MAT 589 Additional Notes on Categories and Functors

Contents

1 Introduction

These are additional notes on categories and functors for this course. Some of the notes are cut-and-pasted from previous courses I taught about basic algebraic objects (semigroups, monoids, groups, acts and actions, associative rings, commutative rings, and modules), elementary language of category theory, and adjoint pairs of functors. Much of the notes are exercises working through the basic results about these definitions.

2 Algebraic Objects

Definition 2.1. A **semigroup** is a pair (G, m) of a set G and a binary relation,

$$m : G \times G \to G,$$

such that m is associative, i.e., the following diagram commutes,

$$\begin{array}{ccc} G \times G \times G & \xrightarrow{m \times \mathrm{Id}_G} & G \times G \\ {\scriptstyle \mathrm{Id}_G \times m}\downarrow & & \downarrow{\scriptstyle m} \\ G \times G & \xrightarrow[m]{} & G \end{array}.$$

The binary operation is equivalent to a set function,

$$L_\bullet : G \to \mathrm{Hom}_{\mathbf{Sets}}(G, G), \quad g \mapsto L_g,$$

such that for every $g, g' \in G$, the composition $L_g \circ L_{g'}$ equals $L_{m(g,g')}$, where $m(g, g')$ is defined to equal $L_g(g')$. When no confusion is likely, the element $m(g, g')$ is often denoted $g \cdot g'$.

For semigroups (G, m) and (G', m') a **semigroup morphism** from the first to the second is a set map

$$u : G \to G',$$

such that the following diagram commutes,

$$\begin{array}{ccc} G \times G & \xrightarrow{u \times u} & G' \times G' \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle m'} \\ G & \xrightarrow[u]{} & G' \end{array}.$$

The set of semigroup morphisms is denoted $\mathrm{Hom}_{\mathbf{Semigroups}}((G, m), (G', m'))$.

Definition 2.2. For a semigroup (G, m), an element e of G of is a **left identity element**, resp. **right identity element**, if for every $g \in G$, g equals $m(e, g)$, resp. g equals $m(e, g)$. An **identity element** is an element that is both a left identity element and a right identity element. A **monoid** is a triple (G, m, e) where (G, m) is a semigroup and e is an identity element. For monoids (G, m, e) and (G', m', e') a **monoid morphism** from the first monoid to the second is a semigroup morphism that preserves identity elements. The set of monoid morphisms is denoted $\mathrm{Hom}_{\mathbf{Monoids}}((G, m, e), (G', m', e'))$.

Example 2.3. For every semigroup (G, m), the **opposite semigroup** is (G, m^{opp}), where $m^{\mathrm{opp}}(g, g')$ is defined to equal $m(g', g)$ for every $(g, g') \in G \times G$. A left identity element of a semigroup is equivalent to a right identity element of the opposite semigroup. In particular, the opposite semigroup of a monoid is again a monoid.

Example 2.4. For every set I and for every collection $(G_\alpha, m_\alpha)_{\alpha \in I}$ of semigroups, for the Cartesian product set $G := \prod_{\alpha \in I} G_\alpha$ with its projections,

$$\mathrm{pr}_\alpha : G \to G_\alpha,$$

there exists a unique semigroup operation m on G such that every projection is a morphism of semigroups. Indeed, for every α, the composition

$$\mathrm{pr}_\alpha \circ m : G \times G \to G_\alpha$$

equals $m_\alpha \circ (\mathrm{pr}_\alpha \times \mathrm{pr}_\alpha)$. There exists an identity element e of (G, m) if and only if there exists an identity element e_α of (G_α, m_α) for every α, in which case e is the unique element such that $\mathrm{pr}_\alpha(e)$ equals e_α for every $\alpha \in I$.

Example 2.5. For every set S, the set $\mathrm{Hom}_{\mathbf{Sets}}(S, S)$ of set maps from S to itself has a structure of monoid where the semigroup operation is set composition, $(f, g) \mapsto f \circ g$, and where the identity element of the monoid is the identity function on S. For every semigroup (G, m), a **left act** of (G, m) on S is a semigroup morphism

$$\rho : (G, m) \to (\mathrm{Hom}_{\mathbf{Sets}}(S, S), \circ).$$

For every ordered pair $((S, \rho), (T, \pi))$ of sets with left G-acts, a **left G-equivariant map** from (S, ρ) to (T, π) is a set function $u : S \to T$ such that $u(\rho(g)s)$ equals $\pi(g)u(s)$ for every $g \in G$ and for every $s \in S$.

