Texts Read. Math., 73 Hindustan Book Agency, New Delhi, 2015.

Mumford, David. *Lectures on curves on an algebraic surface.* With a section by G. M. Bergman Ann. of Math. Stud., No. 59 Princeton University Press, Princeton, NJ, 1966.

Mumford, David. *Abelian varieties.* With appendices by C. P. Ramanujam and Yuri Manin. Corrected reprint of the second (1974) edition Tata Inst. Fund. Res. Stud. Math., 5 Published for the Tata Institute of Fundamental Research, Bombay; byHindustan Book Agency, New Delhi, 2008.

Secondary references.

Atiyah, M. F., and MacDonald, I. G., *Introduction to commutative algebra.* Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969.

Eisenbud, David. *Commutative algebra with a view toward algebraic geometry.* Grad. Texts in Math., 150 Springer-Verlag, New York, 1995.

Matsumura, Hideyuki. *Commutative ring theory.* Translated from the Japanese by M. Reid. Second edition Cambridge Stud. Adv. Math., 8 Cambridge University Press, Cambridge, 1989.

Weibel, Charles. *An introduction to homological algebra.* Cambridge Stud. Adv. Math., 38 Cambridge University Press, Cambridge, 1994

Supporting techniques.

Category theory.

Representable and corepresentable functors. Hulls and cohulls. Yoneda lemma. Limits and colimits. Fiber products and cofiber coproducts (including quotients). Adjoint pairs. Kan extensions.

Homological algebra.

Additive categories. Abelian categories. Delta functors. Derived functors. Products. Balancing Tor and Ext. Spectral sequences. Grothendieck spectral sequence. Leray spectral sequence. Derived categories. Exterior algebra. Koszul complex. Group cohomology.

Commutative algebra.

Rings and modules. Hom and tensor product. Projective, injective and flat modules. Localization. Finite generation and finite presentation. Nakayama's lemma. Nullstellensatz and Jacobson rings. Krull dimension. Noetherian rings. Primary decomposition. Integral domains. Dedekind domains and discrete valuation rings. Fractional ideals and class groups. Integral closure. Going-up and going-down theorems. Depth. Serre's criterion for integrally closed domains. Completion. Artin-Rees lemma. Krull's intersection theorem. Hilbert polynomial and Hilbert-Samuel polynomial. Noether's normalization theorem. Generic freeness. Auslander-Buchsbaum formula. Regular rings. Cohen-Macaulay rings. Gorenstein rings. Local flatness criterion. Excellent rings.

Topology.

Basis and subbasis. Continuous maps. Open maps. Closed maps. Compact maps. Separation axioms. Hausdorff spaces. Presheaves. Stalks. Sheaves. Espace etale. Pushforward sheaf. Inverse image sheaf. Flasque resolutions. Godement resolution. Simplicial sets. Simplicial objects. Simplicial spaces. Higher categories.

Geometry.

Differentials. Pullback of differentials. DeRham complex. Tangent vectors. Foliations and involutive distributions. Frobenius theorem. Unramified immersions. Embeddings. Submersions. Lie groups. Transversality.

Motivation.

Key Ideas.

Locally ringed spaces. Sheaf cohomology. Generalized topological spaces: Grothendieck (pre)topologies, sites. Derived differentials (cotangent complex). Quotients by group actions: geometric invariant theory, algebraic spaces, algebraic stacks, Artin's representability theorems. Parameter spaces. Serre's GAGA (and Grothendieck's generalization). Grothendieck's formal existence theorem. Zariski's main theorem, principle of connectedness, Stein factorization, etc. Serre duality. Grothendieck duality. Semicontinuity. Cohomology and base change. Flatness criteria. Smoothness criteria. Ampleness criteria, etc.

Key Techniques.

Chow's lemma. Devissage. (Quasi)compactness. (Quasi)separatedness. Properness. Projective morphisms. Affine morphisms. (Quasi)coherence. Sorites.

Motivating Questions.

Question 1. For a field K, for a collection $(f_1(t_1,\dots,t_n),\dots,f_c(t_1,\dots,t_n))$ of polynomials f_i in n variables $(t_1,\dots,t_n)$ with coefficients in K, does there an ordered n-tuple of elements of K, $(a_1,\dots,a_n)$, with $f_i(a_1,\dots,a_n)$ equal to zero for every $i=1,\dots,c$? Equivalently, when does a finite type, affine K-scheme have a K-rational point?

Question 2. For a field K, for a collection $(F_1(T_0, T_1, \dots, T_n), \dots, F_c(T_0, T_1, \dots, T_n))$ of homogeneous polynomials F_i of degree $d_i > 0$ in $n+1$ variables $(T_0, T_1, \dots, T_n)$ with coefficients in K, does there exist a "nontrivial" ordered $n+1$-tuple of elements of K, $(a_0, a_1, \dots, a_n) \neq (0, 0, \dots, 0)$, with $F_i(a_0, a_1, \dots, a_n)$ equal to zero for every $i = 1, \dots, c$? Equivalently, when does a projective K-scheme have a K-rational point?

Question 3. For a field K, for a finite group Γ, for an action of Γ on a finite type K-scheme X, is the quotient locally ringed space a K-scheme?

Question 4. For a field K, for a finite group Γ, for an action of Γ on a (quasi-)projective K-scheme X, is the quotient K-scheme also (quasi-)scheme? Which invertible sheaves on X are pullbacks of ample invertible sheaves on the quotient?

Question 5. For a field K, for a finite group Γ whose order is not a multiple of the characteristic of K, for a smooth, quasi-projective K-scheme X with an action of Γ on X via K-morphisms, is the quotient K-smooth?

Question 6. For a field K, for a linear group K-scheme G, for a free action of G on a finite type, affine K-scheme X, is the quotient locally ringed space an affine K-scheme?

Question 7. For a field K, for a reductive linear group K-scheme G, for a K-action of G on a projective K-scheme $(X, \mathcal{O}_X(1))$, what is the maximal G-invariant rational transformation to a projective K-variety $X \supseteq X^{\text{ss}} \twoheadrightarrow Q$ such that $\mathcal{O}_X(d)|_{X^{\text{ss}}}$ is the pullback of an ample invertible sheaf on Q for all sufficiently positive and divisible integers d (possibly X^{ss} is empty)?

Question 8. For a field K, for a K-action of $\mathbf{GL}_n$ on a smooth, affine K-scheme X such that all orbits have equal dimension and such that there exists a finite type geometric quotient, is the quotient K-scheme smooth?

Question 9. For a field K, for a smooth, projective, geometrically connected K-variety X and for an ample invertible sheaf $\mathcal{O}_X(1)$ on X, for which positive integers d is $\mathcal{O}_X(d)$ very ample?

Question 10. For a Noetherian ring R, what are the right derived functors of the left-exact, additive functor $\Gamma(\mathbb{P}^n_R, \widetilde{-})$ on the category of finitely generated, graded modules over the graded ring $S = R[T_0, \dots, T_n] = \bigoplus_{d \in \mathbb{Z}_{\geq 0}} S_d$ defined by $\Gamma(\mathbb{P}^n_R, \widetilde{M}) = \varinjlim_e \text{Hom}_S(S \cdot S_e, M)_0$?

Categories.

There are several useful perspectives for understanding schemes and foundational algebraic geometry. One of these is the perspective on sheaves, ringed spaces, locally ringed spaces and schemes using the language of categories.

Classes.

The definition of categories uses the notion of classes. Classes can be axiomatized as a first-order theory, as done by von Neumann – Bernays – Gödel or by Morse – Kelley. The approach here is a second-order theory using the metalanguage of (first-order) Zermelo – Fraenkel set theory. Technically this describes only the *parametrically definable* classes. For every model of Zermelo – Fraenkel set theory, the parameterically definable classes give one model of class theory (the model most often intended in analysis, algebra, geometry, etc.). We freely use the notions of

Zermelo – Fraenkel set theory. In particular, we use the notion of the Kuratowski ordered pair $(a, b) = \{\{a\}, \{a, b\}\}$ in order the convert predicates of higher arity into predicates of lower arity, i.e., every predicate $p(t_1, t_2, \dots, t_{n-1}, t_n)$ of arity $n \geq 1$ (a "true" natural number) in the first-order language of Zermelo – Fraenkel set theory is equivalent to the following predicate $\widetilde{p}(t)$ of arity 1,

$$\exists t_1 \ \exists t_2 \ \dots \exists t_{n-1} \ \exists t_n \ t = (t_1, (t_2, \dots, (t_{n-1}, t_n) \dots)) \wedge p(t_1, t_2, \dots, t_n).$$

Of course this is a statement about the metalanguage of first-order Zermelo – Fraenkel set theory, and thus it is itself a statement in the meta-metalangue of first-order Zermelo-Fraenkel set theory.

Definition 0.1. A **class** is a Lindenbaum-Tarski equivalence class of pairs $(p(s, t), a)$ of a set a and of a predicate $p(s, t)$ in the first-order language of Zermelo-Fraenkel set theory such that the free variables of $p(s, t)$ are the ordered pair (s, t). Here $(p(s, t), a)$ is **Lindenbaum-Tarski equivalent** to $(p'(s', t'), a')$ if (and only if)

$$\forall b \ \ (p'(a', b) \Leftrightarrow p(a, b)) \, .$$

For every class $[p(s, t), a]$, a set b is a **member** of $[p(s, t), a]$ if (and only if) $p(a, b)$ holds. In this case, we write $b \Subset [p(s, t), a]$.

Definition 0.2. For every class $[p(s, t), a]$, a **subclass** of $[p(s, t), a]$ is a class $[p'(s', t'), a']$ such that every member of $[p'(s', t'), a']$ is a member of $[p(s, t), a,]$, i.e.,

$$\forall b \ \ (p'(a', b) \Rightarrow p(a, b)) \, .$$

In this case, we write $[p'(s', t'), a'] \sqsubseteq [p(s, t), a]$.

Example 0.3. For a tautological predicate $p(s, t)$, say $(s = s) \wedge (t = t)$, for every set a, the class $[(s = s) \wedge (t = t), a]$ is the **von Neumann class**, sometimes called the **von Neumann universe** or the **universal class**, denoted $\mathbf{V}$ or $\text{ob}_{\mathbf{Set}}$. This is the unique class such that every set is a member of the class. Similarly, this is the unique class such that every class is a subclass of $\mathbf{V}$.

Example 0.4. For the predicate $p(s, t)$ that is $t \in s$, for every set a, the class $[t \in s, a]$ is the **class** of a, denoted $\mathbf{Cl}_a$. The members of $\mathbf{Cl}_a$ are the elements of a. For sets a and a', the class $\mathbf{Cl}_a$ is a subclass of $\mathbf{Cl}_a$ if and only if a is a subset of a'. Many axiomatizations of sets and classes identify each set a with the corresponding class $\mathbf{Cl}_a$ (so that the predicate we denote with the word "member" is an extension of the usual predicate for sets denoted with the word "element").

Exercise 0.5. For every class $\mathbf{B}$, for every class $\mathbf{C}$, check that conjunction of predicates produces a class $\mathbf{B} \wedge \mathbf{C}$ whose members are those sets that are simultaneously members of $\mathbf{B}$ and members of $\mathbf{C}$. Check that the subclasses of $\mathbf{B} \wedge \mathbf{C}$ are precisely the classes that are simultaneously subclasses of both $\mathbf{B}$ and $\mathbf{C}$. Also check that $\mathbf{Cl}_b \wedge \mathbf{Cl}_c$ equals $\mathbf{Cl}_{b \cap c}$.

Exercise 0.6. Similarly, check that disjunction produces a class $\mathbf{B} \vee \mathbf{C}$ whose members are those sets that are either a member of $\mathbf{B}$ or a member of $\mathbf{C}$ (or both). Check that a class has $\mathbf{B} \vee \mathbf{C}$ as a subclass if and only if both $\mathbf{B}$ and $\mathbf{C}$ are subclasses. Also check that $\mathbf{Cl}_b \vee \mathbf{Cl}_c$ equals $\mathbf{Cl}_{b \cup c}$.

Exercise 0.7. Finally, check that negation gives a class $\neg\mathbf{B}$ whose members are all sets that are not a member of $\mathbf{B}$. Check that the subclasses of $\neg\mathbf{B}$ are precisely the classes that have no member in common with $\mathbf{B}$. Check that $\neg(\neg\mathbf{B})$ equals $\mathbf{B}$, and check that $\neg(\mathbf{B}\wedge\mathbf{C})$ equals $(\neg\mathbf{B})\vee(\neg\mathbf{C})$. Deduce that $\neg(\mathbf{B}\vee\mathbf{C})$ equals $(\neg\mathbf{B})\wedge(\neg\mathbf{C})$.

As done in Zermelo-Fraenkel set theory, we can encode morphisms between classes as certain kinds of classes.

Definition 0.8. For every class $\mathbf{B}$, a class $\mathbf{E}$ is a **B-class** if (and only if) every member of $\mathbf{E}$ equals (b, c) for some member b of $\mathbf{B}$ and for some set c.

In particular, for every class $\mathbf{B}$, for every class $\mathbf{C}$, the **product class** of $\mathbf{B}$ and $\mathbf{C}$ is the $\mathbf{B}$-class $\mathbf{B}\times\mathbf{C}$ whose members are all sets (b, c) such that b is a member of $\mathbf{B}$ and such that c is a member of $\mathbf{C}$. A subclass of $\mathbf{B}\times\mathbf{C}$ is a **relation** from $\mathbf{B}$ to $\mathbf{C}$.

For every class $\mathbf{B}$, for every $\mathbf{B}$-class $\mathbf{E}$, for every member b of $\mathbf{B}$, a class $\mathbf{E}_{b,\bullet}$ is the **fiber class** of $\mathbf{E}$ over b if (and only if) the members of $\mathbf{E}_b$ are precisely the sets c such that (b, c) is a member of $\mathbf{E}$. The $\mathbf{B}$-class is a **B-set** if (and only if) every fiber class is a class of a set.

For every class $\mathbf{B}$, a $\mathbf{B}$-class $\mathbf{F}$ is a **morphism** of classes from $\mathbf{B}$ if (and only if), for every member b of $\mathbf{B}$, there exists a unique member c of the fiber class $\mathbf{F}_b$. In this case, we write $c = \mathbf{F}(b)$. If also $\mathbf{F}$ is a relation from $\mathbf{B}$ to a class $\mathbf{C}$, then $\mathbf{F}$ is a morphism of classes from $\mathbf{B}$ to $\mathbf{C}$. If for every member c of $\mathbf{C}$ there exists a unique member b of $\mathbf{B}$ such that c equals $\mathbf{F}(b)$, then $\mathbf{F}$ is an **isomorphism** of classes.

For $\mathbf{B}$-classes $\mathbf{D}$ and $\mathbf{E}$, a **morphism** of $\mathbf{B}$-classes from $\mathbf{D}$ to $\mathbf{E}$ is a morphism of classes $\mathbf{F}$ from $\mathbf{D}$ to $\mathbf{E}$ such that for every member b of $\mathbf{B}$ and for every member d of $\mathbf{D}_b$, also $\mathbf{F}(b, d)$ equals (b, e) for some member e of $\mathbf{E}_b$. The induced morphism of classes from $\mathbf{D}_b$ to $\mathbf{E}_b$ is the **morphism of fiber classes**.

Example 0.9. The product class $\mathbf{V}\times\mathbf{V}$ is the subclass of $\mathbf{V}$ consisting of all sets a that are equal to a Kuratowski ordered pair (a', a'') for a set a' and for a set a''. Thus, one relation from $\mathbf{V}$ to itself is the relation of "being an element of", i.e., (a', a'') is in the subclass if and only if a' is an element of a''. Another relation is the relation of subset inclusion, i.e., (a', a'') is in the subclass if and only if a' is a subset of a''.

Example 0.10. For every class $\mathbf{B}$, for every morphism of classes $\mathbf{F}$ from $\mathbf{B}$, there is a $\mathbf{B}$-class $\mathrm{cl}_{\mathbf{B},\mathbf{F}}$ whose members are all ordered pairs (b, c) of a member b of $\mathbf{B}$ and of an element c of the set $\mathbf{F}(b)$.

Exercise 0.11. For every class $\mathbf{B}$, for every morphism of classes $\mathbf{F}$ from $\mathbf{B}$, check that $\mathrm{cl}_{\mathbf{B},\mathbf{F}}$ is a $\mathbf{B}$-set. Conversely, for every $\mathbf{B}$-set $\mathbf{D}$, check that there is a unique morphism of classes $\mathrm{fun}_{\mathbf{B},\mathbf{D}}$ from $\mathbf{B}$ associating to every member b of $\mathbf{B}$ the unique set whose associated class is the fiber class $\mathbf{D}_b$. Check that these two operations determine an equivalence between $\mathbf{B}$-sets and morphisms of classes from $\mathbf{B}$.

Example 0.12. For every set with two elements, say $\mathbb{F}_2 = \{0, 1\}$ where 0 is the empty set and 1 is the singleton set $\{0\}$, for every class $\mathbf{B}$, for every class $\mathbf{C}$, denote by $(\mathbf{B}, \mathbf{C})$ the unique subclass of the product class $(\mathbf{B} \vee \mathbf{C}) \times \mathbf{Cl}_{\mathbb{F}_2}$ whose members are precisely the ordered pairs (x, y) such that either y equals 0 and x is a member of $\mathbf{B}$ or y equals 1 and x is a member of $\mathbf{C}$. From the class $(\mathbf{B}, \mathbf{C})$, we can uniquely recover the first component class $\mathbf{B}$ and the second component class $\mathbf{C}$. Iterating, for every (true) positive integer ℓ, for every ordered ℓ-tuple of classes, we can convert this into a single class from which we can uniquely recover the original ℓ-tuple of classes.

Exercise 0.13. Check the assertions in the previous example. Use this notion to extend to classes the familiar notion of Kuratowski ordered pair of sets. In particular, this allows us to summarize notions that *a priori* depend on a finite ordered n-tuple of classes $t_1, t_2, \ldots, t_{n-1}, t_n$ using a single class $t = (t_1, (t_2, \ldots (t_{n-1}, t_n) \ldots))$.

The notion of composition of functions and relations between sets extends to composition of morphisms and relations between classes. In fact, we also have composition for a notion that is a bit more general than relations.

Definition 0.14. For every class $\mathbf{B}$, for every class $\mathbf{C}$, a **span** from $\mathbf{B}$ to $\mathbf{C}$, or a $(\mathbf{B}, \mathbf{C})$**-span**, is a $\mathbf{B} \times \mathbf{C}$-class $\mathbf{E}$, i.e., every member of $\mathbf{E}$ is of the form $((b, c), d)$ for a member b of $\mathbf{B}$, for a member c of $\mathbf{C}$, and for a set d. Each fiber class $\mathbf{E}_{(b,c)}$ is also denoted by $\mathbf{E}_c^b$. A $(\mathbf{B}, \mathbf{C})$**-span morphism**, respectively $(\mathbf{B}, \mathbf{C})$**-span isomorphism**, is a morphism of $\mathbf{B} \times \mathbf{C}$-classes, resp. an isomorphism of $\mathbf{B} \times \mathbf{C}$-classes.

For every class $\mathbf{B}$, the **identity** of $\mathbf{B}$ is the $\mathbf{B}$-class $\mathrm{Id}_{\mathbf{B}}$ whose members are all sets (b, b) for b a member of $\mathbf{B}$, considered as a morphism of classes from $\mathbf{B}$ to itself. Similarly, the **span identity** of $\mathbf{B}$ to $\mathbf{B}$ is the $\mathbf{B} \times \mathbf{B}$-class $\mathrm{Id}_{s,\mathbf{B}}$ whose members are all sets $((b, b), b)$ for b a member of $\mathbf{B}$, considered as a span from $\mathbf{B}$ to itself. Thus, for every member b of $\mathbf{B}$, the fiber class $(\mathrm{Id}_{s,\mathbf{B}})_b^b$ is the class of the singleton set $\{b\}$, and the fiber class $(\mathrm{Id}_{s,\mathbf{B}})_c^b$ is the empty class if b does not equal c.

For every class $\mathbf{B}$, for every class $\mathbf{C}$, for every class $\mathbf{D}$, for every relation $\mathbf{R}$ from $\mathbf{B}$ to $\mathbf{C}$, for every relation $\mathbf{Q}$ from $\mathbf{C}$ to $\mathbf{D}$, a class $\mathbf{Q} \circ \mathbf{R}$ is the **composition** of $\mathbf{Q}$ and $\mathbf{R}$ if (and only if) the members of $\mathbf{Q} \circ \mathbf{R}$ are all sets (b, d) such that there exists a member c of $\mathbf{C}$ where (b, c) is a member of $\mathbf{R}$ and (c, d) is a member of $\mathbf{Q}$.

More generally, for every span $\mathbf{R}$ from $\mathbf{B}$ to $\mathbf{C}$, for every span $\mathbf{Q}$ from $\mathbf{C}$ to $\mathbf{D}$, the **span composition** $\mathbf{Q} \circ \mathbf{R}$ of $\mathbf{Q}$ and $\mathbf{R}$ is the span from $\mathbf{B}$ to $\mathbf{D}$ such that for every member b of $\mathbf{B}$ and for every member d of $\mathbf{D}$, the members of the fiber class $(\mathbf{Q} \circ \mathbf{R})_d^b$ are all ordered pairs $(c, (q, r))$ of a member c of $\mathbf{C}$ and members q and r of the respective fiber categories $\mathbf{Q}_d^c$ and $\mathbf{R}_c^b$.

For every span $\mathbf{Q}$ from $\mathbf{C}$ to $\mathbf{D}$, there is an isomorphism of $(\mathbf{C}, \mathbf{D})$-spans $\mathsf{r}_{\mathbf{Q}}$ $\mathbf{Q} \circ \mathrm{Id}_{s,\mathbf{C}}$ to $\mathbf{Q}$, respectively $\mathsf{l}_{\mathbf{Q}}$ from $\mathrm{Id}_{s,\mathbf{D}} \circ \mathbf{Q}$ to $\mathbf{Q}$, sending every member $((c, d), (c, (q, c)))$ of $\mathbf{Q} \circ \mathrm{Id}_{s,\mathbf{C}}$ to $((c, d), q)$, respectively sending every member $((c, d), (d, (d, q)))$ of $\mathrm{Id}_{s,\mathbf{C}} \circ \mathbf{R}$ to $((c, d), q)$.

For every span $\mathbf{R}$ from $\mathbf{B}$ to $\mathbf{C}$, for every span $\mathbf{Q}$ from $\mathbf{C}$ to $\mathbf{D}$, and for every span $\mathbf{P}$ from $\mathbf{D}$ to $\mathbf{E}$, there is an isomorphism of $(\mathbf{C}, \mathbf{E})$-spans $\mathsf{a}_{\mathbf{P},\mathbf{Q},\mathbf{R}}$ from $(\mathbf{P} \circ \mathbf{Q}) \circ \mathbf{R}$ to $\mathbf{P} \circ (\mathbf{Q}\mathbf{R})$ that sends every member $((b, e), (c, ((d, (p, q)), r)))$ of $(\mathbf{P} \circ \mathbf{Q}) \circ \mathbf{R}$ to the member $((b, e), (d, (p, (c, (q, r)))))$ of $\mathbf{P} \circ (\mathbf{Q} \circ \mathbf{R})$.

Example 0.15. For the von Neumann class $\mathbf{V}$ of all sets, consider the span $\mathrm{mor}(\mathbf{Set})$ from $\mathbf{V}$ to $\mathbf{V}$ such that for every set b, for every set c, the fiber class over (b,c) is the class whose members are all subsets of $b \times c$ that are graphs of functions from b to c. Of course this is the class of the set $\mathrm{Fun}(b,c)$ of all functions from b to c (considered as a subset of the power set $\mathcal{P}(b \times c)$ of $b \times c$). The span $\mathrm{mor}(\mathbf{Set})$ from $\mathbf{V}$ to itself, together with a composition law, is the $\mathbf{Set}$ of all sets.

Proposition 0.16. *Composition of relations between classes is strictly associative, and the identity relations are strict left-right identities for this composition. Composition of spans is associative up to the specified associator* a*, and the identity spans are left-right identities for this composition up to the left and right unitors* l *and* r*. The associator and unitors satisfy the triangle (coherence) identity and the pentagon (coherence) identity of monoidal categories.*

There is a notion of morphisms of spans. Together with the composition, associator and unitors, spans satisfy the axioms of (a version of) *double category*. Of course spans are classes that may not be sets, so extreme care is necessary in forming any kind of category of spans.

Exercise 0.17. Read about double categories. Formulate and verify the axioms of a double category that are satisfied by the operations above for spans.

Definition 0.18. For every ordered triple $(\mathbf{B}, \mathbf{C}, \mathbf{E})$ of classes $\mathbf{B}$ and $\mathbf{C}$ and a span $\mathbf{E}$ from $\mathbf{B}$ to $\mathbf{C}$, for every ordered triple $(\mathbf{B}', \mathbf{C}', \mathbf{E}')$ of classes $\mathbf{B}'$ and $\mathbf{C}'$ and a span $\mathbf{E}'$ from $\mathbf{B}'$ to $\mathbf{C}'$, a **span cell** from $(\mathbf{B}, \mathbf{C}, \mathbf{E})$ to $(\mathbf{B}', \mathbf{C}', \mathbf{E}')$ is an ordered triple $(s(\mathbf{F}), t(\mathbf{F}), \mathbf{F})$ of a morphism of classes $s(\mathbf{F})$ from $\mathbf{B}$ to $\mathbf{B}'$, of a morphism of classes $t(\mathbf{F})$ from $\mathbf{C}$ to $\mathbf{C}'$, and of a morphism of classes $\mathbf{F}$ from $\mathbf{E}$ to $\mathbf{E}'$ such that for every member $((b,c),e)$ of $\mathbf{E}$, the value $((b',c'),e') = \mathbf{F}((b,c),e)$ satisfies $b' = s(\mathbf{F})(b)$ and $c' = t(\mathbf{F})(c)$.

For every ordered triple $(\mathbf{B}, \mathbf{C}, \mathbf{E})$, the **identity span cell** is $(\mathrm{Id}_{\mathbf{B}}, \mathrm{Id}_{\mathbf{C}}, \mathrm{Id}_{\mathbf{E}})$. For every span cell $(s(\mathbf{F}), t(\mathbf{F}), \mathbf{F})$ from $(\mathbf{B}, \mathbf{C}, \mathbf{E})$ to $(\mathbf{B}', \mathbf{C}', \mathbf{E}')$, and for every span cell $(s(\mathbf{F}'), s(\mathbf{F}'), \mathbf{F}')$ from $(\mathbf{B}', \mathbf{C}', \mathbf{E}')$ to $(\mathbf{B}'', \mathbf{C}'', \mathbf{E}'')$, the **composite span cell** is $(s(\mathbf{F}') \circ s(\mathbf{F}), t(\mathbf{F}') \circ t(\mathbf{F}), \mathbf{F}' \circ \mathbf{F})$.

One advantage of relations, and more generally of spans, over morphisms is that they have *opposites*.

Definition 0.19. For every class $\mathbf{B}$, for every class $\mathbf{C}$, for every relation $\mathbf{R}$ from $\mathbf{B}$ to $\mathbf{C}$, the **opposite relation** $\mathbf{R}^{\mathrm{opp}}$ from $\mathbf{C}$ to $\mathbf{B}$ is the unique subclass of $\mathbf{C} \times \mathbf{B}$ whose members are all ordered pairs (c,b) such that (b,c) is a member of $\mathbf{R}$.

More generally, for every span $\mathbf{E}$ from $\mathbf{B}$ to $\mathbf{C}$, the **opposite span** $\mathbf{E}^{\mathrm{opp}}$ from $\mathbf{C}$ to $\mathbf{B}$ is the $\mathbf{C} \times \mathbf{B}$-class such that for every member b of $\mathbf{B}$ and for every member c of $\mathbf{C}$, the fiber class $(\mathbf{E}^{\mathrm{opp}})^c_b$ equals the fiber class $\mathbf{E}^b_c$.

Exercise 0.20. Formulate the notion of the opposite of a span cell. Check that the opposite span of a span composite is naturally span isomorphic to the span composite of the span opposites of the factors (in the opposite order). Read about *dagger categories*. Formulate and check the axioms of a dagger category that hold for spans.

Categories.

A category is a certain kind of span from a class to itself whose fiber classes are required to be (classes of) sets, and equipped with a composition law from its self-composite to itself that is associative and unital.

Definition 0.21. For every class $\mathbf{O}$, a $(\mathbf{O}, \mathbf{O})$-span $\mathbf{M}$, i.e., a class in which every member is of the form $((a,b),f)$ for members a and b of $\mathbf{O}$ and a set f, is a **O-Hom span** if (and only if), for every member (a,b) of $\mathbf{O} \times \mathbf{O}$, the fiber class $\mathbf{M}_b^a := \mathbf{M}_{(a,b)}$ is the class of a set. Breaking with our earlier convention, we will sometimes also denote this set by $\mathbf{M}_b^a$. More often it is denoted $\mathrm{Hom}_{\mathbf{O},\mathbf{M}}(a,b)$, or just $\mathrm{Hom}(a,b)$ when $\mathbf{O}$ and $\mathbf{M}$ are understood, i.e., the members of $\mathbf{M}$ are sets $((a,b),f)$ for members a and b of $\mathbf{O}$ and elements f of $\mathrm{Hom}(a,b)$.

Example 0.22. Recall the earlier example, where $\mathbf{O}$ is the von Neumann class $\mathbf{V}$ of all sets, the span $\mathrm{mor}(\mathbf{Set})$ from $\mathbf{V}$ to itself is the class of all triples $((a,b),f)$ of a set a, of a set b, and of a function f from a to b. Thus, each Hom set $\mathrm{Hom}_{\mathbf{V},\mathrm{mor}(\mathbf{Set})}(a,b)$ is the set of all functions from a to b (considered as a subset of the power set $\mathcal{P}(a \times b)$ of $a \times b$).

Example 0.23. For another example, again let $\mathbf{O}$ be the von Neumann class $\mathbf{V}$ of all sets, but now let the span $\mathrm{mor}(\mathbf{Rel})$ from $\mathbf{V}$ to itself be the class of all triples $((a,b),R)$ of a set a, of a set b, and of a relation R from a to b, i.e., R is an (arbitrary) subset of $a \times b$. Thus, each Hom set $\mathrm{Hom}_{\mathbf{V},\mathrm{mor}(\mathbf{Rel})}(a,b)$ is the power set of $a \times b$.

Of course, for a Hom span $(\mathbf{O}, \mathbf{M})$, the composite span $\mathbf{M} \circ \mathbf{M}$ is typically **not** a Hom span: for all members a and c of $\mathbf{O}$, the members of $(\mathbf{M} \circ \mathbf{M})_c^a$ are all ordered triples $(b,(g,f))$ of a member b of $\mathbf{O}$, of an element f of the set $\mathrm{Hom}(a,b)$ and of an element g of $\mathrm{Hom}(b,c)$. Since b varies over members of a class (that is typically not a set), the class $(\mathbf{M} \circ \mathbf{M})_c^a$ is typically not a set.

Example 0.24. In the first example above, for every set a, for every set b, the fiber class $(\mathrm{mor}(\mathbf{Set}) \circ \mathrm{mor}(\mathbf{Set}))_c^a$ is the class of all triples $(b,(g,f))$ of a set b, of a function f from a to b, and of a function g from b to c. This class is not a set, but of course the class of all compositions $g \circ f$ is just the set $\mathrm{Hom}(a,b)$ of all functions from a to b.

Definition 0.25. For every class $\mathbf{O}$, for every Hom span $\mathbf{M}$ from $\mathbf{O}$ to itself, a $(\mathbf{O}, \mathbf{M})$-**composition law** is a span morphism $\circ$ from $\mathbf{M} \circ \mathbf{M}$ to $\mathbf{M}$, i.e., a morphism of $\mathbf{O} \times \mathbf{O}$-classes such that, for all members a and c of $\mathbf{O}$, the induced fiber morphism from $(\mathbf{M} \circ \mathbf{M})_c^a$ to $\mathbf{M}_c^a$ sends each member $(b,(g,f))$ of $(\mathbf{M} \circ \mathbf{M})_c^a$ to a member $g \circ f$ of $\mathbf{M}_c^a$.

A composition law is **associative** if (and only if), for all members a, b, c and d of $\mathbf{O}$, for every element (h,g,f) of $\mathrm{Hom}(d,e) \times \mathrm{Hom}(c,d), \mathrm{Hom}(b,c)$, the composition $(h \circ g) \circ f$ equals $h \circ (g \circ f)$ as elements of $\mathrm{Hom}(a,e)$.

An associative composition law is **unital** if (and only if), for every member a of $\mathbf{O}$, there exists an element $\mathrm{Id}_a^{\mathbf{O},\mathbf{M},\circ}$ of $\mathrm{Hom}(a,a)$ such that, for every member b of $\mathbf{O}$, both the left composition with $\mathrm{Id}_a^{\mathbf{O},\mathbf{M},\circ}$ from $\mathrm{Hom}(b,a)$ to itself is the identity, and the right composition with $\mathrm{Id}_a^{\mathbf{O},\mathbf{M},\circ}$ from $\mathrm{Hom}(a,b)$ to itself is the identity.

A **category** is a class $\mathbf{O}$, called the **class of objects**, a $\mathbf{O}$-Hom span $\mathbf{M}$, called the **class of morphisms**, the specification of the **source** and **target** morphisms from $\mathbf{M}$ to $\mathbf{O}$ sending every member $((a,b),f)$ of $\mathbf{M}$ to the member a of $\mathbf{O}$, respectively to the member b of $\mathbf{O}$, and a $(\mathbf{O},\mathbf{M})$-**composition law** $\circ$ that is both associative and unital. An **isomorphism** in a category is a morphism $((a,b),f)$ such that there exists a morphism $((b,a),g)$ such that both $g\circ f$ equals Id_a and $f\circ g$ equals Id_b; in this case we denote g by f^{-1}.

For a category $\mathbf{C}$, the class $\mathbf{O}$ is often denoted $\mathrm{ob}(\mathbf{C})$ and its members are called **C-objects** or **objects** of $\mathbf{C}$. The class $\mathbf{M}$ is often denoted $\mathrm{mor}(\mathbf{C})$, each set $\mathrm{Hom}_{\mathbf{O},\mathbf{M}}(a,b)$ is denoted $\mathbf{C}_b^a$ or $\mathrm{Hom}_{\mathbf{C}}(a,b)$ and its elements are called **C-morphisms** from a to b. The composition law is denoted $\circ^{\mathbf{C}}$, or just $\circ$ when confusion is unlikely. For every object a of $\mathbf{C}$, the left-right identity morphism from a to itself is usually denoted $\mathrm{Id}_a^{\mathbf{C}}$ or Id_a when confusion is unlikely.

Example 0.26. The category **Set** of sets has object class $\mathrm{obj}(\mathbf{Set})$ equal to the von Neumann class / universal class $\mathbf{V}$ of all sets, has morphism class $\mathrm{mor}(\mathbf{Set})$ as in the first example above with fiber class $\mathbf{Set}_b^a$ equal to the (class of the) set of all functions f from a to b, and has the usual composition of functions. The identity functions are the identity morphisms of this category.

Example 0.27. The category **Rel** of relations again has object class equal to $\mathbf{V}$, but has morphism class as in the second example above with fiber class $\mathbf{Rel}_b^a$ equal to the (class of the) power set of $a\times b$. Composition is composition of relations (as defined in the previous section). The identity functions (or their graphs) are the identity morphisms of this category.

There are many different kinds of categories.

Definition 0.28. A category is **small** if (and only if) the object class is the class of a set. A category is a **monoid** if (and only if) the object class is the class of a singleton set. A category is **thin** if (and only if) every nonempty Hom set is a singleton set. A category is a **groupoid** if (and only if) every morphism is an isomorphism. A thin groupoid is a **setoid**. A category is **skeletal** if (and only if) all isomorphic objects are equal. A skeletal setoid is a **discrete category**.

There are many ways to produce new categories from given categories.

Definition 0.29. For every category $\mathbf{C}=(\mathbf{O},\mathbf{M},\circ)$, the **opposite category** is the category $(\mathbf{O},\mathbf{M}^{\mathrm{opp}},\circ^{\mathrm{opp}})$, where $\mathbf{M}^{\mathrm{opp}}$ is the opposite span of $\mathbf{M}$, and where, for every member $((a,b),f)$ of $\mathbf{M}$ and for every member $((b,c),g)$ of $\mathbf{M}$, the opposite composition is defined by

$$((b,a),f)\circ^{\mathrm{opp}}((c,b),g)=((c,a),g\circ f).$$

Definition 0.30. For every category $\mathbf{C}$ and for every category $\mathbf{C}'$, the **product category** $\mathbf{C}\times\mathbf{C}'$ of $\mathbf{C}$ and $\mathbf{C}'$ is the category whose objects are ordered pairs (a,a') of a $\mathbf{C}$-object a and a $\mathbf{C}'$-object a'. For every ordered pair $((a,a'),(b,b'))$, the Hom set in $\mathbf{C}\times\mathbf{C}'$ is

$$(\mathbf{C}\times\mathbf{C}')_{(b,b')}^{(a,a')}=\mathbf{C}_b^a\times(\mathbf{C}')_{b'}^{a'}.$$

Finally, composition is defined componentwise.

Definition 0.31. For every category $\mathbf{C} = (\mathbf{O}, \mathbf{M}, \circ)$, for every subclass $\mathbf{O}'$ of $\mathbf{O}$, the **full subcategory** of $\mathbf{C}$ with objects class $\mathbf{O}'$ is the category $(\mathbf{O}', \mathbf{M}|_{\mathbf{O}'}, \circ')$ where, for all members a and b of $\mathbf{O}'$, the class $(\mathbf{M}|_{\mathbf{O}'})^a_b$ equals $\mathbf{M}^a_b$, and where $\circ'$ is the restriction of $\circ$. More generally, a (not necessarily full) **subcategory** of $\mathcal{C}$ consists of a subclass $\mathbf{O}'$ of $\mathbf{O}$ and a subclass of $\mathbf{M}'$ of $\mathbf{M}|_{\mathbf{O}'}$ that contains all identity morphisms of objects of $\mathbf{O}'$ and that is stable for composition, thus defining a restriction composition on the subcategory.

Example 0.32. The category **Set** of sets is a non-full subcategory of the category **Rel** of all relations. The category of all finite sets is a full subcategory of the category of all sets.

Example 0.33. For every set H together with a binary operation $\bullet$ from $H \times H$ to H that is associative and unital, there exists a monoid whose unique object is, say, the set H itself (perhaps considered as a right act over itself), and whose unique Hom set is H with $\bullet$ giving the binary operation. For every category, for every object of that category, the restriction of composition to the Hom set of that object is a monoid as above. In particular, every category with a unique object is strongly equivalent to the category of the monoid of the unique Hom set with its composition operation.

Example 0.34. In the previous example, if every element of H is invertible, then $(H, \bullet)$ is precisely a group. In this case, the category above is a skeletal groupoid, sometimes denoted BH. For every groupoid, for every object of that groupoid, the restriction of composition to the Hom set of that object is a group H, and the full subcategory whose unique object is the given object is strongly equivalent to BH. In particular, every groupoid with a unique object is strongly equivalent to BH for the unique Hom set H with its composition operation.

Example 0.35. We could "deskeletonize" the previous example by considering the category whose objects are all right acts over the monoid H that are principal homogeneous spaces, and all morphisms are morphisms of right H-acts.

Example 0.36. The category of right principal homogenous spaces for H, as above, is a full subcategory of the category of all right H-acts. Another full subcategory is the category of all right H-acts that are trivial in the sense that every element of H acts identically on the set. This full subcategory is strongly equivalent to the category **Set** of all sets. Of course if H is itself a singleton monoid, then this full subcategory equals the entire category of right H-acts, so that this category is strongly equivalent to **Set**.

Example 0.37. The category **Monoid** has as objects class the class whose members are all ordered pairs $(H, \bullet)$ of a set H together with a unital, associative binary operation $\bullet$ from H to itself, and where morphisms from $(H, \bullet)$ to $(H', \bullet')$ are functions f from H to H' that preserve the identity and preserve the binary operation (i.e., usual morphisms of monoids): $f(e_H)$ equals $e_{H'}$ and $f(h \bullet k)$ equals $f(h) \bullet' f(k)$ for all elements h, k of H. This has a full subcategory **Grp** whose objects are groups. This has a full subcategory **Ab** whose objects are Abelian groups. This has a full subcategory $\mathbb{Q} - mbfMod$ whose objects are Abelian groups such that multiplication by n is a bijection of the group to itself for every nonzero integer n, i.e., the Abelian group is a $\mathbb{Q}$-vector space. This has a full subcategory whose objects are finite-dimensional $\mathbb{Q}$-vector spaces.

Example 0.38. A hybrid of the previous two examples is the category whose objects are all ordered pairs $((H,\bullet),(S,\rho))$ of a monoid $(H,\bullet)$ together with a right H-act $\rho : S \times H \to S$. The morphisms from $((H,\bullet),(S,\rho))$ to $((H',\bullet'),(S',\rho'))$ are all ordered pairs (f,g) of a morphism f of monoids from $(H,\bullet)$ to $(H',\bullet')$ together with a function g from S to S' such that, for every element h of H and every element s of S, the image $g(\rho(s,h))$ equals $\rho'(g(s),f(h))$, i.e., g is a morphism of right H-acts for the induced right H-act on S' obtained from ρ' and f. This hybrid category is an example of the "Grothendieck construction" for fibered categories (one of the basic notions in extending from schemes to stacks).

Example 0.39. For every category $\mathbf{C}$, the **core** of $\mathbf{C}$ is the (usually non-full) subcategory with the same objects, but whose Hom set is the subset of invertible elements in the corresponding Hom set of $\mathbf{C}$. Thus, for a monoid $(H,\bullet)$, the core of the category of $(H,\bullet)$ is $BH^\times$, where $H^\times$ is the group of invertible elements of H. The core of **Set** is the the groupoid of sets where morphisms are bijections. This also equals the core of **Rel**.

Example 0.40. In particular, in the core of the hybrid category, for every object $((H,\bullet),(H,r_H))$ where $(H,\bullet)$ is a group and where (H,r_H) is the right regular H-action on itself, the group of automorphisms of this object is the classical notion of *holomorph* of the group, i.e., the semidirect product of the group with its automorphism group.

Example 0.41. In particular, for each monoid $(H,\bullet)$, for the right regular H-act (H,r_H) of H on itself, the monoid of self-morphisms of $((H,\bullet),(H,r_H))$ in the previous category is the semidirect product of the monoid $(H,\bullet)$ with its monoid of monoid endomorphisms; this is the analogue for monoids of the classical notion of *holomorph* of a group.

Example 0.42. For every set S together with a relation R from S to itself, consider the class of S as a class of objects, and consider R as a span from this class to itself. An associative, unital composition law extending this to a category is unique, if it exists. In fact, this span extends to a category if and only if R is preorder, i.e., if and only if R is both transitive and reflexive. In this case, the corresponding category is small and thin. Every small, thin category is strongly equivalent to the category of a preordered set. The category of a preordered set is skeletal if and only if the preorder is a partial order, i.e., if and only if the relation is transitive, reflexive and asymmetric. Similarly, the category of a preordered set is a groupoid if and only if the relation is an equivalence relation, i.e., if and only if the relation is transitive, reflexive and symmetric. Every preordered set is the pullback of a partial order under a surjection whose associated equivalence relation is the core of the preorder. If we accept the Axiom of Choice, there exists a subset of the original set that surjects isomorphically to the partially ordered set, and this defines a full subcategory of the category of the preordered set that is a skeleton.

Functors.

Definition 0.43. For every category $\mathbf{C} = (\mathbf{O},\mathbf{M},\circ)$, for every category $\mathbf{C}' = (\mathbf{O}',\mathbf{M}',\circ')$, a **covariant functor** from $\mathbf{C}$ to $\mathbf{C}'$ is a span cell $(s(\mathbf{F}),t(\mathbf{F}),\mathbf{F}) = (\mathbf{F}_{\text{obj}},\mathbf{F}_{\text{obj}},\mathbf{F}_{\text{mor}})$ from the span $(\mathbf{O},\mathbf{O},\mathbf{M})$ to the span $(\mathbf{O}',\mathbf{O}',\mathbf{M}')$ that maps identities to identities and that is compatible with composition: for every object a of $\mathbf{C}$, the morphism $\mathbf{F}_{\text{mor}}$ maps $(a, a, \text{Id}_a^{\mathbf{C}})$ to $(a', a', \text{Id}_{a'}^{\mathbf{C}'})$ for $a' = \mathbf{F}_{\text{obj}}(a)$, and for every ordered pair $((a,b), f)$, $((b,c), g)$ of members of $\mathbf{M}$ with images $((a',b'), f')$ and $((b',c'), g')$ under $\mathbf{F}_{\text{mor}}$, also $((a,c), g \circ f)$ has image $((a',c'), g' \circ' f')$.

In particular, the identity span cell is a covariant functor $(\text{Id}_{\mathbf{O}}, \text{Id}_{\mathbf{O}}, \text{Id}_{\mathbf{M}})$ from $\mathbf{C}$ to itself. Also, for every covariant functor $(\mathbf{F}_{\text{obj}}, \mathbf{F}_{\text{obj}}, \mathbf{F}_{\text{mor}})$ from $(\mathbf{O}, \mathbf{M}, \circ)$ to $(\mathbf{O}', \mathbf{M}', \circ')$, and for every covariant functor $(\mathbf{F}'_{\text{obj}}, \mathbf{F}'_{\text{obj}}, \mathbf{F}'_{\text{mor}})$ from $(\mathbf{O}', \mathbf{M}', \circ')$ to $(\mathbf{O}'', \mathbf{M}'', \circ'')$, the composition $(\mathbf{F}'_{\text{obj}} \circ \mathbf{F}_{\text{obj}}, \mathbf{F}'_{\text{obj}} \circ \mathbf{F}_{\text{obj}}, \mathbf{F}'_{\text{mor}} \circ \mathbf{F}_{\text{mor}})$ is a covariant functor from $(\mathbf{O}, \mathbf{M}, \circ)$ to $(\mathbf{O}'', \mathbf{M}'', \circ'')$ called the **composite covariant functor**.

Definition 0.44. For every covariant functor $(\mathbf{F}_{\text{obj}}, \mathbf{F}_{\text{obj}}, \mathbf{F}_{\text{mor}})$ from $(\mathbf{O}, \mathbf{M}, \circ)$ to $(\mathbf{O}', \mathbf{M}', \circ')$, the span cell of opposites spans is a covariant functor of the opposite category, $(\mathbf{F}_{\text{obj}}, \mathbf{F}_{\text{obj}}, \mathbf{F}_{\text{mor}}^{\text{opp}})$ from $(\mathbf{O}, \mathbf{M}^{\text{opp}}, \circ^{\mathbf{opp}})$ to $(\mathbf{O}', (\mathbf{M}')^{\text{opp}}, (\circ')^{\text{opp}})$. This is the **opposite covariant functor**. The opposite covariant functor of the opposite covariant functor equals the original covariant functor.

For every category $\mathbf{C}$, for every category $\mathbf{C}'$, a covariant functor from $\mathbf{C}^{\text{opp}}$ to $\mathbf{C}'$ is then equivalent (up to taking opposites) to a covariant functor from $\mathbf{C}$ to $(\mathbf{C}')^{\text{opp}}$, and these are both (somewhat confusingly) called **contravariant functors** from $\mathbf{C}$ to $\mathbf{C}'$.

Definition 0.45. For every covariant functor $(\mathbf{F}_{\text{obj}}, \mathbf{F}_{\text{obj}}, \mathbf{F}_{\text{mor}})$ from $\mathbf{C} = (\mathbf{O}, \mathbf{M}, \circ)$ to $\mathbf{C}' = (\mathbf{O}', \mathbf{M}', \circ')$, the functor is **full**, respectively **faithful**, **fully faithful**, if for all members a and b of $\mathbf{O}$ with values $a' = \mathbf{F}_{\text{obj}}(a)$ and $b' = \mathbf{F}_{\text{obj}}(b)$, the function $\mathbf{F}_{\text{mor},b}^{a}$ from $\text{Hom}_{\mathbf{C}}(a,b)$ to $\text{Hom}_{\mathbf{C}'}(a',b')$ is surjective, resp. injective, bijective. A functor is **essentially surjective** if every object of $\mathbf{C}'$ is isomorphic to an object of the form $\mathbf{F}_{\text{obj}}(a)$ for some member a of $\mathbf{C}$. A functor that is essentially surjective and fully faithful is a **weak equivalence** of categories.

Definition 0.46. For every category, the **identity functor** from the category to itself maps every object to itself and maps every morphism to itself. For every functor $\mathbf{F} = (\mathbf{F}_{\text{obj}}, \mathbf{F}_{\text{obj}}, \mathbf{F}_{\text{mor}})$ from a category $\mathbf{C}$ to a category $\mathbf{C}'$, for every functor $\mathbf{F}' = (\mathbf{F}'_{\text{obj}}, \mathbf{F}'_{\text{obj}}, \mathbf{F}'_{\text{mor}})$ from the category $\mathbf{C}'$ to a category $\mathbf{C}''$, the **composite functor** is the composite of span cells, $\mathbf{F}' \circ \mathbf{F} = (\mathbf{F}'_{\text{obj}} \circ \mathbf{F}_{\text{obj}}, \mathbf{F}'_{\text{obj}} \circ \mathbf{F}_{\text{obj}}, \mathbf{F}'_{\text{mor}} \circ \mathbf{F}_{\text{mor}})$ that sends every $\mathbf{C}$-object a to $\mathbf{F}'_{\text{obj}}(\mathbf{F}_{\text{obj}}(a))$ and that sends every $\mathbf{C}$-morphism $((a,b), f)$ in $\mathbf{C}_b^a$ to $\mathbf{F}'_{\text{mor}}(\mathbf{F}_{\text{mor}}((a,b), f))$.