For each set S, a **right act** of G on S is a semigroup morphism ρ from (G, m) to the opposite semigroup of $\mathrm{Hom}_{\mathbf{Sets}}(S, S)$. Note, this is equivalent to a left act of the opposite semigroup $G^{\mathbf{opp}}$ on S. For every ordered pair $((S, \rho), (T, \pi))$ of sets with a right G-act, a **right G-equivariant map** is a set function $u : S \to T$ such that $u(s\rho(g))$ equals $u(s)\pi(g)$ for every $g \in G$ and for every $s \in S$. Note, this is equivalent to a left G^{opp}-equivariant map.

For an ordered pair $((G, m), (H, n))$ of semigroups, for each set S, a $G - H$-act on S is an ordered pair (ρ, π) of a left G-act on S, ρ, and a right H-act on S, π, such that $(\rho(g)s)\pi(h)$ equals

$\rho(g)(s\pi(h))$ for every $g \in G$, for every $h \in H$, and for every $s \in S$. This is equivalent to a left act on S by the product semigroup of G and H^{opp}. A $G-H$-equivariant map is a map that is left equivariant for the associated left act by $G \times H^{\text{opp}}$.

For every monoid (G, m, e), a **left action** of (G, m, e) on S is a monoid morphism from (G, m, e) to $\text{Hom}_{\mathbf{Sets}}(S, S)$. There is a category $G - \mathbf{Sets}$ whose objects are pairs (S, ρ) of a set S and a left action of (G, m, e) on S, whose morphisms are left G-equivariant maps, and where composition is usual set function composition. A **right action** is a monoid morphism from (G, m, e) to the opposite monoid of $\text{Hom}_{\mathbf{Sets}}(S, S)$. There is a category $\mathbf{Sets} - G$ whose objects are pairs (S, ρ) of a set S and a right action of (G, m, e) on S, whose morphisms are left G-equivariant maps, and where composition is usual set function composition. Finally, for every ordered pair $((G, m, e), (H, n, f))$ of monoids, a $G-H$-action on S is a $G-H$ act (ρ, π) such that each of ρ and π is an action. There is a category $G - H - \mathbf{Sets}$ whose objects are sets together with a $G-H$-action, whose morphisms are $G-H$-equivariant maps, and where composition is usual set function composition.

Definition 2.6. A semigroup $(G, \cdot)$ is called **left cancellative**, resp. **right cancellative**, if for every f, g, h in G, if $f \cdot g$ equals $f \cdot h$, resp. if $g \cdot f$ equals $h \cdot f$, then g equals h. A semigroup is **cancellative** if it is both left cancellative and right cancellative. A semigroup is **commutative** if for every $f, g \in G$, $f \cdot g$ equals $g \cdot f$, i.e., the identity function from G to itself is a semigroup morphism from G to the opposite semigroup. For an element f of a monoid, a **left inverse**, resp. **right inverse**, is an element g such that $g \cdot f$ equals the identity, resp. such that $f \cdot g$ equals the identity. An **inverse** of f is an element that is both a left inverse and a right inverse. An element f is **invertible** if it has an inverse.

Definition 2.7. A **group** is a monoid such that every element is invertible. The map that associates to each element the (unique) inverse element is the **group inverse map**, $i : G \to G$. If the monoid operation is commutative, the group is **Abelian**. A monoid morphism between groups is a **group homomorphism**, and the set of monoid morphisms between two groups is denoted $\mathbf{Hom}_{\mathbf{Groups}}((G, m, e), (G', m', e'))$. If both groups happen to be Abelian, this is also denoted $\mathbf{Hom}_{\mathbb{Z}-\mathbf{mod}}((G, m, e), (G', m', e'))$. In this case, this set is itself naturally an Abelian group for the operation that associates to a pair (u, v) of group homomorphisms the group homomorphism $u \cdot v$ defined by $(u \cdot v)(g) = m'(u(g), v(g))$.

Definition 2.8. An **associative ring** is an ordered pair $((A, +, 0), L_\bullet)$ of an Abelian group $(A, +, 0)$ and a homomorphism of Abelian groups,

$$L_\bullet : A \to \text{Hom}_{\mathbb{Z}-\mathbf{mod}}(A, A), \quad a \mapsto (L_a : A \to A)$$

such that for every $a, a' \in A$, the composition $L_a \circ L_{a'}$ equals $L_{a \cdot a'}$, where $a \cdot a'$ denotes $L_a(a')$. The set map $L_\bullet$ is equivalent to a biadditive binary operation,

$$\cdot : A \times A \to A, \quad (a, a') = a \cdot a',$$

that is also associative, i.e., for every a, a', a'' in A, the element $(a \cdot a') \cdot a''$ equals $a \cdot (a' \cdot a'')$. In particular, $(A, \cdot)$ is a semigroup. For associative rings $(A, +, 0, \cdot)$ and $(A', +', 0', \cdot')$, a **ring homomorphism** from the first to the second is a set function that is simultaneously a morphism of

Abelian groups from $(A,+,0)$ to $(A',+',0')$ and a morphism of semigroups from $(A,\cdot)$ to $(A',\cdot')$. For every associative ring $(A,+,0,\cdot)$, the **opposite ring** is $(A,+,0,\cdot^{\text{opp}})$.