Proposition 0.47. *Composition of functors is associative, and it is unital for the identity functors.*

Example 0.48. The inclusion of every subcategory into a category is a faithful functor. This functor is full if and only if the subcategory is a full subcategory. The inclusion of a full skeletal subcategory in a category that is essentially surjective is called a **skeleton** of the category. If we assume a strong version of the Axiom of Choice (such as Hilbert's epsilon operator), then every category has a skeleton (this follows from the usual Axiom of Choice if we also work in a fixed Grothendieck universe, which itself requires large cardinal axioms).

Example 0.49. A faithful functor from a category $\mathbf{C}$ to **Set** is a **concrete functor**, and this functor makes $\mathbf{C}$ into a **concrete category**. Most of the categories that arise in analysis, algebra, geometry, etc. are concrete, and typically the concrete functor is a "forgetful functor" that "forgets" some of the structure of the objects of $\mathbf{C}$. For example, the forgetful functor from **Monoid** to **Set** that forgets the binary operation is a concrete functor. Thus, we also get concrete functors by restricting to the full subcategories **Grp**, **Ab** and $\mathbb{Q} - \mathbf{Mod}$.

Example 0.50. For every monoid $(H, \bullet)$, for every monoid $(H', \bullet')$, for every monoid homomorphism f from $(H, \bullet)$ to $(H', \bullet')$, there is a unique functor from the category of $(H, \bullet)$ to the category of $(H', \bullet')$ that maps the unique object to the unique object, and that maps Hom sets via f. Every functor between these categories is of this form for a unique monoid homomorphism f. More generally, for every functor $\mathbf{F}$ from a category $\mathbf{C}$ to a category $\mathbf{C}'$, for every object a of $\mathbf{C}$ with image $a' = \mathbf{F}(a)$, the function $\mathbf{F}_a^a$ from $\mathbf{C}_a^a$ to $(\mathbf{C}')_{a'}^{a'}$ is a monoid homomorphism. Moreover, for every ordered pair (a, b) of objects of $\mathbf{C}$, for the set $\mathbf{C}_b^a$ with its natural left $\mathbf{C}_b^b$-act and its natural right $\mathbf{C}_a^a$-act, for the set $(\mathbf{C}')_{b'}^{a'}$ with the induced left $\mathbf{C}_b^b$-act and right $\mathbf{C}_a^a$-act arising from the monoid homomorphisms $\mathbf{F}_b^b$ and $\mathbf{F}_a^a$, the function $\mathbf{F}_b^a$ from $\mathbf{C}_b^a$ to $(\mathbf{C}')_{b'}^{a'}$ is compatible with the left and right acts.

Example 0.51. Specializing the previous example to the case when $(H, \bullet)$ and $(H', \bullet')$ are groups, the functors from BH to BH' are equivalent to group homomorphisms from $(H, \bullet)$ to $(H', \bullet')$. More generally, every functor between groupoids induces group homomorphisms between automorphism groups of objects and the induced functioms between general Hom sets are compatible with both the left and right actions by these automorphism groups.

Example 0.52. For every category $\mathbf{C}$, for every preordered set (S', R'), every functor from $\mathbf{C}$ to the category of (S', R') is equivalent to a morphism $\mathbf{F}_{\text{obj}}$ from $\text{obj}(\mathbf{C})$ to (the class of) S' that is *nondecreasing*, i.e., for every ordered pair (a, b) of objects of $\mathbf{C}$ such that $\mathbf{C}_b^a$ is nonempty, then $(f(a), f(b))$ is an element of R'.

Example 0.53. For every every category $\mathbf{C}$, for every category $\mathbf{C}'$, for object a' of a category $\mathbf{C}'$ there is a functor $\text{const}_{\mathbf{C}',a}^{\mathbf{C}}$ from $\mathbf{C}$ to $\mathbf{C}'$ that maps every object to a and maps every morphism to Id_a.

Example 0.54. In particular, for the category **Set**, the functor $\mathbf{L} = \text{const}_{\mathbf{Set},\emptyset}^{\mathbf{Set}}$ from **Set** to itself has the special property that $\text{Hom}_{\mathbf{Set}}(\mathbf{L}(a), b)$ is always a singleton set. An object e of a general category $\mathbf{C}$ is an **initial object** if and only if $\mathbf{L} = \text{const}_{\mathbf{C},e}^{\mathbf{C}}$ has this property.

Example 0.55. Similarly, for the category **Set**, for every singleton set, say $\{\emptyset\}$, the functor $\mathbf{R} = \text{const}_{\mathbf{Set},\{\emptyset\}}^{\mathbf{Set}}$ from **Set** to itself has the special property that $\text{Hom}_{\mathbf{Set}}(a, \mathbf{R}(b))$ is always a singleton set. An object f of a general category $\mathbf{C}$ is a **final object** if and only if $\mathbf{R} = \text{const}_{\mathbf{C},f}^{\mathbf{C}}$ has this property. An object that is simultaneously an initial object and a final object is a **zero object**, e.g., the singleton monoid is a zero object in the category of monoids, and simultaneously in its full subcategory of groups, in the full subcategory of Abelian group, in the full subcategory of $\mathbb{Q}$-vector spaces, etc.

Natural Transformations.

Definition 0.56. For every category $\mathbf{C}$, for every category $\mathbf{D}$, for every covariant functor $\mathbf{F}$ from $\mathbf{C}$ to $\mathbf{D}$, and for every covariant functor $\mathbf{G}$ from $\mathbf{C}$ to $\mathbf{D}$, a **natural transformation** from $\mathbf{F}$ to $\mathbf{G}$ is a morphism of classes θ from $\mathrm{ob}_{\mathbf{C}}$ associating to every object a of $\mathbf{C}$ a member θ_a of $\mathbf{D}_{\mathbf{G}(a)}^{\mathbf{F}(a)}$ such that, for every ordered pair (a, b) of objects of $\mathbf{C}$ and for every member u of $\mathbf{C}_b^a$, the $\mathbf{D}$-composite $\theta_b \circ F_b^a(u)$ equals the $\mathbf{D}$-composite $G_b^a(u) \circ \theta_a$.

A natural transformation is a **natural equivalence** (or **natural isomorphism**) if (and only if) the morphism associated to each object is an isomorphism. In particular, for every category $\mathbf{C}$, for every category $\mathbf{D}$, and for every covariant functor $\mathbf{F}$ from $\mathbf{C}$ to $\mathbf{D}$, the **identity natural transformation** from $\mathbf{F}$ to itself is the natural transformation that associates to every object a of $\mathbf{C}$ the identity morphism $\mathrm{Id}_{\mathbf{F}(a)}^{\mathbf{D}}$. This is denoted by $\mathrm{Id}_{\mathbf{F}}^{\mathbf{C},\mathbf{D}}$, or just $\mathrm{Id}_{\mathbf{F}}$ when confusion is unlikely.

For every category $\mathbf{C}$, for every category $\mathbf{D}$, for every ordered triple $(\mathbf{F}, \mathbf{G}, \mathbf{H})$ of covariant functors from $\mathbf{C}$ to $\mathbf{D}$, for every natural transformation θ from $\mathbf{F}$ to $\mathbf{G}$, for every natural transformation η from $\mathbf{G}$ to $\mathbf{H}$, the (vertical) **composite natural transformation** $\eta \circ \theta$ from $\mathbf{F}$ to $\mathbf{H}$ is the natural transformation that associates to every object a of $\mathbf{C}$ the composite morphism $\eta_a \circ \theta_a$ from $\mathbf{F}(a)$ to $\mathbf{H}(a)$.

For every category $\mathbf{C}$, for every category $\mathbf{D}$, for every category $\mathbf{E}$, for every ordered pair $(\mathbf{F}, \mathbf{G})$ of covariant functors from $\mathbf{C}$ to $\mathbf{D}$, for every ordered pair $(\mathbf{H}, \mathbf{I})$ of covariant functors from $\mathbf{D}$ to $\mathcal{E}$, for every natural transformation θ from $\mathbf{F}$ to $\mathbf{G}$, for every natural transformation η from $\mathbf{H}$ to $\mathbf{I}$, the **horizontal composition natural transformation** of η and θ, sometimes called the **Godement product**, is the natural transformation $\eta * \theta : \mathbf{H} \circ \mathbf{F} \to \mathbf{I} \circ \mathbf{G}$ associating to every object a of $\mathbf{C}$ the $\mathbf{E}$-morphism,

$$\eta_{\mathbf{G}(a)} \circ_{\mathbf{C}} \mathbf{H}_{\mathbf{G}(a)}^{\mathbf{F}(a)}(\theta_a) = (\eta * \theta)_a = \mathbf{I}_{\mathbf{G}(a)}^{\mathbf{F}(a)}(\theta_a) \circ \eta_{\mathbf{F}(a)}.$$

This is associative in both θ and η separately.

For every category $\mathbf{C}$, for every category $\mathbf{D}$, for every category $\mathbf{E}$, for every ordered pair $(\mathbf{F}, \mathbf{G})$ of covariant functors from $\mathbf{C}$ to $\mathbf{D}$, for every covariant functor $\mathbf{H}$ from $\mathbf{D}$ to $\mathbf{E}$, for every natural transformation θ from $\mathbf{F}$ to $\mathbf{G}$, the **H-pushforward natural transformation** is $\mathbf{H}_*\theta = \mathrm{Id}_{\mathbf{H}}^{\mathbf{D},\mathcal{E}} * \theta$, associating to every object a of $\mathbf{A}$ the $\mathbf{E}$-morphism $\mathbf{H}_{\mathbf{G}(a)}^{\mathbf{F}(a)}(\theta_a)$. Similarly, for every category $\mathbf{B}$, for every category $\mathbf{C}$, for every category $\mathbf{D}$, for every covariant functor $\mathbf{J}$ from $\mathbf{B}$ to $\mathbf{C}$, for every ordered pair $(\mathbf{F}, \mathbf{G})$ of covariant functors from $\mathbf{C}$ to $\mathbf{D}$, for every natural transformation θ from $\mathbf{F}$ to $\mathbf{G}$, the **J-pullback natural transformation**, $\mathbf{J}^*\theta = \theta * \mathrm{Id}_E^{\mathcal{B},\mathcal{E}}$ associates to every object b of $\mathbf{B}$ the $\mathbf{D}$-morphism $\theta_{\mathbf{J}(b)}$. Of course the Godement product can be expanded in terms of pushforward, pullback and vertical composition,

$$\mathbf{G}^*\eta \circ \mathbf{H}_*\theta = \eta * \theta = \mathbf{I}_*\theta \circ \mathbf{F}^*\eta.$$

Example 0.57. For every monoid $(H, \bullet)$, for every monoid $(H', \bullet')$, for monoid homomorphisms f and g from $(H, \bullet)$ to $(H', \bullet')$, for the associated functors from the category of $(H, \bullet)$ to the category of $(H', \bullet')$, a natural transformation between these functors is an element h' of H' such that for every element h of H, the composite $h' \bullet' f(h)$ equals $g(h) \circ' h'$. In particular, if g equals f, then h' is in the commutator of the image of f. Similarly, if $(H', \bullet')$ is a group, this is the same thing as an element h' such that g equals the composite $\text{inner}_{h'} \circ f$, where $\text{inner}_{h'}$ is the inner automorphism of $(H', \bullet')$ associated to h'. In particular, the natural endomorphisms of every functor f from the category of a monoid to the category BH' of a group $(H', \bullet')$ are equivalent to elements of H' in the centralizer of the image of f.

Example 0.58. Similarly, for every natural transformation θ between functors $\mathbf{F}$ and $\mathbf{G}$ from a category $\mathbf{C}$ to a category $\mathbf{C}'$, for every object a of $\mathbf{C}$ that maps under both $\mathbf{F}$ and $\mathbf{G}$ to a common object a', the monoid homomorphisms $\mathbf{F}^a_a$ and $\mathbf{G}^a_a$ from $\mathbf{C}^a_a$ to $(\mathbf{C}')^{a'}_{a'}$ are *intertwined* by the element θ_a of $(\mathbf{C}')^{a'}_{a'}$ in the sense that $\theta_a \circ \mathbf{F}^a_a(u)$ equals $\mathbf{G}^a_a(u) \circ \theta_a$ for every element u of $\mathbf{C}^a_a$.

Example 0.59. For every functor $\mathbf{F}$ from a category $\mathbf{C}$ to a category $\mathbf{C}'$, if there exists an initial object e' of $\mathbf{C}'$, then there is a unique natural transformation from the constant functor $\text{const}^{\mathbf{C}}_{\mathbf{C}',e'}$ to $\mathbf{F}$ that associates to every object a of $\mathbf{C}$ the unique $\mathbf{C}'$-morphism from e' to $\mathbf{F}(a)$.

Example 0.60. For every functor $\mathbf{F}$ from a category $\mathbf{C}$ to a category $\mathbf{C}'$, if there exists a final object f' of $\mathbf{C}'$, then there is a unique natural transformation from $\mathbf{F}$ to the constant functor $\text{const}^{\mathbf{C}}_{\mathbf{C}',f'}$ that associates to every object a of $\mathbf{C}$ the unique $\mathbf{C}'$-morphism from $\mathbf{F}(a)$ to f'.

Example 0.61. For every category $\mathbf{C}$, for every preordered set (S', R'), for functors $\mathbf{F}$ and $\mathbf{G}$ from $\mathbf{C}$ to the category of (S, R), i.e., nondecreasing morphisms from $\text{obj}(\mathbf{C})$ to (the class of) S', there exists a natural transformation from $\mathbf{F}$ to $\mathbf{G}$ if and only if $(\mathbf{F}(a), \mathbf{G}(a))$ is an element of R' for every object a of $\mathbf{C}$, and then the natural transformation is unique. Thus, there exists a natural transformation from $\mathbf{F}$ to $\mathbf{G}$ if and only if, valuewise $\mathbf{F}_{\text{mor}}$ is "less than or equal to" $\mathbf{G}_{\text{mor}}$.

Definition 0.62. For every small category $\mathbf{C}$, for every category $\mathbf{D}$, every functor from $\mathbf{C}$ to $\mathbf{D}$ is the class of a set, since both $\text{obj}(\mathbf{C})$ and $\text{mor}(\mathbf{C})$ are classes of sets. Thus, there is a class $\text{obj}(\text{Fun}(\mathbf{C}, \mathbf{D})$, sometimes also denoted $\text{obj}([\mathbf{C}, \mathbf{D}])$ or $\text{obj}(\mathbf{D}^{\mathbf{C}})$, whose members are the sets whose classes give functors from $\mathbf{C}$ to $\mathbf{D}$. For every ordered pair $(\mathbf{F}, \mathbf{G})$ of functors from $\mathbf{C}$ to $\mathbf{D}$, again because $\mathbf{C}$ is small, every natural transformation from $\mathbf{F}$ to $\mathbf{G}$ is the class of a set, and the class of all sets whose classes are natural transformations from $\mathbf{F}$ to $\mathbf{G}$ is itself a set. Thus, there is also a span $\text{mor}(\text{Fun}(\mathbf{C}, \mathbf{D})$, sometimes also denoted $\text{mor}([\mathbf{C}, \mathbf{D}])$ or $\text{mor}(\mathbf{D}^{\mathbf{C}})$, from $\text{obj}(\text{Fun}(\mathbf{C}, \mathbf{D}))$ whose fiber class over each ordered pair $(\mathbf{F}, \mathbf{G})$ is a class of a set with members being sets whose associated class is a natural transformation from $\mathbf{F}$ to $\mathbf{G}$. Finally, composition of natural transformations defines a composition that makes this into a category $\text{Fun}(\mathbf{C}, \mathbf{D})$, sometimes also denote $[\mathbf{C}, \mathbf{D}]$ or $\mathbf{D}^{\mathbf{C}}$.

Exercise 0.63. For every small category $\mathbf{C}$, for every small category $\mathbf{D}$, prove that $\text{Fun}(\mathbf{C}, \mathbf{D})$ is a small category.

Definition 0.64. The class of **small categories** is the class $\text{obj}(\mathbf{Cat})$ whose members are sets whose associated class is a small category. The class of **functors of small categories** is the span $\text{mor}(\mathbf{Cat})$ from $\text{obj}(\mathbf{Cat})$ to itself whose fiber class over each pair $(\mathbf{C}, \mathbf{D})$ has for members those sets whose associated class is a functor from $\mathbf{C}$ to $\mathbf{D}$. Composition of functors defines a composition law that completes these classes to a category $\mathbf{Cat}$, the **category of small categories**.

Exercise 0.65. Read about 2-categories. Formulate and prove the assertion that the natural transformations between functors extend the category of small categories to a 2-category.

Exercise 0.66. Formulate and prove the statement that formation of $\text{Fun}(\mathbf{C}, \mathbf{D})$ is covariant in categories $\mathbf{D}$ and is contravariant in small categories $\mathbf{C}$. In particular, for every small category $\mathbf{C}$, prove that the covariant Yoneda functor of $\mathbf{C}$ in $\mathbf{Cat}$ enriches to a functor from $\mathbf{Cat}$ to itself. Similarly, for every small category $\mathbf{D}$, prove that the contravariant Yoneda functor of $\mathbf{D}$ in $\mathbf{Cat}$ enriches to a functor from $\mathbf{Cat}^{\text{opp}}$ to $\mathbf{Cat}$.

Adjoint Pairs.

Definition 0.67. For every category $\mathbf{C}$, for every category $\mathbf{D}$, an **adjoint pair** of covariant functors between $\mathbf{C}$ and $\mathbf{D}$ is an ordered pair $((\mathbf{L}, \mathbf{R}), (\theta, \eta))$ of an ordered pair pair of covariant functors,

$$\mathbf{L} : \mathbf{C} \to \mathbf{D},$$

$$\mathbf{R} : \mathbf{D} \to \mathbf{C},$$

and a pair of natural transformations of covariant functors,

$$\theta : \text{Id}_{\mathbf{C}} \Rightarrow \mathbf{R} \circ \mathbf{L}, \ \theta(a) : a \to \mathbf{R}(\mathbf{L}(a)),$$

$$\eta : \mathbf{L} \circ \mathbf{R} \Rightarrow \text{Id}_{\mathbf{D}}, \ \eta(b) : \mathbf{L}(\mathbf{R}(b)) \to b,$$

such that the following composition of natural transformations equals $\text{Id}_{\mathbf{R}}$, respectively equals $\text{Id}_{\mathbf{L}}$,

$$(*_{\mathbf{R}}) : \mathbf{R} \overset{\theta \circ \mathbf{R}}{\Rightarrow} \mathbf{R} \circ \mathbf{L} \circ \mathbf{R} \overset{\mathbf{R} \circ \eta}{\Rightarrow} \mathbf{R},$$

$$(*_{\mathbf{L}}) : \mathbf{L} \overset{\mathbf{L} \circ \theta}{\Rightarrow} \mathbf{L} \circ \mathbf{R} \circ \mathbf{L} \overset{\eta \circ \mathbf{L}}{\Rightarrow} \mathbf{L}.$$

For every object a of $\mathbf{C}$ and for every object b of $\mathbf{D}$, define set maps,

$$H^{\mathbf{L}}_{\mathbf{R}}(a, b) : \text{Hom}_{\mathbf{D}}(\mathbf{L}(a), b) \to \text{Hom}_{\mathbf{C}}(a, \mathbf{R}(b)),$$

$$(\mathbf{L}(a) \xrightarrow{\phi} b) \mapsto \left(a \xrightarrow{\theta(a)} \mathbf{R}(\mathbf{L}(a)) \xrightarrow{\mathbf{R}(\phi)} \mathbf{R}(b)\right),$$

and

$$H^{\mathbf{R}}_{\mathbf{L}}(a, b) : \text{Hom}_{\mathbf{C}}(a, \mathbf{R}(b)) \to \text{Hom}_{\mathbf{D}}(\mathbf{L}(a), b),$$

$$(a \xrightarrow{\psi} \mathbf{R}(b)) \mapsto \left(\mathbf{L}(a) \xrightarrow{\mathbf{L}(\psi)} \mathbf{L}(\mathbf{R}(b)) \xrightarrow{\eta(b)} b\right).$$

Exercise 0.68. For $\mathbf{L}$, $\mathbf{R}$, θ and η as above, prove that the conditions $(*_{\mathbf{R}})$ and $(*_{\mathbf{L}})$ hold if and only if, for every object a of $\mathbf{C}$ and for every object b of $\mathbf{D}$, the morphisms $H^{\mathbf{L}}_{\mathbf{R}}(a, b)$ and $H^{\mathbf{R}}_{\mathbf{L}}(a, b)$ are inverse bijections.

Exercise 0.69. Prove that both $H^{\mathbf{L}}_{\mathbf{R}}(a, b)$ and $H^{\mathbf{R}}_{\mathbf{L}}(a, b)$ are binatural in a and b.

Exercise 0.70. For functors $\mathbf{L}$ and $\mathbf{R}$, and for binatural inverse bijections $H^{\mathbf{L}}_{\mathbf{R}}(a,b)$ and $H^{\mathbf{R}}_{\mathbf{L}}(a,b)$ between the bifunctors

$$\mathrm{Hom}_{\mathbf{D}}(\mathbf{L}(a),b), \mathrm{Hom}_{\mathbf{C}}(a,\mathbf{R}(b)) : \mathbf{C}\times\mathbf{D} \to \mathbf{Set},$$

prove that there exist unique θ and η extending $\mathbf{L}$ and $\mathbf{R}$ to an adjoint pair such that $H^{\mathbf{L}}_{\mathbf{R}}$ and $H^{\mathbf{R}}_{\mathbf{L}}$ agree with the binatural inverse bijections defined above.

Exercise 0.71. Let $(\mathbf{L},\mathbf{R},\theta,\eta)$ be an adjoint pair as above. Let a covariant functor

$$\widetilde{\mathbf{R}} : \mathbf{D} \to \mathbf{C},$$

and natural transformations,

$$\widetilde{\theta} : \mathrm{Id}_{\mathbf{C}} \Rightarrow \widetilde{\mathbf{R}}\circ\mathbf{L}, \widetilde{\eta} : \mathbf{L}\circ\widetilde{\mathbf{R}} \Rightarrow \mathrm{Id}_{\mathbf{D}},$$

be natural transformations such that $(\mathbf{L},\widetilde{\mathbf{R}},\widetilde{\theta},\widetilde{\eta})$ is also an adjoint pair. For every object b of $\mathbf{D}$, define $\iota(b)$ in $\mathrm{Hom}_{\mathbf{D}}(\mathbf{R}(b),\widetilde{\mathbf{R}}(b))$ to be the image of Id_b under the composition,

$$\mathrm{Hom}_{\mathbf{D}}(b,b) \xrightarrow{\mathrm{Hom}_{\mathbf{D}}(\theta(b),b)} \mathrm{Hom}_{\mathbf{D}}(\mathbf{L}(\mathbf{R}(b)),b) \xrightarrow{H^{\widetilde{\mathbf{R}}}_{\mathbf{L}}(\mathbf{R}(b),b)} \mathrm{Hom}_{\mathbf{D}}(\mathbf{R}(b),\widetilde{\mathbf{R}}(b)).$$

Similarly, define $\kappa(b)$ in $\mathrm{Hom}_{\mathbf{D}}(\widetilde{\mathbf{R}}(b),\mathbf{R}(b))$, to be the image of Id_b under the composition,

$$\mathrm{Hom}_{\mathbf{D}}(b,b) \xrightarrow{\mathrm{Hom}_{\mathbf{D}}(\widetilde{\theta}(b),b)} \mathrm{Hom}_{\mathbf{D}}(\mathbf{L}(\widetilde{\mathbf{R}}(b)),b) \xrightarrow{H^{\mathbf{R}}_{\mathbf{L}}(\widetilde{\mathbf{R}}(b),b)} \mathrm{Hom}_{\mathbf{D}}(\widetilde{\mathbf{R}}(b),\mathbf{R}(b)).$$

Prove that ι and κ are the unique natural transformations of functors,

$$\iota : \mathbf{R} \Rightarrow \widetilde{\mathbf{R}},\ \ \kappa : \widetilde{\mathbf{R}} \Rightarrow \mathbf{R},$$

such that $\widetilde{\theta}$ equals $(\iota\circ\mathbf{L})\circ\theta$, θ equals $(\kappa\circ\mathbf{L})\circ\widetilde{\theta}$, $\widetilde{\eta}$ equals $\eta\circ(\mathbf{L}\circ\iota)$, and η equals $\widetilde{\eta}\circ(\mathbf{L}\circ\kappa)$. Moreover, prove that ι and κ are inverse natural equivalences. In this sense, every extension of a functor $\mathbf{L}$ to an adjoint pair $(\mathbf{L},\mathbf{R},\theta,\eta)$ is unique up to unique natural isomorphisms (ι,κ). Formulate and prove the symmetric statement for all extensions of a functor $\mathbf{R}$ to an adjoint pair $(\mathbf{L},\mathbf{R},\theta,\eta)$ (you could use opposite categories to simplify this).

Exercise 0.72. For every adjoint pair $(\mathbf{L},\mathbf{R},\theta,\eta)$, prove that also $(\mathbf{R}^{\mathrm{opp}},\mathbf{L}^{\mathrm{opp}},\eta^{\mathrm{opp}},\theta^{\mathrm{opp}})$ is an adjoint pair.

Exercise 0.73. Formulate the corresponding notions of adjoint pairs when $\mathbf{L}$ and $\mathbf{R}$ are contravariant functors (just replace one of the categories by its opposite category).

Exercise 0.74. For every ordered triple of categories, $(\mathbf{C},\mathbf{D},\mathcal{E})$ for all covariant functors,

$$\mathbf{L}' : \mathbf{C} \to \mathbf{D}$$

$$\mathbf{R}' : \mathbf{D} \to \mathbf{C},$$

for all natural transformations that form an adjoint pair,

$$\theta' : \mathrm{Id}_{\mathbf{C}} \Rightarrow \mathbf{R}'\mathbf{L}',$$

$$\eta' : \mathbf{L}'\mathbf{R}' \Rightarrow \mathrm{Id}_{\mathbf{D}},$$

for all covariant functors,

$$\mathbf{L}'' : \mathbf{D} \to \mathcal{E},$$
$$\mathbf{R}'' : \mathcal{E} \to \mathbf{D},$$

and for all natural transformations that form an adjoint pair,

$$\theta'' : \mathrm{Id}_{\mathbf{D}} \Rightarrow \mathbf{R}''\mathbf{L}'',$$

$$\eta'' : \mathbf{L}''\mathbf{R}'' \Rightarrow \mathrm{Id}_{\mathcal{E}},$$

define covariant functors

$$\mathbf{L} : \mathbf{C} \to \mathcal{E},\ \ \mathbf{R} : \mathcal{E} \to \mathbf{C}$$

by $\mathbf{L} = \mathbf{L}'' \circ \mathbf{L}'$, $\mathbf{R} = \mathbf{R}' \circ \mathbf{R}''$, define the natural transformation,

$$\theta : \mathrm{Id}_{\mathbf{C}} \Rightarrow \mathbf{R} \circ \mathbf{L},$$

to be the composition of natural transformations,

$$\mathrm{Id}_{\mathbf{C}} \overset{\theta'}{\Rightarrow} \mathbf{R}' \circ \mathbf{L}' \overset{\mathbf{R}'\circ\theta''\circ\mathbf{L}'}{\Rightarrow} \mathbf{R}' \circ \mathbf{R}'' \circ \mathbf{L}'' \circ \mathbf{L}',$$

and define the natural transformation,

$$\eta : \mathbf{L} \circ \mathbf{R} \Rightarrow \mathrm{Id}_{\mathcal{E}},$$

to be the composition of natural transformations,

$$\mathbf{L}'' \circ \mathbf{L}' \circ \mathbf{R}' \circ \mathbf{R}'' \overset{\mathbf{L}''\circ\eta'\circ\mathbf{R}''}{\Rightarrow} \mathbf{L}'' \circ \mathbf{R}'' \overset{\eta''}{\Rightarrow} \mathrm{Id}_{\mathcal{E}}.$$

Prove that $\mathbf{L}$, $\mathbf{R}$, θ and η form an adjoint pair of functors. This is the **composition** of $(\mathbf{L}', \mathbf{R}', \theta', \eta')$ and $(\mathbf{L}'', \mathbf{R}'', \theta'', \eta'')$.

Exercise 0.75. If $\mathbf{C}$ equals $\mathbf{D}$, if $\mathbf{L}'$ and $\mathbf{R}'$ are the identity functors, and if θ' and η' are the identity natural transformations, prove that $(\mathbf{L}, \mathbf{R}, \theta, \eta)$ equals $(\mathbf{L}'', \mathbf{R}'', \theta'', \eta'')$. Similarly, if $\mathbf{D}$ equals $\mathcal{E}$, if $\mathbf{L}''$ and $\mathbf{R}''$ are the identity functors, and if θ'' and η'' are the identity natural transformations, prove that $(\mathbf{L}, \mathbf{R}, \theta, \eta)$ equals $(\mathbf{L}', \mathbf{R}', \theta', \eta')$. Finally, prove that composition of three adjoint pairs is associative.

Example 0.76. Let $\mathbf{C}$ be a category that has a final object f, and let $\mathbf{D}$ be a category that has an initial object e. Let $\mathbf{L}$ be $\mathrm{const}^{\mathbf{C}}_{\mathbf{D},e}$, and let $\mathbf{R}$ be $\mathrm{const}^{\mathbf{D}}_{\mathbf{C},f}$. Thus, $\mathbf{R} \circ \mathbf{L}$ equals $\mathrm{const}^{\mathbf{C}}_{\mathbf{C},f}$, and $\mathbf{L} \circ \mathbf{R}$ equals $\mathrm{const}^{\mathbf{D}}_{\mathbf{D},e}$. Since f is a final object of $\mathbf{C}$, there is a unique natural transformation from every endofunctor of $\mathbf{C}$ to $\mathrm{const}^{\mathbf{C}}_{\mathbf{C},f}$. In particular, there exists a unique natural transformation θ from the identity functor to $\mathrm{const}^{\mathbf{C}}_{\mathbf{C},f}$. Since e is an initial object of $\mathbf{C}$, there is a unique natural transformation from $\mathrm{const}^{\mathbf{D}}_{\mathbf{D},e}$ to every endofunctor of $\mathbf{D}$. In particular, there exists a unique natural transformation η from $\mathrm{const}^{\mathbf{C}}_{\mathbf{C},f}$ to the identity functor. Together, these define an adjoint pair giving binatural bijections for every object a of $\mathbf{C}$ and every object b of $\mathbf{D}$,

$$\mathrm{Hom}_{\mathbf{D}}(\mathrm{const}^{\mathbf{C}}_{\mathbf{D},e}(a), b) \cong \mathrm{Hom}_{\mathbf{C}}(a, \mathrm{const}^{\mathbf{D}}_{\mathbf{C},f}(b)).$$

Example 0.77. Let $(S, \leq)$ and $(S', \leq')$ be partially ordered sets. Let $\mathbf{L}$ be a nondecreasing function from $(S, \leq)$ to $(S', \leq')$ considered as a functor between the associated categories. Let $\mathbf{R}$ be a nondecreasing function from $(S', \leq')$ to $(S, \leq)$ considered as a functor between the associated categories. There exist natural transformations completing this to an adjoint pair if and only if, for every element a of S, for every element a' of S', we have $L(a) \leq' a'$ if and only $a \leq R(a')$. In this case, the natural transformations extending to an adjoint pair are unique.

Definition 0.78. For every category $\mathbf{C}$, for every category $\mathbf{D}$, for every adjoint pair

$$(\mathbf{L} : \mathbf{C} \to \mathbf{D}, \mathbf{R} : \mathbf{D} \to \mathbf{C}, \theta : \mathrm{Id}_{\mathbf{C}} \Rightarrow \mathbf{R} \circ \mathbf{L}, \eta : \mathbf{L} \circ \mathbf{R} \Rightarrow \mathrm{Id}_{\mathbf{D}}),$$

the adjoint pair is a **strict equivalence** from $\mathbf{C}$ to $\mathbf{D}$ if (and only if) both θ is a natural equivalence and η is a natural equivalence.

Exercise 0.79. Prove that identity adjoint pairs are strict equivalences. Prove that the composition adjoint pair of strict equivalences is a strict equivalence. For every strict equivalence from $\mathbf{C}$ to $\mathbf{D}$ as above, prove that also $(\mathbf{R}, \mathbf{L}, \eta^{-1}, \theta^{-1})$ is a strict equivalence from $\mathbf{D}$ to $\mathbf{C}$ that is a left-right inverse of the original strict equivalence.

Exercise 0.80. Prove that each of the functors in a strict equivalence is a weak equivalence. Prove that every composition of weak equivalences is a weak equivalence.

Exercise 0.81. Let $\mathbf{C}$ and $\mathbf{D}$ be strictly small categories. Prove that for every weak equivalence L from $\mathbf{C}$ to $\mathbf{D}$ there exists a strict equivalence (L, R, θ, η) from $\mathbf{C}$ to $\mathbf{D}$, and this strict equivalence is unique up to isomorphism (which is not necessarily unique). Thus, using Hilbert's formulation of the Axiom of Choice, using the Axiom of Choice in von Neumann – Bernays – Gödel theory, or using large cardinal axioms / Grothendieck universes, every weak equivalence should arise (non-uniquely) from a strict equivalence.

The Yoneda Embedding.

Definition 0.82. For every category $\mathbf{C}$, for every object a of $\mathbf{C}$, the set-valued **covariant Yoneda functor** of a from $\mathbf{C}$ maps every $\mathbf{C}$-object b to the set $\mathbf{C}^a_b = \mathrm{Hom}_{\mathbf{C}}(a, b)$. This is also denoted $h^a_{\mathbf{C}}(b)$, or just $h^a(b)$ when confusion is unlikely. Also, for every $\mathbf{C}$-morphism v from b to b', the functor maps u to left-composition with v from $\mathbf{C}^a_b$ to $\mathbf{C}^a_{b'}$. This is denoted $h^a(v)$. Similarly, for every set S, the set-valued functor $S \times h^a$ maps every $\mathbf{C}$-object b to $S \times h^a(b)$ and maps every $\mathbf{C}$-morphism v to $\mathrm{Id}^{\mathbf{Set}}_S \times h^a(v)$ from $S \times h^a(b)$ to $S \times h^a(b')$.

Similarly, for every object b of $\mathbf{C}$, the set-valued **contravariant Yoneda functor** of b is the covariant functor from $\mathbf{C}^{\text{opp}}$ that maps every $\mathbf{C}$-object a to the set $\mathbf{C}^a_b = \mathrm{Hom}_{\mathbf{C}}(a, b)$. This is also denoted $h_{\mathbf{C},b}(a)$, or just $h_b(a)$ when confusion is unlikely. Also, for every $\mathbf{C}$-morphism u from a to a', the functor map u to the right-composition with u from $\mathbf{C}^{a'}_b$ to $\mathbf{C}^a_b$ (note this is contravariant). This is denoted $h_b(u)$. Similarly, for every set S, the set-valued functor $S \times h_b$ maps every $\mathbf{C}$-object a to $S \times h_b(a)$ and maps every $\mathbf{C}$-morphism u to $\mathrm{Id}^{\mathbf{Set}}_S \times h_b(u)$ from $S \times h_b(a')$ to $S \times h_b(a)$.

Exercise 0.83. Check that each of these does preserve identities and composition, so that it is a functor.

Example 0.84. Let $(S, \leq)$ be a partially ordered set. For every element a of S, for every element b of S, the Yoneda functor $h^a(b)$ is a singleton set if and only if $a \leq b$, and otherwise it is empty, i.e., the image in **Set** is either an initial object or a final object. If we define the **support** of such a function to be the subset of S where the image is not the empty set, then the support of h^a is the subset $S_{\geq a}$ of all elements b with $a \leq b$. Similarly, the support of h_b is the subset $S_{\leq b}$ of all elements of b with $a \leq b$.

Example 0.85. For every monoid $(H, \bullet)$, for the unique object (which, recall, is chosen to be H itself considered as a set), the Yoneda functor h^H associates to the unique object (i.e., H) the set H, and associates to each element a of H, considered as a morphism from the unique object to itself, the associated bijection of H of left-multiplication by a, i.e., h^H is the left regular representation of $(H, \bullet)$. Similarly h_H is the right regular representation of $(H, \bullet)$.

Definition 0.86. For every category $\mathbf{C}$, for every $\mathbf{C}$-morphism u from a to a', the **Yoneda natural transformation of covariant functors** from $h^{a'}$ to h^a associates to every object b the set function of right-composition with u from $h^{a'}(b) = \mathbf{C}^{a'}_b$ to $h^a(b) = \mathbf{C}^a_b$. This natural transformation is denoted by h^u. Similarly, for every set S, the natural transformation $\mathrm{Id}^{\mathbf{Set}}_S \times h^u$ maps every set $S \times h^{a'}(b)$ to $S \times h^a(b)$ by $\mathrm{Id}^{\mathbf{Set}}_S \times h^u(b)$.

For every category $\mathbf{C}$, for every $\mathbf{C}$-morphism v from b to b', the **Yoneda natural transformation of contravariant functors** from h_b to $h_{b'}$ associates to every object a of the set function of left-composition with v from $h_b(a)$ to $h_{b'}(a)$. This natural transformation is denoted by h_v. Similarly, for every set S, the natural transformation $\mathrm{Id}^{\mathbf{Set}}_S \times h_v$ maps every set $S \times h_b(a)$ to $S \times h_{b'}(a)$ by $\mathrm{Id}^{\mathbf{Set}}_S \times h_v(a)$.

Exercise 0.87. Check that each of these is a natural transformation of set-valued functors from $\mathbf{C}$.

Exercise 0.88. For every $\mathbf{C}$-morphism u from a to a', for every $\mathbf{C}$-morphism u' from a' to a'', check that $h^u \circ h^{u'}$ equals $h^{u' \circ u}$; thus, also, $(\mathrm{Id}^{\mathbf{Set}}_S \times h^u) \circ (\mathrm{Id}^{\mathbf{Set}}_s \times h^{u'})$ equals $\mathrm{Id}^{\mathbf{Set}}_S \times h^{u' \circ u}$. Conclude *contravariance* of the assignment to every $\mathbf{C}$-object a of the covariant Yoneda functor h^a and to every $\mathbf{C}$-object u of the Yoneda natural transformation h^u.

Exercise 0.89. For every $\mathbf{C}$-morphism v from b to b', for every $\mathbf{C}$-morphism v' from b' to b'', check that $h_{v'} \circ h_v$ equals $h_{v' \circ v}$; thus, also, $(\mathrm{Id}_S^{\mathbf{Set}} \times h_{v'}) \circ (\mathrm{Id}_S^{\mathbf{Set}} \times h_v)$ equals $\mathrm{Id}_S^{\mathbf{Set}} \times h_{v' \circ v}$. Conclude *covariance* of the assignment to every $\mathbf{C}$-object a of the contravariant Yoneda functor h_a and to every $\mathbf{C}$-object v of the Yoneda natural transformation h_v.

Exercise 0.90. For every set-valued functor $\mathbf{F}$ from $\mathbf{C}$, respectively from $\mathbf{C}^{\text{opp}}$, for every set S, for the set-valued functor $S \times \mathbf{F}$ from $\mathbf{C}$, resp. from $\mathbf{C}^{\text{opp}}$, check covariance in S.

Definition 0.91. For every category $\mathbf{B}$, for every set-valued covariant functor $\mathbf{F}$ from $\mathbf{B}^{\text{opp}}$, for every $\mathbf{C}$-object b, for every element γ of the set $\mathbf{F}(b)$, the **Yoneda evaluation natural transformation** from h_b to $\mathbf{F}$ associates to every $\mathbf{C}$-object a the set-function from $h_b(a) = \mathrm{Hom}_{\mathbf{C}}(a, b)$ to $\mathbf{F}(a)$ sending each element w of $\mathrm{Hom}_{\mathbf{C}}(a, b)$ to the image of γ under the set function $\mathbf{F}(w)$ from $\mathbf{F}(b)$ to $\mathbf{F}(a)$. This natural transformation is denoted by $\eta_b^{\gamma,\bullet}(\mathbf{F})$, so that w maps to $\eta_b^{\gamma,\bullet}(\mathbf{F})(w)$. Similarly, $\eta_b(\mathbf{F})$ is the natural transformation from $\mathbf{F}(b) \times h_b$ to $\mathbf{F}$ that associates to every $\mathbf{C}$-object a the set-function from $\mathbf{F}(b) \times h_b(a)$ to $\mathbf{F}(a)$ sending every element (γ, w) to $\eta_b^{\gamma,\bullet}(\mathbf{F})(w)$.

For every category $\mathbf{B}$, for every set-valued covariant functor $\mathbf{F}$ from $\mathbf{B}$, for every $\mathbf{C}$-object a, for every element δ of the set $\mathbf{F}(a)$, the **Yoneda evaluation natural transformation** from h^a to $\mathbf{F}$ associates to every $\mathbf{C}$-object b the set-function from $h^a(b) = \mathrm{Hom}_{\mathbf{C}}(a, b)$ to $\mathbf{F}(b)$ sending each element w of $\mathrm{Hom}_{\mathbf{C}}(a, b)$ to the image of δ under the set function $\mathbf{F}(w)$ from $\mathbf{F}(a)$ to $\mathbf{F}(b)$. This natural transformation is denoted by $\eta^b_{\delta,\bullet}(\mathbf{F})$, so that w maps to $\eta^a_{\gamma,\bullet}(\mathbf{F})(w)$. Similarly, $\eta^a(\mathbf{F})$ is the natural transformation from $\mathbf{F}(a) \times h^a$ to $\mathbf{F}$ that associates to every $\mathbf{C}$-object b the set-function from $\mathbf{F}(a) \times h^a(b)$ to $\mathbf{F}(b)$ sending every element (δ, w) to $\eta^a_{\gamma,\bullet}(\mathbf{F})(w)$.

Exercise 0.92. Check that $\eta_b(\mathbf{F})$ and $\eta^a(\mathbf{F})$ are natural transformations.

Exercise 0.93. For every natural transformation α from $\mathbf{F}$ to $\mathbf{G}$ of set-valued covariant functors from $\mathbf{C}$, check that $\alpha \circ \eta_b(\mathbf{F})$ equals the composition of $\eta_b(\mathbf{G})$ with the natural transformation of functors $\alpha(b) \times \mathrm{Id}_{h_b}$ from $\mathbf{F}(b) \times h_b$ to $\mathbf{G}(b) \times h_b$ induced by the set function $\alpha(b)$ from $\mathbf{F}(b)$ to $\mathbf{G}(b)$. Thus, $\eta_b(\mathbf{F})$ is "covariant" in $\mathbf{F}$.

Lemma 0.94 (Yoneda Lemma). *For every category $\mathbf{C}$, for every covariant set-valued functor $\mathbf{F}$ from $\mathbf{C}^{opp}$, for every $\mathbf{C}$-object b, every natural transformation Γ from h_b to $\mathbf{F}$ is of the form $\eta_b^{\gamma,\bullet}(\mathbf{F})$ for a unique element γ of $\mathbf{F}(b)$, namely the image under Γ of the element $Id_b^{\mathbf{C}}$ of $h_b(b) = Hom_{\mathbf{C}}(b, b)$.*

Exercise 0.95. Formulate and prove the analogous result for covariant set-valued functors from $\mathbf{C}$ and the Yoneda functors h^a.

Definition 0.96. For every set S, the **identity section** is the set function from S to $S \times h_b(b) = S \times \mathrm{Hom}_{\mathbf{C}}(b, b)$ that pairs each element of S with $\mathrm{Id}_b^{\mathbf{C}}$.

Exercise 0.97. Check that the identity section is covariant in S.

Definition 0.98. For every small category $\mathbf{C}$, for every category $\mathbf{D}$, the **functor category $\mathbf{D}^{\mathbf{C}}$** has as objects the covariant functors from $\mathbf{C}$ to $\mathbf{D}$ (each of these is a class associated to a set since $\mathbf{C}$ is small) and has as morphisms the natural transformation between such functors with the identity natural transformations and composition of natural transformations defined earlier.

Exercise 0.99. Check that these operations do form a category.

Definition 0.100. For every small category $\mathbf{C}$, for every $\mathbf{C}$-object b, the set-valued **left Yoneda functor** $\mathbf{L}_b$ from the functor category $\mathbf{Set}^{(\mathbf{C}^{\text{opp}})}$ associates to every set-valued covariant functor $\mathbf{F}$ from $\mathbf{C}^{\text{opp}}$ the set $\mathbf{F}(b)$ and associates to every natural transformation α from $\mathbf{F}$ to $\mathbf{G}$ the set function $\alpha(b)$ from $\mathbf{F}(b)$ to $\mathbf{G}(b)$.

Similarly, the **right Yoneda functor** $\mathbf{R}_b$ from $\mathbf{Set}$ to $\mathbf{Set}^{\mathbf{C}^{\text{opp}}}$ associates to every set S the covariant set-valued functor $S \times h_b$ from $\mathbf{C}^{\text{opp}}$, and associates to every set function f from S to S' the natural transformation $f \times \text{Id}_{h_b}$ from $S \times h_b$ to $S' \times h_b$.

Exercise 0.101. Check that each of these is a functor. Check that the identity section is a natural transformation from the identity functor of $\mathbf{Set}$ to the composite functor $\mathbf{R}_b \circ \mathbf{L}_b$.

Lemma 0.102 (Yoneda Lemma II)**.** *For every small category $\mathbf{C}$, for every $\mathbf{C}$-object b, the left Yoneda functor and the right Yoneda functor extend to an adjoint pair of functors using the natural transformation η_b above and the identity section natural transformation.*

Exercise 0.103. For every small category $\mathbf{C}$, conclude that the Yoneda functor from $\mathbf{C}$ to $\mathbf{Set}^{(\mathbf{C}^{\text{opp}}}$ sending every $\mathbf{C}$-object b to h_b is a fully faithful embedding of categories.

Limits and Colimits

Mostly in this course we use the special cases of products and coproducts. The notation here is meant to emphasize the connection with operations on presheaves and sheaves such as formation of global sections, stalks, pushforward and inverse image.

Definition 0.104. For every small category τ, for every category $\mathcal{C}$, a τ-**family** in $\mathcal{C}$ is a (covariant) functor,

$$\mathcal{F} : \tau \to \mathcal{C}.$$

Precisely, this associates a $\mathcal{C}$-object $\mathcal{F}(U)$ to every τ-object U, and this associates a $\mathcal{C}$-morphism $\mathcal{F}(r) : \mathcal{F}(U) \to \mathcal{F}(V)$ to every τ-morphism $r : U \to V$. Also, $\mathcal{F}(\text{Id}_U)$ equals $\text{Id}_{\mathcal{F}(U)}$. Finally, for every pair of morphisms of τ, $r : U \to V$ and $s : V \to W$, $\mathcal{F}(s) \circ \mathcal{F}(r)$ equals $\mathcal{F}(s \circ r)$.

Definition 0.105. For every small category τ, for every category $\mathcal{C}$, for every ordered pair $(\mathcal{F}, \mathcal{G})$ of τ-families in $\mathcal{C}$, a **morphism** of τ-families from $\mathcal{F}$ to $\mathcal{G}$ is a natural transformation of functors, $\phi : \mathcal{F} \Rightarrow \mathcal{G}$.

Notation 0.106. For every small category τ, for every category $\mathcal{C}$, for every object a of $\mathcal{C}$, denote by

$$\underline{a}_\tau : \tau \to \mathcal{C}$$

the constant functor $\text{const}^\tau_{\mathcal{C},a}$ that sends every object to a and that sends every morphism to Id_a. For every morphism in $\mathcal{C}$, $p : a \to b$, denote by

$$\underline{p}_\tau : \underline{a}_\tau \Rightarrow \underline{b}_\tau$$

the natural transformation that assigns to every object U of τ the morphism $p : a \to b$. Finally, for every object U of τ, denote

$$\Gamma(U, \mathcal{F}) = \mathcal{F}(U), \quad \Gamma(U, \theta) = \theta(U),$$

and for every morphism $r : U \to V$ of τ, denote

$$\Gamma(r, \mathcal{F}) = \mathcal{F}(r).$$

Functor Categories and Section Functors.

Definition 0.107. For every small category τ, for every category $\mathcal{C}$, for all τ-families $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$, and for all morphisms of τ-families, $\theta : \mathcal{F} \to \mathcal{G}$ and $\eta : \mathcal{G} \to \mathcal{H}$, the **composition** of θ and η is the composite natural transformation $\eta \circ \theta : \mathcal{F} \to \mathcal{H}$.

Recall that associated to the small category τ and the category $\mathcal{C}$ there is the functor category $\text{Fun}(\tau, \mathcal{C})$ whose objects are functors and whose morphisms are natural transformations

Exercise 0.108. Prove that

$$\underline{*}_\tau : \mathcal{C} \to \text{Fun}(\tau, \mathcal{C}), \ a \mapsto \underline{a}_\tau, \ p \mapsto \underline{p}_\tau,$$

is a functor that preserves monomorphisms, epimorphisms and isomorphisms. Moreover, for every object U of τ, also

$$\Gamma(U, -) : \text{Fun}(\tau, \mathcal{C}) \to \mathcal{C}, \ \mathcal{F} \mapsto \Gamma(U, \mathcal{F}), \ \theta \mapsto \Gamma(U, \theta),$$

is a functor. Finally, for every morphism $r : U \to V$ of τ, the morphisms $\Gamma(r, -)$ define a natural transformation $\Gamma(U, -) \Rightarrow \Gamma(V, -)$.