Definition 2.9. An **associative, unital ring** is an associative ring such that the multiplication semigroup has an identity element, i.e., there exists a multiplicative identity. An **unital ring homomorphism** is a ring homomorphism that preserves multiplicative identities. For associative, unital rings $(A,+,0,\cdot,1)$ and $(A',+',0',\cdot',1')$, the set of unital ring homomorphisms from the first to the second is denoted $\text{Hom}_{\mathbf{UnitalRings}}((A,+,0,\cdot,1),(A',+',0',\cdot',1'))$, or just $\text{Hom}_{\mathbf{UnitalRings}}(A,A')$ if the identities and operations are understood. In particular, a **commutative, associative, unital ring** is an associative unital ring such that the multiplication monoid is commutative. The set of unital ring homomorphisms between two commutative, associative, unital rings is denoted $\text{Hom}_{\mathbf{CommUnitalRings}}(A,A')$.

Definition 2.10. For every Abelian group $(F,+,0)$, the Abelian group $\text{Hom}_{\mathbb{Z}\text{-}\mathbf{mod}}(F,F)$ of group homomorphisms from the group to itself has a structure of associative, unital ring where the multiplication operation is composition, and where the identity element is the identity homomorphism. For every associative ring $(R,+,0,\cdot)$, a (not necessarily unital) **left module structure** on F for the associative ring R is a morphism of associative rings from R to $\text{Hom}_{\mathbb{Z}\text{-}\mathbf{mod}}(F,F)$. A (not necessarily unital) **right module structure** is a morphism of associative rings from R to the opposite ring of $\text{Hom}_{\mathbb{Z}\text{-}\mathbf{mod}}(F,F)$. For every associative, unital ring $(R,+,0,\cdot,1)$, a (unital) **left module structure** on F for the associative unital ring R is a morphism of associative unital rings from R to $\text{Hom}_{\mathbb{Z}\text{-}\mathbf{mod}}(F,F)$. A (unital) **right module structure** on F for R is a morphism of associative unital rings from R to the the oppsite ring of $\text{Hom}_{\mathbb{Z}\text{-}\mathbf{mod}}(F,F)$. For every left module structure on F of R, the **opposite module** is the equivalent right module structure on F of the opposite ring of R.

For left R-modules F and F',

$$L_\bullet : R \to \text{Hom}_{\mathbb{Z}\text{-}\mathbf{mod}}(F,F), \quad L'_\bullet : R \to \text{Hom}_{\mathbb{Z}\text{-}\mathbf{mod}}(F',F'),$$

a **left R-module morphism** from F to F' is a group homomorphism,

$$\phi : (F,+,0) \to (F',+',0'),$$

such that for every $r \in R$, the following composition functions are equal,

$$\phi \circ L_r, L'_r \circ \phi : F \to F',$$

i.e., $\phi(r\cdot x)$ equals $r\cdot'\phi(x)$ for every $r\in R$ and for every $x\in F$. For right R-modules G and G', a **right R-module morphism** from G to G' is a left R^{opp}-module morphism from the opposite module G^{opp} to the opposite module $(G')^{\text{opp}}$.

3 Categories

Definition 3.1. A **category** $\mathcal{A}$ consists of (i) a "recognition principle" or "axiom list" (possibly depending on auxiliary sets) for determining whether a specified set a is an **object** of this category,

(ii) an assignment, for every ordered pair (a, a') of objects of $\mathcal{A}$, of a specified set $\mathrm{Hom}_{\mathcal{A}}(a, a')$, and
(iii) an assignment, for every ordered triple (a, a', a'') of objects of $\mathcal{A}$, of a specified set function

$$-\circ - : \mathrm{Hom}_{\mathcal{A}}(a', a'') \times \mathrm{Hom}_{\mathcal{A}}(a, a') \to \mathrm{Hom}_{\mathcal{A}}(a, a''), \quad (g, f) \mapsto g \circ f,$$

such that, for every object a of $\mathcal{A}$, there exists an element $\mathrm{Id}_a \in \mathrm{Hom}_{\mathcal{A}}(a, a)$ that is a left-right identity for $\circ$, and such that for every ordered 4-tuple (a, a', a'', a''') of objects of $\mathcal{A}$ and for every ordered triple

$$(g, f, e) \in \mathrm{Hom}_{\mathcal{A}}(a'', a''') \times \mathrm{Hom}_{\mathcal{A}}(a', a'') \times \mathrm{Hom}_{\mathcal{A}}(a, a'),$$

the elements $g \circ (f \circ e)$ and $(g \circ f) \circ e$ in $\mathrm{Hom}_{\mathcal{A}}(a, a''')$ are equal, i.e., $\circ$ is associative. The elements of $\mathrm{Hom}_{\mathcal{A}}(a, a')$ are **morphisms** from a to a' in $\mathcal{A}$. The set function $- \circ -$ is **composition** in $\mathcal{A}$.