Adjointness of Constant / Diagonal Functors and the Global Sections Functor.

Exercise 0.109. For every small category τ, for every category $\mathcal{C}$, if $\mathcal{C}$ has an initial object X, prove that $(\underline{*}_\tau, \Gamma(X, -))$ extends to an adjoint pair of functors.

Definition 0.110. For every small category τ, for every category $\mathcal{C}$, for every τ-family $\mathcal{F}$ in $\mathcal{C}$, a **limit** of the τ-family $\mathcal{F}$ is a natural transformation $\eta : \underline{a}_\tau \Rightarrow \mathcal{F}$ that is final among all such natural transformations, i.e., for every natural transformation $\theta : \underline{b}_\tau \Rightarrow \mathcal{F}$, there exists a unique morphism $t : b \to a$ in $\mathcal{C}$ such that θ equals $\eta \circ \underline{t}_\tau$.

Exercise 0.111. For every small category τ, for every category $\mathcal{C}$, for all τ-families $\mathcal{F}$ and $\mathcal{G}$ in $\mathcal{C}$, for every morphism ϕ of τ-families from $\mathcal{F}$ to $\mathcal{G}$, for all limits $\eta : \underline{a}_\tau \Rightarrow \mathcal{F}$ and $\theta : \underline{b}_\tau \Rightarrow \mathcal{G}$, prove that there exists a unique morphism $f : a \to b$ such that $\theta \circ \underline{p}_\tau$ equals $\phi \circ \eta$. In particular, prove that if a limit of $\mathcal{F}$ exists, then it is unique up to unique isomorphism. Thus, for every object a of $\mathcal{C}$, the identity transformation $\theta_a : \underline{a}_\tau \to \underline{a}_\tau$ is a limit of $\underline{a}_\tau$.

Adjointness of Constant / Diagonal Functors and Limits.

Definition 0.112. A category $\mathcal{C}$ is **complete** if (and only if), for every small category τ, every τ-family has a limit (which is then unique up to unique isomorphism by the previous exercise).

For every complete category $\mathcal{C}$, some version of the Axiom of Choice (e.g., Hilbert's epsilon operator) produces a rule Γ_τ that assigns to every τ-family $\mathcal{F}$ an object $\Gamma_\tau(\mathcal{F})$ and a natural transformation $\eta_{\mathcal{F}} : \underline{\Gamma_\tau(\mathcal{F})}_\tau \to \mathcal{F}$ that is a limit. (In many concrete categories, there is an explicit "construction" of such a rule.)

Exercise 0.113. For every small category τ, for every complete category $\mathcal{C}$, and for every rule Γ_τ as above, prove that there is an extension to a functor,

$$\Gamma_\tau : \mathrm{Fun}(\tau, \mathcal{C}) \to \mathcal{C},$$

and a natural transformation of functors

$$\eta : \underline{*}_\tau \circ \Gamma_\tau \Rightarrow \mathrm{Id}_{\mathrm{Fun}(\tau,\mathcal{C})}.$$

Moreover, the rule sending every object a of $\mathcal{C}$ to the identity natural transformation θ_a is a natural transformation $\theta : \mathrm{Id}_{\mathcal{C}} \Rightarrow \Gamma_\tau \circ \underline{*}_\tau$. The quadruple $(\underline{*}_\tau, \Gamma, \theta, \eta)$ is an adjoint pair of functors. In particular, the limit functor Γ_τ preserves monomorphisms and sends injective objects of $\mathrm{Fun}(\tau, \mathcal{C})$ to injective objects of $\mathcal{C}$.

Adjointness of Colimits and Constant / Diagonal Functors.

Exercise 0.114. For every small category τ, for every category $\mathcal{C}$, if $\mathcal{C}$ has a final object O, prove that $(\Gamma(O, -), \underline{*}_\tau)$ extends to an adjoint pair of functors.

Definition 0.115. For every small category τ, for every category $\mathcal{C}$, for every τ-family $\mathcal{F}$ in $\mathcal{C}$, a **colimit** of the τ-family $\mathcal{F}$ is a natural transformation $\theta : \mathcal{F} \Rightarrow \underline{a}_\tau$ that is final among all such natural transformations, i.e., for every natural transformation $\eta : \mathcal{F} \Rightarrow \underline{b}_\tau$, there exists a unique morphism $h : a \to b$ in $\mathcal{C}$ such that $\underline{h}_\tau \circ \theta$ equals η.

Exercise 0.116. For every small category τ, for every category $\mathcal{C}$, for all τ-families $\mathcal{F}$ and $\mathcal{G}$ in $\mathcal{C}$, for every morphism ϕ of τ-families from $\mathcal{F}$ to $\mathcal{G}$, for all colimits $\theta : \mathcal{F} \Rightarrow \underline{a}_\tau$ and $\eta : \mathcal{G} \Rightarrow \underline{b}_\tau$, prove that there exists a unique morphism $f : a \to b$ such that $\underline{f}_\tau \circ \theta$ equals $\eta \circ \phi$. In particular, prove that if a colimit of $\mathcal{F}$ exists, then it is unique up to unique isomorphism. Thus, for every object a of $\mathcal{C}$, the identity transformation $\theta_a : \underline{a}_\tau \to \underline{a}_\tau$ is a colimit of $\underline{a}_\tau$. Finally, repeat the previous results with colimits in place of limits. Deduce that colimits (if they exist) preserve epimorphisms and projective objects. (You can use opposite categories to reduce most of this to the case of limits.)

Functoriality in the Source.

Definition 0.117. For every complete category $\mathcal{C}$, for every functor x from a small category σ to a small category τ, for every τ-family $\mathcal{F}$, the x-**pullback** $\mathcal{F}_x$ of $\mathcal{F}$ is the composite functor $\mathcal{F} \circ x$, which is a σ-family. For every morphism of τ-families, say ϕ from $\mathcal{F}$ to $\mathcal{G}$, the x-**pullback** ϕ_x from the σ-family $\mathcal{F}_x$ to $\mathcal{G}_x$ is $\phi \circ x$, which is a morphism of σ-families.

Exercise 0.118. For every complete category $\mathcal{C}$, for every functor x from a small category σ to a small category τ, prove that x-pullback defines a functor

$$*_x : \mathrm{Fun}(\tau, \mathcal{C}) \to \mathrm{Fun}(\sigma, \mathcal{C}).$$

For the identity functor $\mathrm{Id}_\tau : \tau \to \tau$, prove that Id_τ-pullback is the identity functor from $\mathrm{Fun}(\tau, \mathcal{C})$ to itself. For every functor y from a small category ρ to σ, prove that $x \circ y$-pullback equals the composite $*_y \circ *_x$. In this sense, deduce that pullback is contravariant in x.

Definition 0.119. For every complete category $\mathcal{C}$, for every small category σ, for every small category τ, for all functors x and x' from σ to τ, and for every natural transformation n from x to x', the **associated morphism** of σ-families is the natural transformation $\mathcal{F}_n$ from $\mathcal{F}_x$ to $\mathcal{F}_{x'}$ that sends every σ-object V to the morphism $\mathcal{F}(n(V))$ from $\mathcal{F}(x(V))$ to $\mathcal{F}(x'(V))$.

Exercise 0.120. Prove that $\mathcal{F}_n$ is a morphism of σ-families. Also, for every morphism of τ-families, ϕ from $\mathcal{F}$ to $\mathcal{G}$, prove that $\phi_{x'} \circ \mathcal{F}_n$ equals $\mathcal{G}_n \circ \phi_x$. Thus, the operation $*_n$ is a natural transformation from the functor $*_x$ to $*_{x'}$. For the identity natural transformation Id_x from x to itself, also $*_{\mathrm{Id}_x}$ is the identity natural transformation of $*_x$. Finally, for every functor x'' from σ to τ, and for every natural transformation m from x' to x'', the morphism of σ-families $\mathcal{F}_{m \circ n}$ equals $\mathcal{F}_m \circ \mathcal{F}_n$. In this sense, the operation $*_x$ is also compatible with natural transformations. In particular, if (x, y, θ, η) is an adjoint pair of functors, then also $(*_y, *_x, *_\theta, *_\eta)$ is an adjoint pair of functors.

Fiber Categories The following notion of *fiber category* is a special case of the notion of *2-fiber product* of functors of categories. Let $x : \sigma \to \tau$ be a functor; this is also called a *category over* τ. For every object U of τ, a $\sigma_{x,U}$*-object* is a pair $(V, r : x(V) \to U)$ of an object V of σ and a τ-isomorphism $r : x(V) \to U$. For two objects $\sigma_{x,U}$-objects (V, r) and (V', r') of $\sigma_{x,U}$, a $\sigma_{x,U}$-*morphism* from (V, r) to (V', r') is a morphism of σ, $s : V \to V'$, such that $r' \circ x(s)$ equals r. **Prove** that Id_V is a $\sigma_{x,U}$-morphism from (V, r) to itself; more generally, the $\sigma_{x,U}$-morphisms from (V, r) to (V, r) are precisely the σ-morphisms $s : V \to V$ such that $x(s)$ equals $\mathrm{Id}_{x(V)}$. For every pair of $\sigma_{x,U}$-morphisms, $s : (V, r) \to (V', r')$ and $s' : (V', r') \to (V'', r'')$, **prove** that $s' \circ s$ is a $\sigma_{x,U}$-morphism from (V, r) to (V'', r''). Conclude that these rules form a category, denoted $\sigma_{x,U}$. **Prove** that the rule $(V, r) \mapsto V$ and $s \mapsto s$ defines a faithful functor,

$$\Phi_{x,U} : \sigma_{x,U} \to \sigma,$$

and $r : x(V) \to U$ defines a natural isomorphism $\theta_{x,U} : x \circ \Phi_{x,U} \Rightarrow \underline{U}_{\sigma_{x,U}}$. Finally, for every category σ', for every functor $\Phi' : \sigma' \to \sigma$, and for every natural isomorphism $\theta' : x \circ \Phi' \Rightarrow \underline{U}_{\sigma'}$, **prove** that there exists a unique functor $F : \sigma' \to \sigma_{x,U}$ such that Φ' equals $\Phi_{x,U} \circ F$ and θ' equals $\theta_{x,U} \circ F$. In this sense, $(\Phi_{x,U}, \theta_{x,U})$ is final among pairs (Φ', θ') as above.

For every pair of functors $x, x_1 : \sigma \to \tau$, and for every natural *isomorphism* $n : x \Rightarrow x_1$, for every $\sigma_{x_1,U}$-object $(V, r_1 : x_1(V) \to U)$, **prove** that $(V, r_1 \circ n_V : x(V) \to U)$ is an object of $\sigma_{x,U}$. For every morphism in $\sigma_{x_1,U}$, $s : (V, r_1) \to (V', r_1')$, **prove** that s is also a morphism $(V, r_1 \circ n_V) \to (V', r_1' \circ n_{V'})$. Conclude that these rules define a functor,

$$\sigma_{n,U} : \sigma_{x_1,U} \to \sigma_{x,U}.$$

Prove that this functor is a *strict equivalence* of categories: it is a bijection on Hom sets (as for all equivalences), but it is also a bijection on objects (rather than merely being essentially surjective). **Prove** that $\sigma_{n,U}$ is functorial in n, i.e., for a second natural isomorphism $m : x_1 \Rightarrow x_2$, prove that $\sigma_{m \circ n,U}$ equals $\sigma_{n,U} \circ \sigma_{m,U}$.

For every pair of functors, $x : \sigma \to \tau$ and $y : \rho \to \tau$, and for every functor $z : \sigma \to \rho$ such that x equals $y \circ z$ equals x, for every $\sigma_{x,U}$-object (V, r), **prove** that $(z(V), r)$ is a $\rho_{y,U}$-object. For every $\sigma_{x,U}$-morphism $s : (V, r) \to (V', r')$, **prove** that $z(s)$ is a $\rho_{y,U}$-morphism $(z(V), r) \to (z(V'), r')$. **Prove** that $z(\mathrm{Id}_V)$ equals $\mathrm{Id}_{z(V)}$, and **prove** that z preserves composition. Conclude that these rules define a functor,

$$z_U : \sigma_{x,U} \to \rho_{y,U}.$$

Prove that this is functorial in z: $(\mathrm{Id}_\sigma)_U$ equals $\mathrm{Id}_{\sigma_{x,U}}$, and for a third functor $w : \pi \to \tau$ and functor $z' : \rho \to \pi$ such that y equals $w \circ z'$, then $(z' \circ z)_U$ equals $z'_U \circ z_U$. For an object (W, r_W) of $\rho_{y,U}$, for each object $((V, r_V), q : Z(V) \to W)$ of $(\sigma_{x,U})_{z,(W,r_W)}$, define the *associated* object of $\sigma_{z,W}$ to be (V, q). For an object $((V', r_{V'}), q' : Z(V') \to W)$ of $(\sigma_{x,U})_{z,(W,r_W)}$, for every morphism $s : (V, r_V) \to (V', r_{V'})$ such that q equals $q' \circ z(s)$, define the *associated* morphism of $\sigma_{z,W}$ to be s. **Prove** that this defines a functor

$$\widetilde{z}_{U,(W,r_W)} : (\sigma_{x,U})_{z_U,(W,r_W)} \to \sigma_{z,W}.$$

Prove that this functor is a strict equivalence of categories. **Prove** that this equivalence is functorial in z. Finally, for two functors $z, z_1 : \sigma \to \rho$ such that x equals both $y \circ z$ and $y \circ z_1$, and for a natural transformation $m : z \Rightarrow z_1$, for every object $(V, r : x(V) \to U)$ of $\sigma_{x,U}$, **prove** that m_V is a morphism in $\rho_{y,U}$ from $(z(V), r)$ to $(z_1(V), r)$. Moreover, for every morphism in $\sigma_{x,U}$, $s : (V, r) \to (V', r')$, **prove** that $m_{V'} \circ z(s)$ equals $z_1(s) \circ m_V$. Conclude that this rule is a natural transformation $m_U : z_U \Rightarrow (z_1)_U$. **Prove** that this is functorial in m. If m is a natural isomorphism, **prove** that also m_U is a natural isomorphism, and the strict equivalence $(m_U)_{(W,r_W)}$ is compatible with the strict equivalence m_W. Finally, **prove** that $m \mapsto m_U$ is compatible with precomposition and postcomposition of m with functors of categories over τ.

(vii)(Colimits and Limits along an Essentially Surjective Functor) Let $x : \sigma \to \tau$ be a functor of small categories. **Prove** that every fiber category $\sigma_{x,U}$ is small. Next, assume that x is *essentially surjective*, i.e., for every object U of τ, there exists a $\sigma_{x,U}$-object (V, r). Let $y : \tau \to \sigma$ be a functor, and let $\alpha : \mathrm{Id}_\sigma \Rightarrow y \circ x$ be a natural transformation. **Prove** that this extends to an adjoint pair of functors (x, y, α, β) if and only if for every object V of σ, the morphism $x(\alpha_V) : x(V) \to x(y(x(V)))$ is an isomorphism and $(y(x(V)), x(\alpha_V)^{-1})$ is a final object of the fiber category $\sigma_{x,x(V)}$. (Conversely, up to some form of the Axiom of Choice, there exists y and α extending to an adjoint pair if and only if every fiber category $\sigma_{x,U}$ has a final object.) For every adjoint pair (x, y, α, β), also $(*_y, *_x, *_\alpha, *_\beta)$ is an adjoint pair. More generally, no longer assume that there exists y and α, yet let L_x be a rule that assigns to every object $\mathcal{F}$ of $\mathbf{Fun}(\sigma, \mathcal{C})$ an object $L_x(\mathcal{F})$ of $\mathbf{Fun}(\tau, \mathcal{C})$ and a natural transformation,

$$\theta_{\mathcal{F}} : \mathcal{F} \to *_x \circ L_x(\mathcal{F}),$$

of objects in $\mathbf{Fun}(\sigma, \mathcal{C})$. For every object U of τ, this defines a natural transformation

$$\theta_{\mathcal{F},x,U} : \mathcal{F} \circ \Phi_{x,U} \Rightarrow L_x(\mathcal{F}) \circ \underline{U}_{\sigma_{x,U}},$$

of objects in $\mathbf{Fun}(\sigma_{x,U}, \mathcal{C})$. Assume that each $(L_x(\mathcal{F})(U), \theta_{\mathcal{F},x,U})$ is a colimit of $\mathcal{F} \circ \Phi_{x,U}$. **Prove** that this extends uniquely to a functor,

$$L_x : \mathbf{Fun}(\sigma, \mathcal{C}) \to \mathbf{Fun}(\tau, \mathcal{C}),$$

and a natural transformation

$$\theta_x : \mathrm{Id}_{\mathbf{Fun}(\sigma,\mathcal{C})} \Rightarrow *_x \circ L_x.$$

Moreover, for every $\mathcal{G}$ in $\mathbf{Fun}(\tau, \mathcal{C})$, the identity morphism,

$$\mathrm{Id}_{\mathcal{G}} : \mathcal{G} \circ x \circ \Phi_{x,U} \to \mathcal{G} \circ \underline{U}_{\sigma_{x,U}},$$

factors uniquely through a $\mathcal{C}$-morphism $L_x(\mathcal{G} \circ x)(U) \to \mathcal{G}(U)$. **Prove** that this defines a morphism $\eta_{\mathcal{G}} : L_x(\mathcal{G} \circ x) \to \mathcal{G}$ in $\mathbf{Fun}(\tau, \mathcal{C})$. **Prove** that is a natural transformation,

$$\eta : L_x \circ *_x \Rightarrow \mathrm{Id}_{\mathbf{Fun}(\tau,\mathcal{C})}.$$

Prove that $(L_x, *_x, \theta, \eta)$ is an adjoint pair of functors. (Using some version of the Axiom of Choice, if every $\mathcal{F} \circ \Phi_{x,U}$ admits a colimit, then there exists a Γ^x and θ as above.)

Next, as above, let $x : \sigma \to \tau$ be a functor of small catgories that is essentially surjective. Let $y : \tau \to sigma$ be a functor, and let $\beta : y \circ x \Rightarrow \mathrm{Id}_\sigma$ be a natural transformation. **Prove** that this extends to an adjoint pair of functors (x, y, α, β) if and only if for every object V of σ, the morphism $x(\beta_v) : x(y(x(V))) \to x(V)$ is an isomorphism and $(y(x(V)), x(\beta_v))$ is an initial object of the fiber category $\sigma_{x,x(V)}$. (Conversely, up to some form of the Axiom of Choice, there exists y and β extending to an adjoint pair if and only if every fiber category $\sigma_{x,U}$ has an initial object.) For every adjoint pair $(y, x, \alpha, beta)$ also $(*_x, *_y, *_\alpha, *_\beta)$ is an adjoint pair. More generally, no longer assume that there exists y and β, yet let R_x be a rule that assigns to every object $\mathcal{F}$ of $\mathbf{Fun}(\sigma, \mathcal{C})$ an object $R_x(\mathcal{F})$ of $\mathbf{Fun}(\tau, \mathcal{C})$ and a natural transformation,

$$\eta_{\mathcal{F}} : *_x \circ R_x(\mathcal{F}) \to \mathcal{F},$$

of objects in $\mathbf{Fun}(\sigma, \mathcal{C})$. For every object U of τ, this defines a natural transformation

$$\eta_{\mathcal{F},x,U} : R_x(\mathcal{F}) \circ \underline{U}_{\sigma_{x,U}} \Rightarrow \mathcal{F} \circ \Phi_{x,U},$$

of objects in $\mathbf{Fun}(\sigma_{x,U},\mathcal{C})$. Assume that each $(R_x(\mathcal{F})(U),\eta_{\mathcal{F},x,U})$ is a limit of $\mathcal{F}\circ\Phi_{x,U}$. **Prove** that this extends uniquely to a functor,

$$R_x : \mathbf{Fun}(\sigma,\mathcal{C}) \to \mathbf{Fun}(\tau,\mathcal{C}),$$

and a natural transformation,

$$\eta : *_x \circ R_x \Rightarrow \mathrm{Id}_{\mathbf{Fun}(\sigma,\mathcal{C})}.$$

Moreover, for every $\mathcal{G}$ in $\mathbf{Fun}(\tau,\mathcal{C})$, the identity morphism,

$$\mathrm{Id}_{\mathcal{G}} : \mathcal{G}\circ \underline{U}_{\sigma_{x,U}} \Rightarrow \mathcal{G}\circ x\circ \Phi_{x,U},$$

factors uniquely through a $\mathcal{G}(U)\to\mathcal{C}$-morphism $R_x(\mathcal{G}\circ x)(U)$. **Prove** that this defines a morphism $\theta_{\mathcal{G}} : \mathcal{G}\to R_x(\mathcal{G}\circ x)$ in $\mathbf{Fun}(\tau,\mathcal{C})$. **Prove** that this is a natural transformation,

$$\theta : \mathrm{Id}_{\mathbf{Fun}(\tau,\mathcal{C})} \Rightarrow R_x\circ *_x.$$

Prove that $(*_x, R_x, \theta, \eta)$ is an adjoint pair of functors. (Using some version of the Axiom of Choice, if every $\mathcal{F}\circ\Phi_{x,U}$ admits a colimit, then there exists a R_x and η as above.)

(viii)(Adjoints Relative to a Full, Upper Subcategory) In a complementary direction to the previous case, let $x:\sigma\to\tau$ be an embedding of a full subcategory (thus, x is essentially surjective if and only if x is an equivalence of categories). In this case, the functor

$$*_x : \mathbf{Fun}(\tau,\mathcal{C}) \to \mathbf{Fun}(\sigma,\mathcal{C})$$

is called *restriction*. Assume further that σ is *upper* (a la the theory of partially ordered sets) in the sense that every morphism of τ whose source is an object of σ also has target an object of σ. Assume that $\mathcal{C}$ has an initial object, $\odot$. Let $\mathcal{G}$ be a σ-family of objects of $\mathcal{C}$. Also, let $\phi:\mathcal{G}\to\mathcal{H}$ be a morphism of σ-families. For every object U of τ, if U is an object of σ, then define ${}_x\mathcal{G}(U)$ to be $\mathcal{G}(U)$, and define ${}_x\phi(U)$ to be $\phi(U)$. For every object U of τ that is not an object of σ, define ${}_x\mathcal{G}(U)$ to be $\odot$, and define ${}_x\phi(U)$ to be $\mathrm{Id}_{\odot}$. For every morphism $r:U\to V$, if U is an object of σ, then r is a morphism of σ. In this case, define ${}_x\mathcal{G}(r)$ to be $\mathcal{G}(r)$. On the other hand, if U is not an object of σ, then $\mathcal{G}(U)$ is the initial object $\odot$. In this case, define ${}_x\mathcal{G}(r)$ to be the unique morphism ${}_x\mathcal{G}(U)\to{}_x\mathcal{G}(V)$. **Prove** that ${}_x\mathcal{G}$ is a τ-family of objects, i.e., the definitions above are compatible with composition of morphisms in τ and with identity morphisms. Also **prove** that ${}_x\phi$ is a morphism of τ-families. **Prove** that ${}_x\mathrm{Id}_{\mathcal{G}}$ equals $\mathrm{Id}_{{}_x\mathcal{G}}$. Also, for a second morphism of σ-families, $\psi:\mathcal{H}\to\mathcal{I}$, **prove** that ${}_x(\psi\circ\phi)$ equals ${}_x\psi\circ{}_x\phi$. Conclude that these rules form a functor,

$${}_x* : \mathbf{Fun}(\sigma,\mathcal{C}) \to \mathbf{Fun}(\tau,\mathcal{C}).$$

Prove that $({}_x*, *_x)$ extends to an adjoint pair of functors. In particular, conclude that $*_x$ preserves epimorphisms and ${}_x*$ preserves monomorphisms.

Next assume that $\mathcal{C}$ is an Abelian category that satisfies (AB3). For every τ-family $\mathcal{F}$, for every object U of τ, define $\theta_{\mathcal{F}}(U):\mathcal{F}(U)\to{}^x\mathcal{F}(U)$ to be the cokernel of $\mathcal{F}(U)$ by the direct sum of the images of

$$\mathcal{F}(s):\mathcal{F}(T)\to\mathcal{F}(U),$$

for all morphisms $s : T \to U$ with V *not* in σ (possibly empty, in which case $\theta_{\mathcal{F}}(U)$ is the identity on $\mathcal{F}(U)$). In particular, if U is not in σ, then ${}^x\mathcal{F}(U)$ is zero. For every morphism $r : U \to V$ in τ, **prove** that the composition $\theta_{\mathcal{F}}(V) \circ \mathcal{F}(r)$ equals ${}^x\mathcal{F}(r) \circ \theta_{\mathcal{F}}(U)$ for a unique morphism

$$ {}^x\mathcal{F}(r) : {}^x\mathcal{F}(U) \to {}^x\mathcal{F}(V). $$

Prove that ${}^x\mathcal{F}(\mathrm{Id}_U)$ is the identity morphism of ${}^x\mathcal{F}(U)$. **Prove** that $r \mapsto {}^x\mathcal{F}(r)$ is compatible with composition in τ. Conclude that ${}^x\mathcal{F}$ is a τ-family, and $\theta_{\mathcal{F}}$ is a morphism of τ-families. For every morphism $\phi : \mathcal{F} \to \mathcal{E}$ of τ-families, for every object U of τ, **prove** that $\theta_{\mathcal{E}}(U) \circ \phi(U)$ equals ${}^x\phi(U) \circ \theta_{\mathcal{F}}(U)$ for a unique morphism

$$ {}^x\phi(U) : {}^x\mathcal{F}(U) \to {}^x\mathcal{E}(U). $$

Prove that the rule $U \mapsto {}^x\phi(U)$ is a morphism of τ-families. **Prove** that ${}^x\mathrm{Id}_{\mathcal{F}}$ is the identity on ${}^x\mathcal{F}$. Also **prove** that $\phi \mapsto {}^x\phi$ is compatible with composition. Conclude that these rules define a functor

$$ {}^x* : \mathbf{Fun}(\tau, \mathcal{C}) \to \mathbf{Fun}(\tau, \mathcal{C}). $$

Prove that the rule $\mathcal{F} \mapsto \theta_{\mathcal{F}}$ is a natural transformation $\mathrm{Id}_{\mathbf{Fun}(\tau,\mathcal{C})} \Rightarrow {}^x*$. **Prove** that the natural morphism of τ-families,

$$ {}^x\mathcal{F} \to {}_x(({}^x\mathcal{F})_x), $$

is an isomorphism. Conclude that there exists a unique functor,

$$ *^x : \mathbf{Fun}(\tau, \mathcal{C}) \to \mathbf{Fun}(\sigma, \mathcal{C}), $$

and a natural isomorphism ${}^x* \Rightarrow {}_x(*^x)$. **Prove** that $(*^x, {}_x*, \theta)$ extends to an adjoint pair of functors. In particular, conclude that ${}_x*$ preserves epimorphisms and $*^x$ preserves monomorphisms.

Finally, drop the assumption that $\mathcal{C}$ has an initial object, but assume that σ is upper, assume that σ has an initial object, W_σ, and assume that there is a functor

$$ y : \tau \to \sigma $$

and a natural transformation $\theta : \mathrm{Id}_\tau \Rightarrow x \circ y$, such that for every object U of τ, the unique morphism $W_\sigma \to y(U)$ and the morphism $\theta_U : U \to y(U)$ make $y(U)$ into a coproduct of W_σ and U in τ. For simplicity, for every object U of σ, assume that $\theta_U : U \to y(U)$ is the identity Id_U (rather than merely being an isomorphism), and for every morphism $r : U \to V$ in σ, assume that $y(r)$ equals r. Thus, for every object V of σ, the identity morphism $y(V) \to V$ defines a natural transformation $\eta : y \circ x \Rightarrow \mathrm{Id}_\sigma$. **Prove** that (y, x, θ, η) is an adjoint pair of functors. Conclude that $(*_x, *_y, *_\theta, *_\eta)$ is an adjoint pair of functors. In particular, conclude that $*_x$ preserves monomorphisms and $*_y$ preserves epimorphisms.

(ix)(Compatibility of Limits and Colimits with Functors) Denote by 0 the "singleton category" 0 with a single object and a single morphism. **Prove** that $\Gamma(0, -)$ is an equivalence of categories. For an arbitrary category τ, for the unique natural transformation $\hat{\tau} : \tau \to 0$, **prove** that $*_{\hat{\tau}}$ equals the composite $\underline{*}_\tau \circ \Gamma(0,-)$ so that $\underline{*}_\tau$ is an example of this construction. In particular, for every functor $x : \sigma \to \tau$, **prove** that $(\underline{a}_\tau)_x$ equals $\underline{a}_\sigma$. If $\eta : \underline{a}_\tau \Rightarrow \mathcal{F}$ is a limit of a τ-family $\mathcal{F}$, and if $\theta : \underline{b}_\sigma \Rightarrow \mathcal{F}_x$ is a limit of the associated σ-family $\mathcal{F}_x$, then **prove** that there is a unique morphism $h : a \to b$ in $\mathcal{C}$ such that η_x equals $\theta \circ \underline{p}_\sigma$. If there are right adjoints Γ_τ of $\underline{*}_\tau$ and Γ_σ of $\underline{*}_\sigma$, conclude that there exists a unique natural transformation

$$\Gamma_x : \Gamma_\tau \Rightarrow \Gamma_\sigma \circ *_x$$

so that $\eta_{\mathcal{F}_x} \circ \underline{\Gamma_x(\mathcal{F})}_\sigma$ equals $(\eta_{\mathcal{F}})_x$. **Repeat** this construction for colimits.

(x)(Limits / Colimits of a Concrete Category) Let σ be a small category in which the only morphisms are identity morphisms: identify σ with the underlying set of objects. Let $\mathcal{C}$ be the category **Sets**. For every σ-family $\mathcal{F}$, **prove** that the rule

$$\Gamma_\sigma(\mathcal{F}) := \prod_{U \in \Sigma} \Gamma(U, \mathcal{F})$$

together with the morphism

$$\eta_{\mathcal{F}} : \underline{\Gamma_\sigma(\mathcal{F})}_\sigma \Rightarrow \mathcal{F},$$
$$\eta_{\mathcal{F}}(V) = \mathrm{pr}_V : \prod_{U \in \Sigma} \Gamma(U, \mathcal{F}) \to \Gamma(V, \mathcal{F}),$$

is a limit of $\mathcal{F}$. Next, for every small category τ, define σ to be the category with the same objects as τ, but with the only morphisms being identity morphisms. Define $x : \sigma \to \tau$ to be the unique functor that sends every object to itself. Define $\Gamma_\tau(\mathcal{F})$ to be the subobject of $\Gamma_\sigma(\mathcal{F}_x)$ of data $(f_U)_{U \in \Sigma}$ such that for every morphism $r : U \to V$, $\mathcal{F}(r)$ maps f_U to f_V. **Prove** that with this definition, there exists a unique natural transformation $\eta_{\mathcal{F}} : \underline{\Gamma_\tau(\mathcal{F})}_\tau \Rightarrow \mathcal{F}$ such that the natural transformation $\underline{\Gamma_\tau(\mathcal{F})}_\sigma \Rightarrow \underline{\Gamma_\sigma(\mathcal{F}_x)} \Rightarrow \mathcal{F}_x$ equals $(\eta_{\mathcal{F}})_x$. **Prove** that $\eta_{\mathcal{F}}$ is a limit of $\mathcal{F}$. Conclude that **Sets** has all small limits. Similarly, for associative, unital rings R and S, **prove** that the forgetful functor

$$\Phi : R - S - \mathrm{mod} \to \mathbf{Sets}$$

sends products to products. Let $\mathcal{F}$ be a τ-family of $R-S$-modules. **Prove** that the defining relations for $\Gamma_\tau(\Phi \circ \mathcal{F})$ as a subset of $\Gamma_\sigma(\Phi \circ \mathcal{F})$ are the simultaneous kernels of $R-S$-module homomorphisms. Conclude that there is a natural $R-S$-module structure on $\Gamma_\tau(\Phi \circ \mathcal{F})$, and use this to **prove** that $R-S$-mod has all limits.

(xi)(Functoriality in the Target) For every functor of categories,

$$H : \mathcal{C} \to \mathcal{D},$$

for every τ-family $\mathcal{F}$ in $\mathcal{C}$, **prove** that $H \circ \mathcal{F}$ is a τ-family in $\mathcal{D}$. For every morphism of τ-families in $\mathcal{C}$, $\phi : \mathcal{F} \Rightarrow \mathcal{G}$, **prove** that $H \circ \phi$ is a morphism of τ-families in $\mathcal{D}$. **Prove** that this defines a functor

$$H_\tau : \mathbf{Fun}(\tau, \mathcal{C}) \to \mathbf{Fun}(\tau, \mathcal{D}).$$

For the identity functor $\mathrm{Id}_{\mathcal{C}}$, **prove** that $(\mathrm{Id}_{\mathcal{C}})_\tau$ is the identity functor. For $I : \mathcal{D} \to \mathcal{E}$ a functor of categories, **prove** that $(I \circ H)_\tau$ is the composite $I_\tau \circ H_\tau$. In this sense, deduce that H_τ is functorial in H.

For two functors, $H, I : \mathcal{C} \to \mathcal{D}$, and for a natural transformation $N : H \Rightarrow I$, for every τ-family $\mathcal{F}$ in $\mathcal{C}$, define $N_\tau(\mathcal{F})$ to be

$$N \circ \mathcal{F} : H \circ \mathcal{F} \Rightarrow I \circ \mathcal{F}.$$

Prove that $N_\tau(\mathcal{F})$ is a morphism of τ-families in $\mathcal{D}$. For every morphism of τ-families in $\mathcal{C}$, $\phi : \mathcal{F} \to \mathcal{G}$, **prove** that $N_\tau(\mathcal{G}) \circ H_\tau(\phi)$ equals $I_\tau(\phi) \circ N_\tau(\mathcal{F})$. In this sense, conclude that N_τ is a natural transformation $H_\tau \Rightarrow I_\tau$. For the identity natural transformation $\mathrm{Id}_H : H \Rightarrow H$, **prove** that $(\mathrm{Id}_H)_\tau$ is the identity natural transformation of H_τ. For a second natural transformation $M : I \Rightarrow J$, **prove** that $(M \circ N)_\tau$ equals $M_\tau \circ N_\tau$. In this sense, deduce that $(-)_\tau$ is also compatible with natural transformations.

(xii)(Reductions of Limits to Finite Systems for Concrete Categories) A category is *cofiltering* if for every pair of objects U and V there exists a pair of morphisms, $r : W \to U$ and $s : W \to V$, and for every pair of morphisms, $r, s : V \to U$, there exists a morphism $t : W \to V$ such that $r \circ t$ equals $s \circ t$ (both of these are automatic if the category has an initial object X). Assume that the category $\mathcal{C}$ has limits for all categories τ with finitely many objects, and also for all small cofiltering categories. For an arbitrary small category τ, define $\widehat{\tau}$ to be the small category whose objects are finite full subcategories σ of τ, and whose morphisms are inclusions of subcategories, $\rho \subset \sigma$, of τ. **Prove** that $\widehat{\tau}$ is cofiltering. Let $\mathcal{F}$ be a τ-family in $\mathcal{C}$. For every finite full subcategory $\sigma \subset \tau$, denote by $\mathcal{F}_\sigma$ the restriction as in (f) above. By hypothesis, there is a limit $\eta_\sigma : \underline{\widehat{\mathcal{F}}(\sigma)}_\sigma \Rightarrow \mathcal{F}_\sigma$. Moreover, by (g), for every inclusion of full subcategories $\rho \subset \sigma$, there is a natural morphism in $\mathcal{C}$, $\widehat{\mathcal{F}}(\rho) \to \widehat{\mathcal{F}}(\sigma)$, and this is functorial. Conclude that $\widehat{\mathcal{F}}$ is a $\widehat{\tau}$-family in $\mathcal{C}$. Since $\widehat{\tau}$ is filtering, there is a limit

$$\eta_{\widehat{\mathcal{F}}} : \underline{a}_{\widehat{\tau}} \Rightarrow \widehat{\mathcal{F}}.$$

Prove that this defines a limit $\eta_{\mathcal{F}} \underline{a}_\tau \Rightarrow \mathcal{F}$.

Finally, use this to **prove** that limits exist in each of the following categories: the category of (not necessarily Abelian) groups, the category of Abelian groups, the category of associative, unital (not necessarily commutative) rings, the category of commutative rings, and the category of $R-S$-bimodules (where R and S are associative, unital rings).

(xiii)(bis, Colimits) Repeat the steps above for colimits in place of limits. Use this to **prove** that colimits exist in each of the following categories: the category of (not necessarily Abelian) groups, the category of Abelian groups, the category of associative, unital (not necessarily commutative) rings, the category of commutative rings, and the category of $R-S$-bimodules (where R and S are associative, unital rings).

Practice with Limits and Colimits Exercise. In each of the following cases, say whether the given category (a) has an initial object, (b) has a final object, (c) has a zero object, (d) has finite products, (e) has finite coproducts, (f) has arbitrary products, (g) has arbitrary coproducts, (h) has arbitrary limits (sometimes called *inverse limits*), (i) has arbitrary colimits (sometimes called *direct limits*), (j) coproducts / filtering colimits preserve monomorphisms, (k) products / cofiltering limits preserve epimorphisms.

(i) The category **Sets** whose objects are sets, whose morphisms are set maps, whose composition is usual composition, and whose identity morphisms are usual identity maps.

(ii) The opposite category $\mathbf{Sets}^{\text{opp}}$.

(iii) For a given set S, the category whose objects are elements of the set, and where the only morphisms are the identity morphisms from an element to that same element. What if the set is the empty set? What if the set is a singleton set?

(iv) For a partially ordered set $(S, \preceq)$, the category whose objects are elements of S, and where the Hom set between two elements x, y of S is a singleton set if $x \preceq y$ and empty otherwise. What if the partially ordered set $(S, \preceq)$ is a **lattice**, i.e., every finite subset (resp. arbitrary subset) has a least upper bound and has a greatest lower bound?

(v) For a monoid $(M, \cdot, 1)$, the category with only one object whose Hom set, with its natural composition and identity, is $(M, \cdot, 1)$. What is M equals $\{1\}$?

(vi) For a monoid $(M, \cdot, 1)$ and an action of that monoid on a set, $\rho : M \times S \to S$, the category whose objects are the elements of S, and where the Hom set from x to y is the subset $M_{x,y} = \{m \in M | m \cdot x = y\}$. What if the action is both transitive and faithful, i.e., S equals M with its left regular representation?

(vii) The category **PtdSets** whose objects are pairs (S, s_0) of a set S and a specified element s_0 of S, i.e., *pointed sets*, whose morphisms are set maps that send the specified point of the domain to the specified point of the target, whose composition is usual composition, and whose identity morphisms are usual identity maps.

(viii) The category **Monoids** whose objects are monoids, whose morphisms are homomorphisms of monoids, whose composition is sual composition, and whose identity morphisms are usual identity maps.

(ix) For a specified monoid $(M, \cdot, 1)$, the category whose objects are pairs (S, ρ) of a set S and an action $\rho : M \times S \to S$ of M on S, whose morphisms are set maps compatible with the action, whose composition is usual composition, and whose identity morphisms are usual identity maps.

(x) The full subcategory **Groups** of **Monoids** whose objects are groups. Does the inclusion functor preserve coproducts, resp. products? Does the inclusion functor preserve monomorphisms, resp. epimorphisms?

(xi) The full subcategory $\mathbb{Z}-\text{mod}$ of **Groups** whose objects are Abelian groups. Does the inclusion functor preserve coproducts, resp. products? Does the inclusion functor preserve monomorphisms, resp. epimorphisms?

(xii) The full subcategory **FiniteGroups** of **Groups** whose objects are finite groups. Are coproducts, resp. products, in the subcategory also coproducts, resp. products, in the larger category **Groups**? Does the inclusion functor preserve monomorphisms, resp. epimorphisms?

(xiii) The full subcategory $\mathbb{Z}-\text{mod}_{\text{tor}}$ of $\mathbb{Z}-\text{mod}$ consisting of torsion Abelian groups, i.e., every element has finite order (allowed to vary from element to element). Are coproducts, resp. products, preserved by the inclusion functor? Are monomorphisms, resp. epimorphisms preserved?

(xiv) The category **Rings** whose objects are associative, unital rings, whose morphisms are homomorphisms of rings (preserving the multiplicative identity), whose composition is the usual composition, and whose identity morphisms are the usual identity maps. **Hint.** For the coproduct of two associative, unital rings $(R', +, 0, \cdot', 1')$ and $(R'', +, 0, \cdot'', 1'')$, first form the coproduct $R' \oplus R''$ of $(R', +, 0)$ and $(R'', +, 0)$ as a $\mathbb{Z}$-module, then form the total tensor product ring $T^{\bullet}_{\mathbb{Z}}(R' \oplus R'')$ as in the previous problem set. For the two natural maps $q' : R' \hookrightarrow T^1_{\mathbb{Z}}(R' \oplus R'')$ and $q'' : R'' \hookrightarrow T^1_{\mathbb{Z}}(R' \oplus R'')$ form the left-right ideal $I \subset T^{\bullet}_{\mathbb{Z}}(R' \oplus R'')$ generated by $q'(1') - 1$, $q''(1'') - 1$, $q'(r' \cdot' s') - q'(r') \cdot q'(s')$, and $q''(r'' \cdot'' s'') - q''(r'') \cdot q''(s'')$ for all elements $r', s' \in R'$ and $r'', s'' \in R''$. Define

$$p : T^1_{\mathbb{Z}}(R' \oplus R'') \to R,$$

to be the quotient by I. Prove that $p \circ q' : R' \to R$ and $p \circ q'' : R'' \to R$ are ring homomorphisms that make R into a coproduct of R' and R''.

(xv) The full subcategory **CommRings** of **Rings** whose objects are commutative, unital rings. Does the inclusion functor preserve coproducts, resp. products? Does the inclusion functor preserve monomorphisms, resp. epimorphisms?

(xvi) The full subcategory **NilCommRings** of **CommRings** whose objects are commutative, unital rings such that every noninvertible element is nilpotent. Does the inclusion functor preserve coproducts, resp. products? (Be careful about products!) Does the inclusion functor preserve monomorphisms, resp. epimorphisms?

(xvii) Let R and S be associative, unital rings. Let $R-\text{mod}$, resp. $\text{mod}-S$, $R-S-\text{mod}$, be the category of left R-modules, resp. right S-modules, $R-S$-bimodules. Does the inclusion functor from $R-S-\text{mod}$ to $R-\text{mod}$, resp. to $\text{mod}-S$, preserve coproduct, products, monomorphisms and epimorphisms?

(xviii) Let $(I, \preceq)$ be a partially ordered set. Let $\mathcal{C}$ be a category. An $(I, \preceq)$-*system* in $\mathcal{C}$ is a datum

$$c = ((c_i)_{i \in I}, (f_{i,j})_{(i,j) \in I \times I, i \preceq j})$$

where every c_i is an object of $\mathcal{C}$, where for every pair $(i, j) \in I \times I$ with $i \preceq j$, $c_{i,j}$ is an element of $\text{Hom}_{\mathcal{C}}(c_i, c_j)$, and satisfying the following conditions: (a) for every $i \in I$, $c_{i,i}$ equals Id_{c_i}, and (b) for every triple $(i, j, k) \in I$ with $i \preceq j$ and $j \preceq k$, $c_{j,k} \circ c_{i,j}$ equals $c_{i,k}$. For every pair of $(I, \preceq)$-systems in $\mathcal{C}$, $c = ((c_i)_{i \in I}, (c_{i,j})_{i \preceq j})$ and $c' = ((c'_i)_{i \in I}, (c'_{i,j})_{i \preceq j})$, a morphism $g : c \to c'$ is defined to be a datum $(g_i)_{i \in I}$ of morphisms $g_i \in \text{Hom}_{\mathcal{C}}(c_i, c'_i)$ such that for every $(i, j) \in I \times I$ with $i \preceq j$, $g_j \circ c_{i,j}$ equals $c'_{i,j} \circ g_i$. Composition of morphisms g and g' is componentwise $g'_i \circ g_i$, and identities are $\text{Id}_c = (\text{Id}_{c_i})_{i \in I}$. This category is $\text{Fun}((I, \preceq), \mathcal{C})$, and is sometimes referred to as the category of $(I, \preceq)$-*presheaves*. Assuming $\mathcal{C}$ has finite coproducts, resp. finite products, arbitrary coproducts, arbitrary products, a zero object, kernels, cokernels, etc., what can you say about $\text{Fun}((I, \preceq), \mathcal{C})$?

(xix) Let $\mathcal{C}$ be a category that has arbitrary products. Let $(I, \preceq)$ be a partially ordered set whose associated category as in (iv) has finite coproducts and has arbitrary products. The main example is when $I = \mathfrak{U}$ is the collection of all open subsets U of a topology on a set X, and where $U \preceq V$ if $U \supseteq V$. Then coproduct is intersection and product is union. Motivated by this case, an *covering* of an element i of I is a collection $\underline{j} = (j_\alpha)_{\alpha \in A}$ of elements j_α of I such that for every α, $i \preceq j_\alpha$, and such that i is the product of $(j_\alpha)_{\alpha \in A}$ in the sense of (iv). In this case, for every $(\alpha, \beta) \in A \times A$, define $j_{\alpha,\beta}$ to be the element of I such that $j_\alpha \preceq j_{\alpha,\beta}$, such that $j_\beta \preceq j_{\alpha,\beta}$, and such that $j_{\alpha,\beta}$ is a coprodcut of (j_α, j_β). An $(I, \preceq)$-presheaf $c = ((c_i)_{i \in I}, (c_{i,j})_{i \preceq j})$ is an $(I, \preceq)$-*sheaf* if for every element i of I and for every covering $\underline{j} = (j_\alpha)_{\alpha \in A}$, the following diagram in $\mathcal{C}$ is *exact* in a sense to be made precise,

$$c_i \xrightarrow{q} \prod_{\alpha \in A} c_{j_\alpha} \overset{p'}{\rightrightarrows} p'' \prod_{(\alpha,\beta) \in A \times A} c_{j_{\alpha,\beta}}.$$

For every $\alpha \in A$, the factor of q,

$$\mathrm{pr}_\alpha \circ q : c_i \to c_{j_\alpha},$$

is defined to be c_{i,j_α}. For every $(\alpha, \beta) \in A \times A$, the factor of p',

$$\mathrm{pr}_{\alpha,\beta} \circ p' : \prod_{\gamma \in A} c_{j_\gamma} \to c_{j_{\alpha,\beta}},$$

is defined to be $c_{j_\alpha, j_{\alpha,\beta}} \circ \mathrm{pr}_\alpha$. Similarly, $\mathrm{pr}_{\alpha,\beta} \circ p''$ is defined to be $c_{j_\beta, j_{\alpha,\beta}} \circ \mathrm{pr}_\beta$. The diagram above is *exact* in the sense that q is a monomorphism in $\mathcal{C}$ and q is a fiber product in $\mathcal{C}$ of the pair of morphisms (p', p''). The category of $(I, \preceq)$ is the full subcategoryof the category of $(I, \preceq)$-presheaves whose objects are $(I, \preceq)$-sheaves. Does this subcategory have coproducts, products, etc.? Does the inclusion functor preserve coproducts, resp. products, monomorphisms, epimorphisms? Before considering the general case, it is probably best to first consider the case that $\mathcal{C}$ is $\mathbb{Z} - \mathrm{mod}$, and then consider the case that $\mathcal{C}$ is **Sets**.

Sheaves.

Topological Spaces.

Definition 0.121. For every set X, a subset τ of the power set $\mathcal{P}(X)$ of X is a **topology** (by open subsets) of X if (and only if) all of the following hold.

(i) Both $\emptyset$ and X are elements of τ.

(ii) For all elements U and V of τ, also $U \cap V$ is an element of τ.

(iii) For every subset of τ, the union of the subset is an element of τ, i.e., for every subset S of τ, also τ contains as an element the union of all subsets of X that are elements of S.

An ordered pair (X, τ) is a **topological space** if (and only if) τ is a topology on X.

For every ordered pair $((X, \tau), (X', \tau'))$ of topological spaces, for every function f from X to X', the function f is a **continuous map** from (X, τ) to (X', τ') if (and only if), for every element U' of τ', the f-preimage $f^{\mathrm{pre}} U'$ is an element of τ.

Proposition 0.122. *For every topological space (X, τ), the identity set function Id_X from X to itself is a continuous map from (X, τ) to (X, τ). For every ordered triple $((X, \tau), (X', \tau'), (X'', \tau''))$ of topological spaces, for every ordered pair (f, f') of a continous map f from (X, τ) to (X', τ') and a continuous map f' from (X', τ') to (X'', τ''), the composition $f' \circ f$ is a continuous map from (X, τ) to (X'', τ''). Thus, these notions define a category* ***Top*** *whose objects are topological spaces and whose morphisms are continuous maps. The forgetful functor pr_1 that associates to every topological space (X, τ) the set X is faithful.*

Example 0.123. For every set X, the subset $\{\emptyset, X\}$ of $\mathcal{P}(X)$ is a topology on X, called the **indiscrete topology** on X. For every set X, the entire set $\mathcal{P}(X)$ is a topology on X, called the **discrete topology**.