Definition 3.2. For a category $\mathcal{A}$, for an ordered pair (a, a') of objects of $\mathcal{A}$, for an ordered pair of elements

$$(g, f) \in \mathrm{Hom}_{\mathcal{A}}(a, a') \times \mathrm{Hom}_{\mathcal{A}}(a', a),$$

if the composition $g \circ f \in \mathrm{Hom}_{\mathcal{A}}(a, a)$ equals Id_a, then g is a **left inverse** of f in $\mathcal{A}$ and f is a **right inverse** of g in $\mathcal{A}$. If g is both a left inverse of f and a right inverse of f, then g is an **inverse** of f in $\mathcal{A}$. An **isomorphism** in $\mathcal{A}$ is a morphism in $\mathcal{A}$ that has an inverse in $\mathcal{A}$.

Definition 3.3. For a category $\mathcal{A}$, an **initial object**, respectively a **terminal object** (or final object), is an object a such that for every object a', the set $\mathrm{Hom}_{\mathcal{A}}(a, a')$, resp. the set $\mathrm{Hom}_{\mathcal{A}}(a', a)$, is a singleton set. An object that is simultaneously an initial object and a terminal object is called a **zero object**.

Example 3.4. The category **Sets** has as objects all sets. For every ordered pair of sets, the associated set of morphisms in **Sets** is defined to be the set of all set functions from the first set to the second set. The composition in **Sets** is usual composition of functions. A set function has a left inverse, respectively a right inverse, an inverse, if and only if the set function is injective, resp. surjective, bijective. The empty set is an initial object. Every singleton set is a final object.

Example 3.5. For every category $\mathcal{A}$, for every object a of A, there is a monoid $H^a_a := \mathrm{Hom}_{\mathcal{A}}(a, a)$ whose semigroup operation is the categorical composition and whose monoid identity element is the categorical identity morphism of a. This is the **$\mathcal{A}$-monoid** of the object a. For every ordered pair (a, a') of objects of $\mathcal{A}$, the set $H^a_{a'} := \mathrm{Hom}_{\mathcal{A}}(a, a')$ the categorical composition defines a set map,

$$H^{a'}_{a'} \times H^a_{a'} \times H^a_a \to H^a_{a'}, \quad (u', f, u) \mapsto u' \circ f \circ u.$$

This is a $H^{a'}_{a'} - H^a_a$-action on $H^a_{a'}$. This is the **$\mathcal{A}$-action** of $H^{a'}_{a'} - H^a_a$ on $H^a_{a'}$. Finally, for every ordered triple (a, a', a'') of objects, the composition binary operation,

$$H^{a'}_{a''} \times H^a_{a'} \to H^a_{a''},$$

is a $H^{a''}_{a''} - H^a_a$-equivariant map that is $H^{a'}_{a'}$-balanced, i.e., for every $u'' \in H^{a''}_{a''}$, for every $g \in H^{a'}_{a''}$, for every $u' \in H^{a'}_{a'}$, for every $f \in H^a_{a'}$, and for every $u \in H^a_a$, we have,

$$u'' \circ (g \circ f) = (u'' \circ g) \circ f, \quad (g \circ u') \circ f = g \circ (u' \circ f), \quad (g \circ f) \circ u = g \circ (f \circ u).$$

This is the **$\mathcal{A}$-equivariant binary operation**.

Example 3.6. For every monoid, there is a category with a single object whose unique categorical monoid is the specified monoid. Every category with a single object is (strictly) equivalent to such a category for a monoid (unique up to non-unique isomorphism).

Example 3.7. For every monoid (G, m, e), for every set S together with a left G-action ρ, there is an associated category, sometimes denoted $[(S, \rho)/G]$, whose objects are the elements of S, and such that for every ordered pair $(s, s') \in S \times S$ the set of morphisms is

$$G_{s'}^{s} := \{g \in G | \rho(g)s = s'\}.$$

For every ordered triple $(s, s', s'') \in S \times S \times S$, the semigroup operation defines a binary operation,

$$G_{s''}^{s'} \times G_{s'}^{s} \to G_{s''}^{s}, \quad (g', g) \mapsto g'g.$$

The morphism of an element $g \in G_{s'}^{s}$ is left invertible, respectively right invertible, invertible, in this category if and only if the element g of the monoid is left invertible, resp. right invertible, invertible. This category has an initial object if and only if the left G-action is left G-equivariantly isomorphic to the left regular representation of the monoid G on itself, in which case every invertible element is an initial object. For the left regular representation, the category has a final object if and only if the monoid is a group (every element is invertible), in which case every object is both initial and final.

Example 3.8. For every ordered pair of monoids (G, G'), for every ordered pair (M, M') where M is a set with a specified $G' - G$-action and where M is a set with a specified $G - G'$-action, for every ordered pair of biequivariant and balanced binary operations,

$$\circ_{M',M} : M' \times M \to G, \quad \circ_{M,M'} : G \times M' \to G',$$

that are associative, i.e., for all $f, f_1, f_2 \in M$ and for all $f', f_1', f_2' \in M'$,

$$(f_1 \circ_{M,M'} f') \cdot f_2 = f_1 \cdot (f' \circ_{M',M} f_2), \quad (f_1' \circ_{M',M} f) \cdot f_2' = f_1' \cdot (f \circ_{M,M'} f_2'),$$

there is a category $\mathcal{A}$ with precisely two objects a and a' such that the categorical monoid G_a^a equals G, such that the categorical monoid $G_{a'}^{a'}$ equals G', such that the categorical $G' - G$-set $G_{a'}^{a}$ equals M, such that the categorical $G - G'$-set $G_a^{a'}$ equals M' and such that the composition binary relations are the specified binary operations $\circ_{M',M}$ and $\circ_{M,M'}$. Every category with precisely two objects is (strictly) equivalent to such a category for some datum as above, $(G, G', M, M', \circ_{M',m}, \circ_{M,M'})$.

Example 3.9. Continuing the previous example, let (S, S') be an ordered pair of sets, let

$$\rho : G \to \mathrm{Hom}_{\mathbf{Sets}}(S, S), \quad \rho' : G' \to \mathrm{Hom}_{\mathbf{Sets}}(S', S'),$$

be an ordered pair of left actions so that $\mathrm{Hom}_{\mathbf{Sets}}(S, S')$ has an induced $G' - G$ action and $\mathrm{Hom}_{\mathbf{Sets}}(S', S)$ has an induced $G - G'$ action. Let

$$\mu : M \to \mathrm{Hom}_{\mathbf{Sets}}(S, S'), \quad \mu' : M' \to \mathrm{Hom}_{\mathbf{Sets}}(S', S),$$

be an ordered pair of a $G' - G$ equivariant map and a $G - G'$ equivariant map that are compatible with the composition maps, i.e., for every $f \in M$ and for every $f' \in M'$,

$$\mu'(f') \circ \mu(f) = \rho(f' \circ_{M',M} f), \quad \mu(f) \circ \mu'(f') = \rho'(f \circ_{M,M'} f').$$

There is a category $[(S, S', \rho, \rho', \mu, \mu')/(G, G', M, M', \circ_{M',M}, \circ_{M,M'})]$ whose objects are elements s of S and elements s' of S', such that for every pair of elements $(s_1, s_2) \in S \times S$, resp. $(s'_1, s'_2) \in S' \times S'$, the morphisms are $G^{s_1}_{s_2}$, resp. $(G')^{s'_1}_{s'_2}$, as in $[(S,\rho)/G]$, resp. as in $[(S',\rho')/G']$, and such that for every $s \in S$ and for every $s' \in S'$, the morphisms from s to s', resp. the morphisms from s' to s, are those elements m of M with $\mu(m)s = s'$, resp. those elements m' of M' with $\mu'(m')s' = s$. The compositions are defined in the evident way.

Example 3.10. For every monoid M, for the associated category $\mathcal{A}$ with one object a whose monoid of self-morphisms equals M, the category $\mathrm{Hom}\mathcal{A}$ has objects (a, a, f) for every $f \in M$. For an ordered pair $(f, g) \in M \times M$, the set of morphisms from (a, a, f) to (a, a, g) equals the set of ordered pairs $(q, q') \in M \times M$ such that $g \cdot q$ equals $q' \cdot f$.

Example 3.11. For every semigroup (G, m), for every set S with a left G-act ρ on S, the identity function from S to itself is a left G-equivariant map from (S,ρ) to (S,ρ). Also, for every ordered triple $((S,\rho),(T,\pi),(U,\lambda))$ of sets with a left G-act, the composition of each left G-equivariant map from (S,ρ) to (T,π) with a left G-equivariant map from (T,π) to (U,λ) is a left G-equivariant map from (S,ρ) to (U,λ). Thus, there is a category $G - \mathbf{Act}$ whose objects are sets with a left G-act, (S,ρ), where for every ordered pair $((S,\rho),(T,\pi))$ of sets with a left G-act, $\mathrm{Hom}_{G-\mathbf{Act}}((S,\rho),(T,\pi))$ is the subset of $\mathrm{Hom}_{\mathbf{Sets}}(S,T)$ of left G-equivariant maps, and where composition is the usual set function composition. Similarly, there is a category $\mathbf{Act} - G$ whose objects are sets with a right G-act, (S,ρ), where for every ordered pair $((S,\rho),(T,\pi))$ of sets with a right G-act, $\mathrm{Hom}_{\mathbf{Act}-G}((S,\rho),(T,\pi))$ is the subset of $\mathrm{Hom}_{\mathbf{Sets}}(S,T)$ of right G-equivariant maps, and where composition is the usual set function composition. Finally, for every ordered pair $((G,m),(H,n))$ of semigroups, there is a category $G - H - \mathbf{Act}$ whose objects are sets S with a $G - H$-act, whose morphisms are $G - H$-equivariant maps, and where composition is usual set function composition.