Proposition 0.124. *The functor from the category of sets to the category of topological spaces that associates to every set X the topological space $(X, \{\emptyset, X\})$ is a right adjoint to the forgetful functor pr_1. The functor that associates to every set X the topological space $(X, \mathcal{P}(X))$ is a left adjoint to the forgetful functor.*

Definition 0.125. For every topological space (X, τ), a **presheaf** $\mathcal{F}$ of sets on (X, τ) is an assignment to every element U of τ of a set $\mathcal{F}(U)$ and an assignment to every ordered pair (U, V) of elements of τ satisfying $U \subseteq V$ of a set function $\text{res}^V_{\mathcal{F},U}$ from $\mathcal{F}(V)$ to $\mathcal{F}(U)$ such that both of the following hold.

(i) For every element U of τ, the function $\text{res}^U_{\mathcal{F},U}$ equals the identity function on $\mathcal{F}(U)$.

(ii) For every ordered triple (U, V, W) of elements of τ satisfying $U \subseteq V \subseteq W$, the composition $\text{res}^V_{\mathcal{F},U} \circ \text{res}^W_{\mathcal{F},V}$ equals $\text{res}^W_{\mathcal{F},U}$.

For every topological space (X, τ), for every ordered pair $(\mathcal{F}, \mathcal{F}')$ of presheaves of sets on (X, τ), a **morphism** ϕ of presheaves of sets on (X, τ) from $\mathcal{F}$ to $\mathcal{F}'$ is an assignment to every element U of τ of a function ϕ_U from $\mathcal{F}(U)$ to $\mathcal{F}'(U)$ such that for every ordered pair (U, V) of elements of τ satisfying $U \subseteq V$, the composition $\text{res}^V_{\mathcal{F}',U} \circ \phi_V$ equals the composition $\phi_U \text{res}^V_{\mathcal{F},U}$.

Proposition 0.126. *For every topological space (X, τ), for every presheaf $\mathcal{F}$ of sets on (X, τ), the assignment to every element U of τ of the identity function from $\mathcal{F}(U)$ to $\mathcal{F}(U)$ is a morphism of presheaves of sets on (X, τ). For every ordered triple $(\mathcal{F}, \mathcal{F}', \mathcal{F}'')$ of presheaves of sets on (X, τ), for every ordered pair (ϕ, ϕ') of a morphism ϕ of presheaves of sets on (X, τ) from $\mathcal{F}$ to $\mathcal{F}'$ and a morphism ϕ' of presheaves of sets on (X, τ) from $\mathcal{F}'$ to $\mathcal{F}''$, the assignment to every element U of τ of the composition $\phi'_U \circ \phi_U$ is a morphism of presheaves of sets on (X, τ) from $\mathcal{F}$ to $\mathcal{F}''$. Thus, these notions define a category $\boldsymbol{PSh}_{X,\tau}$.*

Proposition 0.127. *For every topological space (X, τ), for every element U of τ, the assignment to every presheaf $\mathcal{F}$ of sets on (X, τ) of the set $\mathcal{F}(U)$ and the assignment, for every ordered pair $(\mathcal{F}, \mathcal{F}')$ of presheaves of sets on (X, τ) and every morphism ϕ of presheaves of sets on (X, τ) from $\mathcal{F}$ to $\mathcal{F}'$ of the function ϕ_U is a functor $\Gamma(U, -)$ from the category of presheaves of sets on (X, τ) to* *the category of sets. For every topological space (X, τ), for every ordered pair (U, V) of elements of τ satisfying $U \subseteq V$, the assignment to every presheaf $\mathcal{F}$ of sets on (X, τ) of the function $res^V_{\mathcal{F},U}$ is a natural transformation Γ^V_U of set-valued functors on $\boldsymbol{PSh}(X, \tau)$ from the functor $\Gamma(V, -)$ to the functor $\Gamma(U, -)$. For every ordered triple (U, V, W) of elements of τ satisfying $U \subseteq V \subseteq W$, the composition of natural transformations $\Gamma^V_U \circ \Gamma^W_V$ equals the natural transformation Γ^W_U.*

The partially ordered set $(\tau, \subseteq)$ does not "remember" the set X. In fact, the set X determines another structure on $(\tau, \subseteq)$: the data of which collections of open subsets of an open set U form covers of U. This, in turn, is the key structure in defining the full subcategory of *sheaves* in the larger category of presheaves.

Definition 0.128. For every partially ordered set that has meets of all finite subsets, or, more generally, for every category $\mathcal{C}$ with finite fiber products, a **Grothendieck pretopology** on $\mathcal{C}$ is a specification for every object T of $\mathcal{C}$ of which sets of morphisms in $\mathcal{C}$ to T, $\mathfrak{U} = (f_\lambda : U_\lambda \to T)_{\lambda\in\Lambda}$, are *covering families* satisfying the following axioms.

(i) Every family consisting of a single isomorphism is a covering family.

(ii) For every covering family $\mathfrak{U}$ of an object T and for every morphism $h : S \to T$, every pullback family $h^*\mathfrak{U} = (S \times_T U_\lambda \xrightarrow{\mathrm{pr}_1} S)_{\lambda\in\Lambda}$ is a covering family of S.

(iii) For every covering family $(f_\lambda : U_\lambda \to T)_{\lambda\in\Lambda}$ of an object T, for every covering family $(f_{\lambda,\mu} : U_{\lambda,\mu} \to U_\lambda)_{\mu\in M_\lambda}$ of the object U_λ for each $\lambda \in \Lambda$, the family $(f_\lambda \circ f_{\lambda,\mu} : U_{\lambda,\mu} \to T)_{\lambda\in\Lambda,\mu\in M_\lambda}$ is a covering family of T.

For covering families $\mathfrak{U} = (f_\lambda : U_\lambda \to T)_{\lambda\in\Lambda}$ and $\mathfrak{V} = (g_\mu : V_\mu \to T)_{\mu\in M}$ of T, a **refinement** from $\mathfrak{V}$ to $\mathfrak{U}$ is a pair $e_\bullet := (r, (e_\mu)_{\mu\in M})$ of a set function $r : M \to \Lambda$ and a collection $(e_\mu : V_\mu \to U_{r(\mu)})_{\mu\in M}$ of morphisms of $\mathcal{C}$ such that $f_{r(\mu)} \circ e_\mu$ equals g_μ for every $\mu \in M$. Composition of covering families are defined in the usual way.

Example 0.129. For every topological space (X, τ), for every element T of τ, for every subset $\mathfrak{U} = (U_\lambda \subseteq T)_{\lambda\in\Lambda}$ of the subspace topology on T, the subset $\mathfrak{U}$ is a (X, τ)-covering of T if (and only if) T equals the union of the subset.

Definition 0.130. For every category $\mathcal{C}$ with a specified Grothendieck pretopology, for every contravariant functor $\mathcal{F}$ from $\mathbf{C}$ to $\mathbf{Set}$, the contravariant functor is a **sheaf** on $\mathbf{C}$ for the Grothendieck pretopology if (and only if), for every object T of $\mathbf{C}$, for every covering family $\mathfrak{U} = (f_\lambda : U_\lambda \to T)_{\lambda\in\Lambda}$, the following diagram of sets is "left-exact",

$$\mathcal{F}(T) \to \prod_{\lambda\in\Lambda} \mathcal{F}(U_\lambda) \rightrightarrows \prod_{(\lambda,\mu)\in\Lambda\times\Lambda} \mathcal{F}(U_\lambda \ \times_T U_\mu),$$

in the sense that for every element $(s_\lambda)_{\lambda\in\Lambda}$ in the codomain of the first arrow whose two images under the second arrows $(\mathrm{pr}_1^* s_\lambda)_{(\lambda,\mu)\in\Lambda\times\Lambda}$ and $(\mathrm{pr}_2^* s_\mu)_{(\lambda,\mu)\in\Lambda\times\Lambda}$ are equal, there exists a unique element s in $\mathcal{F}(T)$ whose image under the first arrow equals $(s_\lambda)_{\lambda\in\Lambda}$.

The category of sheaves on $\mathbf{C}$ with the specified Grothendieck pretopology is the full subcategory of the category of all set-valued contravariant functors on $\mathbf{C}$ whose objects are the sheaves.

Example 0.131. For every topological space (X, τ), for every presheaf of sets $\mathcal{F}$ on (X, τ), the presheaf ia a sheaf if (and only if), for every element T of τ, for every (X, τ)-covering $\mathfrak{U} = (U_\lambda \subseteq T)_{\lambda \in \Lambda}$ of T, for every element $(s_\lambda)_{\lambda \in \Lambda}$ of $\prod_{\lambda \in \Lambda} \mathcal{F}(U_\lambda)$ whose two images $(\text{res}^{U_\lambda}_{\mathcal{F}, U_\lambda \cap U_\mu}(s_\lambda))_{(\lambda,\mu) \in \Lambda \times \Lambda}$ and $(\text{res}^{U_\mu}_{\mathcal{F}, U_\lambda \cap U_\mu}(s_\mu))_{(\lambda,\mu) \in \Lambda \times \Lambda}$ are equal as elements of $\prod_{(\lambda,\mu) \in \Lambda \times \Lambda} \mathcal{F}(U_\lambda \cap U_\mu)$, there exists a unique element s in $\mathcal{F}(T)$ such that $(s_\lambda)_{\lambda \in \Lambda}$ equals $(\text{res}^T_{\mathcal{F}, U_\lambda}(s))_{\lambda \in \Lambda}$. The category $\mathbf{Sh}(X, \tau)$ of sheaves is the full subcategory of the category $\mathbf{PSh}(X, \tau)$ of set-valued presheaves whose objects are sheaves of sets on (X, τ).

In general for a Grothendieck pretopology on a category, there are set-theoretic issues with forming a left adjoint of the full embedding of the category of sheaves in the category of sheaves. However, for sheaves of sets on a topological space (X, τ), there are several explicit constructions. One of these uses stalks and the associated Godement sheaf of a presheaf; the associated sheaf is a certain subsheaf of the Godement sheaf.

Definition 0.132. For every topological space (X, τ), for every element p of X, the **neighborhood system** of (X, τ) at p is the subset τ_p of τ of all elements U of τ such that p is an element of U. For every topological space (X, τ), for every element p of X, for every presheaf $\mathcal{F}$ of sets on (X, τ), the **neighborhood** presheaf of $\mathcal{F}$ at p is the restriction of $\mathcal{F}$ to the neighborhood system τ_p of (X, τ) at p. A **compatible family** of $\mathcal{F}$ at p is an ordered pair $(S, (r^U)_{U \in \tau_p})$ of a set S and a system $(r^U)_{U \in \tau_p}$ assigning to every element U of τ_p a function r^U from $\mathcal{F}(U)$ to S such that for every ordered pair (U, V) of elements of τ_p satisfying $U \subseteq V$, the composition $r^U \circ \text{res}^V_{\mathcal{F}, U}$ equals r^V. The **stalk** of $\mathcal{F}$ at p is the ordered pair $(\mathcal{F}_p, (\text{res}^U_{\mathcal{F},p})_{U \in \tau_p})$ where $\mathcal{F}_p$ is the quotient of the disjoint union $\sqcup_{U \in \tau_p} \mathcal{F}(U) = \{(U, s) | U \in \tau_p,\ s \in \mathcal{F}(U)\}$ by the equivalence relation $\sim$ such that $(U, s) \sim (V, t)$ if and only if there exists and element W in τ_p such that $\text{res}^U_{\mathcal{F}, U \cap V \cap W}(s)$ equals $\text{res}^V_{\mathcal{F}, U \cap V \cap W}(t)$. Denoting the quotient function by $q : \sqcup_{U \in \tau_p} \mathcal{F}(U) \to \mathcal{F}_p$, for every element V of τ_p, the function $\text{res}^V_{\mathcal{F},p}$ equals the precomposition of q with the function that sends every element s of $\mathcal{F}(V)$ to the element (V, s) of $\sqcup_{U \in \tau_p} \mathcal{F}(U)$.

Proposition 0.133. *For every topological space (X, τ), for every element p of X, for every presheaf $\mathcal{F}$ of sets on (X, τ), the ordered pair $(\mathcal{F}_p, (res^U_{\mathcal{F},p})_{U \in \tau_p})$ is a compatible family of $\mathcal{F}$ at p. In fact, it is an initial compatible family, i.e., for every compatible family $(S, (r^U)_{U \in \tau_p})$ of $\mathcal{F}$ at p, there exists a unique function r from $\mathcal{F}_p$ to S such that, for every element U of τ_p, the composition $r \circ res^U_{\mathcal{F},p}$ equals r^U.*

Corollary 0.134. *For every topological space (X, τ), for every element p of X, for every ordered pair $(\mathcal{F}, \mathcal{F}')$ of presheaves of sets on (X, τ), for every morphism ϕ of presheaves of sets on (X, τ) from $\mathcal{F}$ to $\mathcal{F}'$, there exists a unique morphism ϕ_p from $\mathcal{F}_p$ to $\mathcal{F}'_p$ such that, for every element U of τ_p, the composition $\phi_p \circ res^U_{\mathcal{F},p}$ equals $res^U_{\mathcal{F}',p} \circ \phi_U$.*

Corollary 0.135. *For every topological space (X, τ), for every element p of X, there is a functor from **PSh**(X, τ) to **Set** associating to every presheaf of sets on X the associated stalk, and associating to every morphism of presheaves of sets the associated morphism of stalks.*

Definition 0.136. For every topological space (X, τ), for every element p of X, for every set S, the **point pushforward** of S at p is the presheaf $\iota_*^p S$ on (X, τ) associating S to every element U of τ_p, and associating the singleton $\{S\}$ to every element U of $\tau \setminus \tau_p$. For every element U of τ_p, for every element V of τ satisfying $U \subseteq V$, the restriction function res_U^V is defined to be Id_S. For every element U of $\tau \setminus \tau_p$, for every element V of τ satisfying $U \subseteq V$, the restriction function res_U^V is defined to be the unique constant function to $\iota_*^p S(U) = \{S\}$.

For every topological space (X, τ), for every element p of S, for every ordered pair (S, S') of sets, for every function t from S to S', the **morphism** of point pushforwards from $\iota_*^p S$ to $\iota_*^p S'$ associates t to every element U of τ_p, and associates the unique constant function to $\iota_*^p S'(U) = \{S'\}$ to every element U of $\tau \setminus \tau_p$.

Proposition 0.137. *For every topological space (X, τ), for every element p of X, the point pushforward is a functor from **Set** to $\boldsymbol{PSh}(X, \tau)$. Moreover, this functor is right adjoint to the stalk functor. In particular, the point pushforward is an equivalence to its essential image.*

There is analogue of this adjoint pair for all elements of X simultaneously.

Definition 0.138. For every set X, a **discrete system** of sets over X is a set function S with domain X associating to every element p of X a set S_p. For every ordered pair (S, S') of discrete systems of sets over X, a **morphism** of discrete systems of sets over X from S to S' is a set function u with domain X associating to every element p of X a set function u_p from S_p to S'_p. For every ordered triple (S, S', S'') of discrete systems of sets over X, for every ordered pair (u, u') of a morphism u of discrete systems of sets over X from S to S' and of a morphism u' of discrete systems of sets over X from S' to S'', the **composition** of u' with u is the morphism $u' \circ u$ of discrete systems of sets over X from S to S'' associating to every element p of X the composition $u'_p \circ u_p$ from S_p to S''_p.

For every topological space (X, τ), for every presheaf of sets $\mathcal{F}$ for (X, τ), the **associated discrete system** of sets on X is the set function with domain X that associates to every element p of X the stalk $\mathcal{F}_p$. For every topological space (X, τ), for every ordered pair $(\mathcal{F}, \mathcal{F}')$ of presheaves of sets for (X, τ), for every morphism v of presheaves of sets on (X, τ) from $\mathcal{F}$ to $\mathcal{F}'$, the **associated morphism** of discrete systems of sets on X is the set function with domain X that associates to every element p of X the induced morphism of stalks v_p from the stalk $\mathcal{F}_p$ to the stalk $\mathcal{F}'_p$.

Proposition 0.139. *For every topological space (X, τ), the operations above define a functor from the category of presheaves of sets on (X, τ) to the category of discrete systems of sets on X. When restricted to the full subcategory of sheaves of sets on (X, τ), this functor is faithful.*

There is a right adjoint of this functor.

Definition 0.140. For every topological space (X, τ), for every discrete system S of sets on X, the **Godement sheaf** of S on (X, τ) is the sheaf on (X, τ) associating to every τ-open subset U of X the product set $\prod_{p \in U} S_p$ and associating to every inclusion $U \subseteq V$ of τ-open subsets of X

the unique set function $\prod_{p\in V} S_p \to \prod_{p\in U} S_p$ such that for every element q in U, the composite of this set function with the projection $\mathrm{pr}_q : \prod_{p\in U} S_p \to S_q$ equals the projection $\mathrm{pr}_q : \prod_{p\in V} S_p \to S_q$. The Godement sheaf is denoted by GS.

For every topological space (X,τ), for every ordered pair (S,S') of discrete systems of sets on X, for every morphism u of discrete systems of sets on X from S to S', the **associated morphism** of Gu of Godement sheaves is the unique morphism of sheaves on (X,τ) such that for every τ-open subset V of X, the induced morphism Gu_V from $GS(V)$ to $GS'(V)$ is the unique set function $\prod_{p\in V} S_p \to \prod_{p\in V} S'_p$ such that for every element q in V, the composite of this set function with projection $\mathrm{pr}_q : \prod_{p\in V} S'_p \to S'_q$ equals the composite of u_q with the projection $\mathrm{pr}_q : \prod_{p\in V} S_p \to S_p$.

Proposition 0.141. *For every topological space (X,τ), the operations above define a functor from the category of discrete systems of sets over X to the category of sheaves of sets over (X,τ). This functor is right adjoint to the functor associating to every sheaf of sets is associated discrete system.*

Definition 0.142. For every topological space (X,τ), for every presheaf of sets $\mathcal{F}$ on (X,τ), the associated **Godement sheaf** of $\mathcal{F}$ is the Godement sheaf of the associated discrete system, i.e., the sheaf associating to every element U of τ the set $\prod_{p\in U} \mathcal{F}_p$ with projections for the restriction maps (as above). This sheaf is denoted $G\mathcal{F}$.

For every element U of τ, for every element p of U, for every element $(s_q)_{q\in U}$ of $G\mathcal{F}(U)$, the element is **regular** at p if (and only if) there exists an element V of τ_p and an element s of $\mathcal{F}(U\cap V)$ such that, for every element q of $U\cap V$, the element s_q equals the stalk at q of s.

For every element U of τ, for every element $(s_q)_{q\in U}$ of $G\mathcal{F}(U)$, the element is **regular** on U if (and only if), for every element p of U, the element $(s_q)_{q\in U}$ is regular at p. The subset of all regular elements of $G\mathcal{F}(U)$ is denoted by $\mathcal{F}^+(U)$. The assignment to every element U of τ of the subset $\mathcal{F}^+(U)$ of $G\mathcal{F}(U)$ is the **associated sheaf** of $\mathcal{F}$, sometimes called the **sheafification** of $\mathcal{F}$.

Proposition 0.143. *For every topological space (X,τ), for every presheaf of sets $\mathcal{F}$ on (X,τ), the associated sheaf of $\mathcal{F}$ is a subsheaf of $G\mathcal{F}$. Moreover, the natural morphism of presheaves from $\mathcal{F}$ to $G\mathcal{F}$ factors through this subsheaf $\mathcal{F}^+$. Finally, this defines a functor $(-)^+$ from $\boldsymbol{PSh}(X,\tau)$ to $\boldsymbol{Sh}(X,\tau)$ that is left adjoint to the full embedding of $\boldsymbol{Sh}(X,\tau)$ to $\boldsymbol{PSh}(X,\tau)$.*

This method can be repeated for any category with a Grothendieck pretopology that has "enough points," at least up to set-theoretic issues.

Remark 0.144. For every topological space (X,τ), for every filtered cocomplete category $\mathcal{A}$ (i.e., it has filtered colimits) together with a faithful set-valued functor to **Set** that is filtered cocontinuous (i.e., it preserves filtered colimits), we can repeat this construction to obtain the associated sheaf of every presheaf of objects of $\mathcal{A}$ on (X,τ). This is the case for the usual forgetful set-valued functor on the category $\mathcal{A}$ of monoids, of groups, of Abelian groups, of associative rings, and of commutative, unital rings.

Beginning with the work of Leray, but especially in the foundational results of Cartan (in complex geometry) and Serre (in algebraic geometry), the notion of a sheaf of commutative, unital rings on a topological space became the essential ingredient in explaining the basic notions of algebraic geometry.

Definition 0.145. An ordered pair $((X,\tau),\mathcal{R})$ of a topological space (X,τ) and a sheaf of commutative, unital rings $\mathcal{R}$ on (X,τ) is a **ringed space**. For every ringed space $((X,\tau),\mathcal{R})$, for every ringed space $((X',\tau'),\mathcal{R}')$, and for every ordered pair $(f,f^{\#})$ of a continuous map f from (X,τ) to (X',τ') and a morphism of sheaves of commutative, unital rings $f^{\#}$ from $\mathcal{R}'$ to $f_*\mathcal{R}$, the ordered pair $(f,f^{\#})$ is a **morphism of ringed spaces** from $((X,\tau),\mathcal{R})$ to $((X',\tau'),\mathcal{R}')$. For every ordered triple $(((X,\tau),\mathcal{R}),((X',\tau'),\mathcal{R}'),((X'',\tau''),\mathcal{R}''))$ of ringed spaces, for every ordered pair $((f,f^{\#}),(f',(f')^{\#}))$ of a morphism $(f,f^{\#})$ of ringed spaces from $((X,\tau),\mathcal{R})$ to $((X',\tau'),\mathcal{R}')$ and a morphism $(f',(f')^{\#})$ of ringed spaces from $((X',\tau'),\mathcal{R}')$ to $((X'',\tau''),\mathcal{R}'')$, the **composition morphism** of ringed spaces from $((X,\tau),\mathcal{R})$ to $((X'',\tau''),\mathcal{R}'')$ is the ordered pair of the composition continuous map $f'\circ f$ from (X,τ) to (X'',τ'') together with the composition morphism of sheaves of commutative, unital rings $f'_*f^{\#}\circ(f')^{\#}$.

Proposition 0.146. *These notions define a category of ringed spaces, i.e., composition is associative and every ringed space has a (left-right) identity morphism to itself. This category is denoted here by* ***RS****.*

Definition 0.147. For every ringed space $((X,\tau),\mathcal{R})$, denote the topological space (X,τ) by $\mathrm{pr}_1((X,\tau),\mathcal{R})$. For every morphism $(f,f^{\#})$ from a ringed space $((X,\tau),\mathcal{R})$ to a ringed space $((X',\tau'),\mathcal{R}')$, denote the continous map f from (X,τ) to (X',τ') by $\mathrm{pr}_1(f,f^{\#})$.

Proposition 0.148. *These notions define a covariant functor* pr_1 *from the category* ***RS*** *to the category* ***Top*** *of topological spaces, i.e., it preserves composition and identity morphisms.*

This covariant functor is a "forgetful functor," and often that is the term used to denote this particular functor. There are many different functors in the opposite direction that are equivalences to a full subcategory of the category of ringed spaces or locally ringed spaces. We describe a few below. This requires a few other useful notions.

Example 0.149. For every zero ring, i.e., $Z=\{0\}$ with the unique addition and multiplication binary operations, denote by Spec Z the ringed space $((\emptyset,\mathcal{P}(\emptyset)),\widetilde{Z})$ where $\mathcal{P}(\emptyset)$ is the unique topology on the empty set (namely the power set of the empty set), and where $\widetilde{Z}$ is the unique sheaf of rings on $(\emptyset,\tau)$ such that the ring of sections over the empty subset is Z.

Proposition 0.150. *For every zero ring Z, the ringed space Spec Z is an initial object of the category of ringed spaces. Moreover, for every ringed space $((X,\tau),\mathcal{R})$, there exists a morphism of ringed spaces from $((X,\tau),\mathcal{R})$ to Spec Z if and only if X equals the empty set, in which case $((X,\tau),\mathcal{R})$ equals Spec Z' for $Z'=\mathcal{R}(X)$.*

Definition 0.151. For every topological space (X, τ), for every sheaf of Abelian groups $\mathcal{A}$ on (X, τ), the Abelian group $\mathcal{A}(X)$ is the **group of global sections** of $\mathcal{A}$ on (X, τ). This is sometimes also denoted by $\Gamma(X, \mathcal{A})$ or $H^0(X, \mathcal{A})$. For every topological space (X, τ), for every ordered pair $(\mathcal{A}, \mathcal{A}')$ of sheaves of Abelian groups on (X, τ), for every morphism ϕ of sheaves of Abelian groups on (X, τ), the morphism of Abelian groups $\phi(X)$ from $\mathcal{A}(X)$ to $\mathcal{A}'(X)$ is the **morphism of groups of global sections**. This is sometimes also denoted by $\Gamma(X, \phi)$ or $H^0(X, \phi)$.