Example 3.12. For every monoid (G, m, e), for every set S with a left G-action ρ on S, the identity function from S to itself is a left G-equivariant map from (S,ρ) to (S,ρ). Also, for every ordered triple $((S,\rho),(T,\pi),(U,\lambda))$ of sets with a left G-action, the composition of each left G-equivariant map from (S,ρ) to (T,π) with a left G-equivariant map from (T,π) to (U,λ) is a left G-equivariant map from (S,ρ) to (U,λ). Thus, there is a category $G - \mathbf{Sets}$ whose objects are sets with a left G-action, (S,ρ), where for every ordered pair $((S,\rho),(T,\pi))$ of sets with a left G-action, $\mathrm{Hom}_{G-\mathbf{Sets}}((S,\rho),(T,\pi))$ is the subset of $\mathrm{Hom}_{\mathbf{Sets}}(S,T)$ of left G-equivariant maps, and where composition is the usual set function composition. Similarly, there is a category $\mathbf{Sets} - G$ whose objects are sets with a right G-action, (S,ρ), where for every ordered pair $((S,\rho),(T,\pi))$ of sets with a right G-action, $\mathrm{Hom}_{\mathbf{Sets}-G}((S,\rho),(T,\pi))$ is the subset of $\mathrm{Hom}_{\mathbf{Sets}}(S,T)$ of right G-equivariant maps, and where composition is the usual set function composition. Finally, for every ordered pair $((G, m, e), (H, n, f))$ of monoids, there is a category $G - H -$ **Sets** whose objects are sets S with a $G - H$-action, whose morphisms are $G - H$-equivariant maps, and where composition is usual set function composition.

Example 3.13. The category **Semigroups**, respectively **Monoids**, **Groups**, **Rings**, **UnitalRings**, **CommUnitalRings**, has as objects all semigroups, respectively all monoids (semigroups that have an identity element), all groups, all associative, unital rings, all associative, commutative, unital rings. For every ordered pair of objects, the set of morphisms in each of these categories is the set of all set maps between the objects that preserve the algebraic operations (and identity elements, when these are part of the structure). Composition is usual composition of set maps. In each of these categories, a morphism is an isomorphism if and only if it is a bijection, in which case the set-theory inverse of the bijection is also the inverse in the category. Each of these categories has a terminal object consisting of any object whose underlying point set is a singleton set. The trivial object is also an initial object, hence a zero object, in **Monoids** and **Groups**. The commutative, unital ring $\mathbb{Z}$ is an initial object in **UnitalRings** and **CommUnitalRings**.

Example 3.14. For every associative, unital ring A, the category $A -$ **mod**, resp. **mod** $- A$, is the category whose objects are left A-modules, resp. right A-modules, and whose morphisms are homomorphisms of left A-modules, resp. of right A-modules. Composition is usual composition of set functions. The zero module is both an initial object and a terminal object, i.e., a zero object.

Definition 3.15. For a commutative, unital ring R, an $R-$**mod enriched category** is a category $\mathcal{A}$ together with a specified structure of (left-right) R-module on each set of morphisms such that each composition set map is R-bilinear.

Definition 3.16. For every category $\mathcal{A}$, the **arrow category of of $\mathcal{A}$** is the category $\mathcal{A}^{\rightarrow}$ whose objects are ordered triples (a_0, a_1, f) of objects a_0 and a_1 of $\mathcal{A}$ and an element $f \in \mathrm{Hom}_{\mathcal{A}}(a_0, a_1)$, such that for every ordered pair $((a_0, a_1, f), (a'_0, a'_1, f'))$ of objects the set of morphisms is

$$\mathrm{Hom}_{\mathcal{A}^{\rightarrow}}((a_0, a_1, f), (a'_0, a'_1, f')) = \{(q_0, q_1) \in \mathrm{Hom}_{\mathcal{A}}(a_0, a'_0) \times \mathrm{Hom}_{\mathcal{A}}(a_1, a'_1) | f' \circ q_0 = q_1 \circ f\},$$

and for every ordered triple of objects, $((a_0, a_1, f), (a'_0, a'_1, f'), (a''_0, a''_1, f''))$, for every morphism (q_0, q_1) from (a_0, a_1, f) to (a'_0, a'_1, f'), and for every morphism (q'_0, q'_1) from (a'_0, a'_1, f') to (a''_0, a''_1, f''), the composition $(q'_0, q'_1) \circ (q_0, q_1)$ is defined to be $(q'_0 \circ q_0, q'_1 \circ q_1)$.

Definition 3.17. For every category, the **opposite category** has the same objects, but the set of morphisms from a first object to a second object in the opposite category is defined to be the set of morphisms from the second object to the first object in the original category. With this definition, composition in the opposite category is defined to be composition in the original category, but in the opposite order. For every object, the associated categorical monoid of that object in the opposite category equals the opposite monoid of the categorical monoid in the original category. For every ordered pair of objects, the categorical biaction for the opposite category is the opposite biaction of the categorical biaction of the original category.

Example 3.18. For every commutative, unital ring R, for every category enriched over $R-\mathbf{mod}$, every categorical monoid has an associated structure of an associative, unital, central R-algebra such that the algebra product is the monoid operation. Conversely, for every central R-algebra, there is a category enriched over R-$\mathbf{mod}$ with precisely one object whose central R-algebra of self-morphisms is the specified central R-algebra. Also, for the opposite category enriched over $R-\mathbf{mod}$, every central R-algebra of self-morphisms of an object is the opposite central R-algebra of that in the original category.

Example 3.19. For every commutative, unital ring R, for every category $\mathcal{A}$ enriched over $R-\mathbf{mod}$, for every ordered pair (a,a') of objects of $\mathcal{A}$ with the associated central R-algebra structures on the monoids H_a^a and $H_{a'}^{a'}$, categorical composition defines an associated structure of R-central $H_{a'}^{a'}-H_a^a$-bimodule on $H_{a'}^a$ inducing the categorical $H_{a'}^{a'}-H_a^a$-action. Also, for every ordered triple (a,a',a'') of objects of $\mathcal{A}$, the composition binary operation defines an R-central $H_{a''}^{a''}-H_a^a$-bimodule homomorphism,

$$H_{a''}^{a'}\otimes_{H_{a'}^{a'}} H_{a'}^a \to H_{a''}^a.$$

Conversely, for every ordered pair of central R-algebras (H,H'), for every ordered pair (S,T) of an R-central $H'-H$ bimodule S, i.e., a left $H'\otimes_R H^{\text{opp}}$-module, and an R-central $H-H'$ bimodule T, for every ordered pair of balanced bimodule homomorphisms,

$$\circ_{T,S}: T\otimes_{H'} S\to H, \quad \circ_{S,T}: S\otimes_H T\to H',$$

that are associative, there is a category $\mathcal{A}$ enriched over $R-\mathbf{mod}$ with precisely two objects a and a' such that the categorical central R-algebra H_a^a equals H, such that the categorical central R-algebra $H_{a'}^{a'}$ equals H', such that the categorical R-central $H'-H$ bimodule $H_{a'}^a$ equals S, such that categorical R-central $H-H'$ bimodule $H_a^{a'}$ equals T, and such that the composition binary operationss are the specified binary operations $\circ_{G,F}$ and $\circ_{F,G}$.

Also, for the opposite category enriched over $R-\mathbf{mod}$, the R-central algebras of self-morphisms of an object are replaced by their opposites, and the opposite of the R-central $H'-H$ bimodule structure on $H_{a'}^a$ is the categorical R-central $H^{\text{opp}}-(H')^{\text{opp}}$ bimodule structure of the opposite category.

Example 3.20. For every partially ordered set $(S,\leq)$, there is a category whose objects are the elements of S, and such that for every ordered pair $(s,s')\in S\times S$, the set of morphisms is empty unless $s\leq s'$, in which case the set of morphisms is a singleton set. There is a unique composition law consistent with these sets of morphisms. The opposite category is the category associated to the opposite partially ordered set $(S,\geq)$.

Definition 3.21. For a category $\mathcal{A}$, a **subcategory** of $\mathcal{A}$ is a category $\mathcal{B}$ such that every object of $\mathcal{B}$ is an object of $\mathcal{A}$, such that for every ordered pair (b,b') of objects of $\mathcal{B}$, the set $\text{Hom}_{\mathcal{B}}(b,b')$ is a subset of $\text{Hom}_{\mathcal{A}}(b,b')$, and such that for every ordered triple (b,b',b'') of objects of $\mathcal{B}$, the composition in $\mathcal{B}$ is the restriction of composition in $\mathcal{A}$. A subcategory $\mathcal{B}$ of $\mathcal{A}$ is **full** if for every ordered pair (b,b') of objects of $\mathcal{B}$, the subset $\text{Hom}_{\mathcal{B}}(b,b')$ equals all of $\text{Hom}_{\mathcal{A}}(b,b')$.