Proposition 0.152. *For every topological space (X, τ), the notions above define a covariant functor $\Gamma(X, -)$ from the category of sheaves of Abelian groups on (X, τ) to the category of Abelian groups, i.e., it preserves composition and identity morphisms.*

Definition 0.153. For every Abelian group A, for every topological space (X, τ), the **sheaf of locally constant functions** from (X, τ) to A is the unique sheaf of commutative, unital rings on (X, τ) that associates to every nonempty open U the commutative, unital ring of all locally constant functions from U to A, i.e., all continuous from from U (with its subspace topology) to A with the discrete topology. This sheaf is denoted by $\underline{A}_{X,\tau}$, or just $\underline{A}_X$ when τ is understood. The group of sections of $\underline{A}_{X,\tau}$ on $\emptyset$ is the singleton subgroup of the zero in A (for definiteness). For every ordered pair (A, A') of Abelian groups, for every morphism of Abelian groups u from A to A', there is a unique induced morphism of Abelian groups $\underline{u}_{X,\tau}$ from $\underline{A}_{X,\tau}$ to $\underline{A}'_{X,\tau}$ defined by post-composition of locally constant functions to A with u.

Proposition 0.154. *Together, these operations define a functor from the category of Abelian groups to the category of sheaves of Abelian groups on (X, τ). This functor is left adjoint to the functor of global sections. This functor is an exact functor. It is faithful if and only if X is nonempty. It is fully faithful (and thus an equivalence to its essential image) if and only if X is nonempty and connected.*

If A is a commutative, unital ring, then also $\underline{A}_{X,\tau}$ has a unique structure of sheaf of commutative, unital ring such that, for every element p of X, the identification of A with the stalk $(\underline{A}_{X,\tau})_p$ is an isomorphism of commutative, unital rings. Thus, the functor above also defines a functor from the category of commutative, unital rings to the category of sheaves of commutative, unital rings on (X, τ), and this functor is left adjoint to the functor of global sections.

Proposition 0.155. *The functor that associates to every commutative, unital ring A the ringed space on the one-point topological space (say the singleton set whose unique element is $\{0_A\}$ for definiteness) with global section ring A is left adjoint to the (covariant) functor of global sections of the sheaf of rings from the category of ringed spaces to the opposite of the category of commutative, unital rings.*

More generally, for every topological space (X', τ'), for every commutative, unital ring A, for every ringed space $((X, \tau), \mathcal{R})$, for every continuous map f from (X, τ) to (X', τ'), extensions of f to a morphism $(f, f^{\#})$ of ringed spaces from $((X, \tau), \mathcal{R})$ to $((X', \tau'), \underline{A}_{X',\tau'})$ are equivalent to morphisms of commutative, unital rings from A to $\mathcal{R}(X)$. In particular, if (X', τ') is a one-point space, then morphisms of ringed spaces from $((X, \tau), \mathcal{R})$ to $((X', \tau'), \underline{A}_{X',\tau'})$ are equivalent to morphisms of commutative, unital rings from A to $\mathcal{R}(X)$.

Corollary 0.156. *The functor that associates to every topological space (X,τ) the ringed space $((X,\tau),\underline{\mathbb{Z}}_{X,\tau})$ is left adjoint to the forgetful functor pr_1. In particular, this is equivalence from the category of topological spaces to its essential image.*

The category of ringed spaces has some good categorical properties. In particular, it has all finite (fiber) products.

Definition 0.157. For every ordered triple $(((X,\tau),\mathcal{R}),((Y,\sigma),\mathcal{S}),((Z,\rho),\mathcal{T}))$ of ringed spaces, for every ordered pair $((f,f^{\#}),(g,g^{\#}))$ of a morphism $(f,f^{\#})$ of ringed spaces from $((X,\tau),\mathcal{R})$ to $((Z,\rho),\mathcal{T})$ and of a morphism $(g,g^{\#})$ of ringed spaces from $((Y,\sigma),\mathcal{S})$ to $((Z,\rho),\mathcal{T})$, the **fiber product** is the ringed space $((X\times_{f,Z,g}Y,\xi),\mathrm{pr}_1^{-1}\mathcal{R}\otimes_{h^{-1}\mathcal{T}}\mathrm{pr}_2^{-1}\mathcal{S})$, where ξ is the product topology on $X\times Y$ restricted to the subset $X\times_{f,Z,g}Y=(f\times g)^{-1}(\Delta_Z)$, and where h is the common composition $f\circ\mathrm{pr}_1=g\circ\mathrm{pr}_2$ from $X\times_{f,Z,g}Y$ to Z.

Thus, the category of ringed spaces has some good categorical properties relative to the category of topological spaces and to the category of commutative, unital rings. However, as geometric objects, the category of ringed spaces are defective in two ways related to multiplicative inverse of sections of the sheaf of commutative, unital rings.

Definition 0.158. For every ringed space $((X,\tau),\mathcal{R})$, for every element s of $\mathcal{R}(X)$, for every τ-open subset D of X, the subset D is the **invertible open** of s if (and only if), for every τ-open subset U of X, the restriction $\mathrm{res}^X_{\mathcal{R},U}(s)$ is multiplicatively invertible in $\mathcal{R}(U)$ if and only if U is a subset of D. In this case, we denote D by $D_{X,\mathcal{R}}(s)$. For every ringed space $((X,\tau),\mathcal{R})$, for every τ-closed subset C of X, the **associated ideal sheaf** of C in $((X,\tau),\mathcal{R})$ is the unique ideal sheaf in $\mathcal{R}$ that is everywhere locally generated by all local sections of $\mathcal{R}$ such that C is disjoint from the invertible open of s (where s is defined). In this case, we denote the ideal sheaf by $\mathcal{I}_{((X,\tau),\mathcal{R}),C}$, or just $\mathcal{I}_C$ if $((X,\tau),\mathcal{R})$ is understood.

First, for every closed subset C of X, the support of the Abelian sheaf $\mathcal{R}/\mathcal{I}_C$ may not contain C. Indeed, this already occurs for the ringed space $((\{0_A\},\mathcal{P}(\{0_A\})),A)$ that represents the (contravariant) functor $\mathrm{Hom}_{\mathbf{CRing}}(A,\Gamma(-))$ on the category of ringed spaces when C equals the entire set $\{0_A\}$ and A is not a local ring. Second, given a morphism $(f,f^{\#})$ from a ringed space $((X,\tau),\mathcal{R})$ to a ringed space $((X',\tau'),\mathcal{R}')$ and given an element s' of $\mathcal{R}'(X')$, even if $f^{\#}(s')$ is multiplicatively invertible, nonetheless s' may fail to be multiplicatively invertible, i.e., the f-preimage of $D_{X',\mathcal{R}'}(s')$ need not equal $D_{X,\mathcal{R}}(f^*s')$. Both of these defects are solved by passing to the (non-full) subcategory of locally ringed spaces.

Definition 0.159. For every ringed space $((X,\tau),\mathcal{R})$, the ringed space is a **locally ringed space** if (and only if), for every element p of X, the stalk $\mathcal{R}_p$ is a (nonzero) local ring. In that case, the quotient of $\mathcal{R}_p$ by the unique maximal ideal is the **residue field** of $((X,\tau),\mathcal{R})$, denoted by $\kappa_{(X,\tau),\mathcal{R}}(p)$, or just $\kappa(p)$ when $((X,\tau),\mathcal{R})$ is understood.

For every ordered pair $(((X,\tau),\mathcal{R}),((X',\tau'),\mathcal{R}'))$ of locally ringed spaces, for every morphism $(f,f^{\#})$ of ringed spaces from $((X,\tau),\mathcal{R})$ to $((X',\tau'),\mathcal{R}')$, the ordered pair $(f,f^{\#})$ is a **morphism**

of locally ringed spaces if (and only if), for every element p of X, the induced morphism of stalks $\mathcal{R}'_{f(p)} \to \mathcal{R}_p$ is a local homomorphism, i.e., the inverse image of the group of invertible elements of $\mathcal{R}_p$ equals the group of invertible elements of $\mathcal{R}'_{f(p)}$.

For every locally ringed space $((X,\tau),\mathcal{R})$, for every closed subset C of X, the support of $\mathcal{R}/\mathcal{I}_C$ contains the closed subset C. For the inverse image sheaf of $\mathcal{R}/\mathcal{I}_C$ on C, the ordered pair $(\iota_X^C, q^C_{(X,\tau),\mathcal{R}}$ is a morphism of locally ringed spaces, where ι_X^C is the inclusion from C (with its subspace topology) into (X,τ), and where $q^C_{(X,\tau),\mathcal{R}}$ is the unique quotient morphism of sheaves of rings on (X,τ) from $\mathcal{R}$ to $\mathcal{R}/\mathcal{I}_C$. Thus, for every closed subset C of a locally ringed space, there is a "natural" structure of locally ringed space on C such that the inclusion function extends to a morphism of locally ringed spaces. This is called the **induced reduced** structure of locally ringed space on C.

Also, for every morphism of locally ringed spaces $(f,f^\#)$ from a locally ringed space $((X,\tau),\mathcal{R})$ to a locally ringed space $((X',\tau'),\mathcal{R}')$, for every element s' of $\mathcal{R}'(X')$, the f-preimage of $D_{(X',\tau'),\mathcal{R}'}(s')$ equals $D_{(X,\tau),\mathcal{R}}(f^*s')$. Thus, for every open subset U' of X', for every element s' of $\mathcal{R}'(U')$, the f-preimage of the subset $D_{U',\mathcal{R}'|_{U'}}(s')$ equals $D_{f^{\text{pre}}(U'),\mathcal{R}|_{f^{\text{pre}}(U')}}(f^*s')$, i.e., f-preimage defines a morphism of partially ordered sets from the set of opens of the form $D_{U',\mathcal{R}'|_{U'}}(s')$ in U' to the set of opens of the form $D_{f^{\text{pre}}(U'),\mathcal{R}|_{f^{\text{pre}}(U')}}(s)$ in $f^{\text{pre}}(U')$. This is essential for compatibility of some later constructions of sheaves using the notion of $\mathcal{B}$-sheaves for a basis $\mathcal{B}$ of a topology, namely the basis $\mathcal{B}$ of the Zariski topology on U' defined whose elements are sets of the form $D_{U',\mathcal{R}'|_{U'}}(s')$.

Note that the initial object $(\emptyset,\{0\})$ of the category of ringed spaces is a locally ringed space, hence also an initial object in the category of locally ringed spaces. However, for a one-point topological space (X',τ') and a commutative, unital ring A, the ringed space $((X',\tau'),\underline{A}_{X',\tau'})$ is a locally ringed space if and only if A is itself a local ring. Moreover, even when A is a local ring, the contravariant set-valued Yoneda functor of the locally ringed space $((X',\tau'),\underline{A}_{X',\tau'})$ represents the contravariant set-valued functor $((X,\tau),\mathcal{R}) \mapsto \text{Hom}_{\mathbf{CRing}}(A,\mathcal{R}(X))$ if and only if the unique maximal ideal of A is also the unique prime ideal of A, i.e., every element of the maximal ideal is nilpotent. In particular, this holds if A is a field.

Example 0.160. For every field k (in which the additive identity 0_k is required to be distinct from the multiplicative identity 1_k), the singleton set whose unique element is $\{0_k\}$ has a unique topology, namely the entire power set of the singleton set. The functor that sends every sheaf of commutative, unital rings on this topological space to its commutative, unital ring of global sections is an equivalence of categories. Denote by $\mathcal{O}_{\text{Spec }k}$ the unique sheaf of commutative, unital rings whose global sections ring is k (and whose section ring on the empty set is $\{0_k\}$, for definiteness). This is a locally ringed space denoted Spec k. For every field extension $u : k \hookrightarrow K$, the ordered pair $(\text{const},\widetilde{u})$ is a morphism of locally ringed spaces from Spec K to Spec k consisting of the constant morphism from $\{\{0_K\}\}$ to $\{\{0_k\}\}$ on one-point topological spaces, and such that $\widetilde{u}$ is the unique morphism of sheaves of commutative, unital rings on $\{\{0_k\}\}$ whose induced morphism of commutative, unital rings on global sections is the field extension u. This is a morphism of locally ringed spaces.

The operations above define an equivalence of categories from the category of fields and field extensions to the full subcategory of the category of locally ringed spaces whose underlying topological space is a one-point space and whose ring of global sections is a field. In particular, for a finite extension of fields, $k \hookrightarrow K$, the group of automorphisms of Spec K as a locally ringed space over Spec k is canonically isomorphic to the group of automorphisms of the field K as a k-extension. So Galois groups can be extracted from this equivalence of categories.

Similarly, a morphism of ringed spaces from Spec k to a ringed space $((X,\tau),\mathcal{R})$ is equivalent to an element p of X (the image of the unique element of the one-point space), a prime ideal $\mathfrak{p}$ of the stalk $\mathcal{R}_p$, and a field extension $\text{Frac}(\mathcal{R}_p/\mathfrak{p}) \hookrightarrow k$. Compare this with local morphisms from Spec k to a locally ringed space $((X,\tau),\mathcal{R})$: an element p of X and a field extension $\kappa(p) \hookrightarrow k$. This suggests a method to construct a right adjoint of the faithful (non-full) functor from **LRS** to **RS**. Apparently this goes back to Chevalley, but it was described explictly by Monique Hakim. I learned of this from Danny Gillam.

Definition 0.161. For every ringed space $((X,\tau),\mathcal{R})$, the **associated locally ringed space** has underlying point set X^{loc} equal to the set of all pair $(p,\mathfrak{p})$ of an element p of X and a prime ideal $\mathfrak{p}$ of the stalk $\mathcal{R}_p$. For every element U of τ, for every element s of $\mathcal{R}(U)$, the **basic subset** $D(U,s)$ of X^{loc} is the set of all pairs $(p,\mathfrak{p})$ of an element p of U and a prime ideal $\mathfrak{p}$ of $\mathcal{R}_p$ that does not contain s_p. Note that $D(X,1)$ equals X^{loc}, that $D(\emptyset,0)$ equals the empty set, and, for every ordered pair $(D(U,s),D(V,t))$ of basic subsets, the intersection $D(U,s)\cap D(V,t)$ equals $D(U\cap V, s|_{U\cap V}\cdot t|_{U\cap V})$ (by the definition of prime ideal). Thus, these sets form the basis for a topoloy on X^{loc}, the **associated topology** τ^{loc}. The projection pr_1 from X^{loc} to X sending each $(p,\mathfrak{p})$ to p is continuous for these topologies. For every basic set $D(U,s)$, the **associated ring** is $\mathcal{R}(U)[1/s]$ with the natural ring homomorphism $\mathcal{R}(U)\to\mathcal{R}(U)[1/s]$.

Proposition 0.162. *For every ringed space $((X,\tau),\mathcal{R})$ there is a presheaf of rings on (X^{loc},τ^{loc}) extending the assignment above such that the natural ring homomorphisms above define a morphism of presheaves of rings from $pr_1^{-1}\mathcal{R}$ to this presheaf. For the associated sheaf $\mathcal{R}^{loc}$ of this presheaf, the ringed space $((X^{loc},\tau^{loc}),\mathcal{R}^{loc})$ is a locally ringed space. For every morphism of ringed spaces $(f,f^\#)$ from a ringed space $((X,\tau),\mathcal{R})$ to a ringed space $((T,\sigma),\mathcal{S})$, the associated morphism from $((X^{loc},\tau^{loc}),\mathcal{R}^{loc})$ to $((Y^{loc},\sigma^{loc}),\mathcal{S}^{loc})$ is a morphism of locally ringed spaces. The rule associating $((X^{loc},\tau^{loc}),\mathcal{R}^{loc})$ to every ringed space $((X,\tau),\mathcal{R})$ is a right adjoint of the faithful (non-full) functor from **LRS** to **RS**.*

Example 0.163. For every commutative, unital ring A, for the ringed space $((\{0_A\},\mathcal{P}(\{0_A\})),A)$, the associated locally ringed space Spec A has as point set the prime spectrum of A, i.e., the set of all prime ideals of A, has for topology the Zariski topology on the prime spectrum, i.e., the topology generated by basic subsets $D(s)$ for all elements s of A, and has for sheaf of commutative, unital rings the unique extension to all of τ the assignment to each basic subset $D(s)$ of the ring $A[1/s]$. The contravariant Yoneda functor $\text{Hom}_{\mathbf{LRS}}(-,\text{Spec }A)$ represents the contravariant set-valued functor on **LRS** assigning $\text{Hom}_{\mathbf{CRing}}(A,\mathcal{R}(X))$ to each locally ringed space $((X,\tau),\mathcal{R})$.

Schemes are those locally ringed spaces that are locally isomorphic to one of the locally ringed spaces above.

Definition 0.164. For every ringed space $((X, \tau), \mathcal{R})$ for every element U of τ, **induced ringed space structure** on U is $((U, \tau \cap \mathcal{P}(U)), \mathcal{R}|_U)$, where $\tau \cap \mathcal{P}(U)$ is the subspace topology on U and where $\mathcal{R}|_U$ is the restriction to U of the sheaf $\mathcal{R}$. Associated to the continuous inclusion function ι from U to X, by adjointness of pushforward and pullback, there is a natural morphism $\iota^\#$ of sheaves from $\mathcal{R}$ to $\iota_*(\mathcal{R}|_U)$. The pair $(\iota, \iota^\#)$ is a morphism of ringed spaces. If $((X, \tau), \mathcal{R})$ is a locally ringed space, then $(\iota, \iota^\#)$ is a morphism of locally ringed spaces.

Definition 0.165. For every locally ringed space $((X, \tau), \mathcal{R})$, the locally ringed space is a **scheme** if (and only if), for every element p of X, there exists an element U of τ_p such that the natural morphism of locally ringed spaces from $((U, \tau \cap \mathcal{P}(U)), \mathcal{R}|_U)$ to Spec $\mathcal{R}(U)$ is an isomorphism, i.e., the locally ringed space is locally isomorphic to locally ringed spaces of the form Spec A.

In fact, the first locally ringed spaces that were intensively studied were not of this form. Rather they were "spaces with functions."

Example 0.166. For every field k, for every topological space (X', τ'), for the sheaf $\underline{k}_{X',\tau'}$ of locally constant functions on X' to k, the ordered pair $((X', \tau'), \underline{k}_{X',\tau'})$ is a locally ringed space. For every continuous map f of topological spaces from (X, τ) to (X', τ'), the morphism $\underline{k}_f$ is the unique morphism of sheaves of commutative, unital rings from $\underline{k}_{X',\tau'}$ to $f_*\underline{k}_{X,\tau}$ defined by pre-composition with f. The ordered pair $(f, \underline{k}_f)$ is a morphism of locally ringed spaces from the locally ringed space $((X, \tau), \underline{k}_{X,\tau})$ to $((X', \tau'), \underline{k}_{X',\tau'})$.

Proposition 0.167. *For every field k, the covariant functor from the category of topological spaces to the category of locally ringed spaces over Spec k associating to every topological space (X, τ) the locally ringed space $((X, \tau), \underline{k}_{X,\tau})$ with its natural morphism to Spec k is right adjoint to the forgetful functor pr_1 from the category of locally ringed spaces over Spec k to the category of topological spaces.*

A bit more generally, for every topological space (X', τ'), for every element p of X', the composition to the stalk, $k \to \underline{k}_{X',\tau'}(X') \to (\underline{k}_{X',\tau'})_p$, is an isomorphism. Also, for every locally ringed space $((X, \tau), \mathcal{R})$, a morphism of locally ringed spaces from $((X, \tau), \mathcal{R})$ to $((X', \tau'), \underline{k}_{X',\tau'})$ is equivalent to a continuous map f from (X, τ) to (X', τ') together with a morphism of commutative, unital rings from k to $\mathcal{R}(X)$.

This shows that there are also many subcategories of the category of locally ringed spaces that are equivalent to the category of topological spaces, e.g., the functor $\underline{\mathbb{R}}$ gives one equivalence of the category of topological spaces to a full subcategory of the category of locally ringed spaces over Spec $\mathbb{R}$. One more such equivalence uses Godement's notion of sheaf of discontinuous sections.

Definition 0.168. For every field k, a locally ringed space $((X, \tau), \mathcal{R})$ over Spec k is a **space with k-functions** if (and only if) both, for every element p of X, the induced ring homomorphism from k to the residue field $\kappa(p)$ is an isomorphism so that we may identify the residue field $\kappa(p)$ of the stalk $\mathcal{R}_p$ with k, and for the induced morphism from the discrete system of $\mathcal{R}$ to the constant discrete system $\underline{k}_X$, the adjoint morphism from $\mathcal{R}$ to the Godement sheaf $G\underline{k}_X$ is injective, i.e., every local section of $\mathcal{R}$ is uniquely determined by all of its "values" in k.

Proposition 0.169. *For the full subcategory of all locally ringed spaces over Spec k whose objects are all spaces with k-functions, the restriction to this full subcategory of the forgetful functor pr_1 is faithful, i.e., every morphism $(f, f^{\#})$ of locally ringed spaces over Spec k between spaces with functions is uniquely determined by the continuous function f.*

Definition 0.170. For every topological space (X, τ), the sheaf $C^0_{(X,\tau),\mathbb{R}}$ is the subsheaf of the sheaf $G\underline{\mathbb{R}}_X$ associating to each τ-open subset U of X the subset $C^0_{\mathbb{R}}(U)$ of all functions from U (with its subspace topology) to $\mathbb{R}$ with the Euclidean topology that are continuous.

Proposition 0.171. *The functor that associates to every topological space (X, τ) the space with $\mathbb{R}$-functions $((X, \tau), C^0_{(X,\tau),\mathbb{R}})$ is an equivalence to its essential image whose inverse is the (restriction) of the forgetful morphism pr_1, i.e., for every continuous function f from a topological space (X, τ) to a topological space (X', τ'), there is a unique morphism of sheaves of rings $f^{\#}$ from $C^0_{(X',\tau'),\mathbb{R}}$ to $f_*C^0_{(X,\tau),\mathbb{R}}$ such that $(f, f^{\#})$ is a morphism of locally ringed spaces, namely the morphism of precomposition with f.*

There is another such equivalence that then leads to a fully faithful embedding of the category of all differentiable manifolds into the category of locally ringed spaces over Spec $\mathbb{R}$ and to a fully faithful embedding of the category of all complex analytic spaces into the category of locally ringed spaces over Spec $\mathbb{C}$.

Example 0.172. For every topological space (X, τ), denote by $C^0_{X,\tau}$ the sheaf of commutative, unital $\mathbb{R}$-algebras that associates to every open subset U of X the commutative, unital $\mathbb{R}$-algebra of all continuous functions from U (with its subspace topology) to $\mathbb{R}$ with the usual Euclidean topology. For every continuous map f from a topological space (X, τ) to a topological space (X', τ'), denote by C^0_f the morphism of sheaves of commutative, unital $\mathbb{R}$-algebras from $C^0_{X',\tau'}$ to $f_*C^0_{X,\tau}$ obtained by pre-composition with f.

For every topological space (X, τ), the ringed space $((X, \tau), C^0_{X,\tau})$ is a locally ringed space. For every continous map f from a topological space (X, τ) to a topological space (X', τ'), the ordered pair (f, C^0_f) is a morphism of locally ringed spaces from $((X, \tau), C^0_{X,\tau})$ to $((X', \tau'), C^0_{X',\tau'})$. As with the functor $\underline{\mathbb{R}}$, this defines a fully faithful embedding of the category of topological spaces into the category of locally ringed spaces over Spec $\mathbb{R}$. Moreover, this gives rise to an embedding of the category of differentiable manifolds.

Example 0.173. For every topological space (X, τ) together with a maximal atlas of charts that makes (X, τ) into a differentiable manifold, for every open subset U of X, the commutative, unital $\mathbb{R}$-subalgebra of all infinitely differentiable functions from U to $\mathbb{R}$ is denoted by $C^\infty_X(U)$ (where now the maximal atlas of charts is implicit). This defines a subsheaf of $\mathbb{R}$-algebras of the sheaf $C^0_{X,\tau}$. For every (infinitely) differentiable function f from a differentiable manifold X to a differentiable manifold X', the morphism C^0_f restricts to a morphism of sheaves of $\mathbb{R}$-algebras from $C^\infty_{X'}$ to $f_*C^\infty_X$.

The assignment to every differentiable manifold X of the locally ringed space $((X,\tau), C_X^\infty)$ defines a fully faithful embedding of the category of differentiable manifolds into the category of locally ringed spaces over Spec $\mathbb{R}$.

Example 0.174. For every differentiable manifold X together with a structure of complex manifold, for every open subset U of X, the commutative, unital $\mathbb{C}$-subalgebra of $\mathbb{C} \otimes_{\mathbb{R}} C_X^\infty(U)$ whose elements are all holomorphic functions from U to $\mathbb{C}$ is a sheaf of $\mathbb{C}$-subalgebras. These subsheaves are compatible with holomorphic maps between complex manifolds.

These assignments define a fully faithful embedding of the category of complex manifolds into the category of locally ringed spaces over Spec $\mathbb{C}$.