Similarly, for a commutative, unital ring R and a category $\mathcal{A}$ enriched over $R-\mathbf{mod}$, an $R-\mathbf{mod}$ **enriched subcategory** is a subcategory $\mathcal{B}$ of $\mathcal{A}$ such that every subset $\mathrm{Hom}_{\mathcal{B}}(b,b')$ of $\mathrm{Hom}_{\mathcal{A}}(b,b')$ is an R-submodule.

Example 3.22. For every monoid M, for the associated category with one object whose categorical monoid equals M, the subcategories are precisely the categories with one object associated to the submonoids of M. For every commutative, unital ring R, for every R-central algebra A, for the associated category enriched over $R-\mathbf{mod}$ that has precisely one object whose categorical central R-algebra equals A, the $R-\mathbf{mod}$ enriched subcategories are precisely those associated to R-subalgebras of A. For every partially ordered set $(S,\le)$, the subcategories of the associated category are precisely the categories of pairs $(T,\le_T)$ of a subset T of S and a partial ordering $\le_T$ on T such that the inclusion map is order-preserving, $(T,\le_T)\to(S,\le)$. The subcategory is full if and only if $\le_T$ is the restriction of $\le$ to T.

4 Functors

Definition 4.1. For every pair of categories $\mathcal{A}$ and $\mathcal{B}$, a **covariant functor** F from $\mathcal{A}$ to $\mathcal{B}$ is defined to be a rule that associates to every object a of $\mathcal{A}$ an object $F(a)$ of $\mathcal{B}$ and that associates to every ordered pair of objects (a,a') of $\mathcal{A}$ a set map

$$F_{a,a'}:\mathrm{Hom}_{\mathcal{A}}(a,a')\to\mathrm{Hom}_{\mathcal{B}}(F(a),F(a')),$$

such that for every object a of $\mathcal{A}$, $F_{a,a}(\mathrm{Id}_a)$ equals $\mathrm{Id}_{F(a)}$, and such that for every triple of objects (a,a',a'') of $\mathcal{A}$,

$$F_{a,a''}(g\circ f)=F_{a',a''}(g)\circ F_{a,a'}(f),\quad \forall(g,f)\in\mathrm{Hom}_{\mathcal{A}}(a',a'')\times\mathrm{Hom}_{\mathcal{A}}(a,a').$$

The functor is **faithful**, resp. **fully faithful**, if every set map $F_{a,a'}$ is injective, resp. bijective. The functor is **essentially surjective** if every object of $\mathcal{B}$ is isomorphic to $F(a)$ for an object of $\mathcal{A}$. The functor is an **equivalence** if it is fully faithfuly and essentially surjective.

A **contravariant functor** from $\mathcal{A}$ to $\mathcal{B}$ is a covariant functor from the opposite category $\mathcal{A}^{\mathrm{opp}}$ to $\mathcal{B}$.

Definition 4.2. For every triple of categories $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, for every covariant functor F from $\mathcal{A}$ to $\mathcal{B}$ and for every covariant functor G from $\mathcal{B}$ to $\mathcal{C}$, the **composition functor** $G\circ F$ from $\mathcal{A}$ to $\mathcal{C}$ is the covariant functor associating to every object a of $\mathcal{A}$ the object $G(F(a))$ of $\mathcal{C}$, and associating to every ordered pair of objects (a,a') of $\mathcal{A}$, the composition set map,

$$G_{F(a),F(a')}\circ F_{a,a'}:\mathrm{Hom}_{\mathcal{A}}(a,a')\to\mathrm{Hom}_{\mathcal{B}}(F(a),F(a'))\mathrm{Hom}_{\mathcal{C}}(G(F(a)),G(F(a'))).$$

For every category $\mathcal{A}$, the **identity functor** from $\mathcal{A}$ to $\mathcal{A}$ is the rule associating every object to itself, and sending each set of morphisms to itself by the identity set map.

Definition 4.3. For every triple of categories $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, for every pair of covariant functors, $F : \mathcal{A} \to \mathcal{C}$ and $G : \mathcal{B} \to \mathcal{C}$, the **comma category**, $F \downarrow G$, has as objects ordered triples (a, b, u) of an object a of $\mathcal{A}$, an object b of $\mathcal{B}$, and a $\mathcal{C}$-morphism $u : F(a) \to G(b)$. For an ordered pair of objects, $((a, b, u), (a', b', u'))$, a morphism in the comma category is an ordered pair (q, r) of $q \in \mathrm{Hom}_{\mathcal{A}}(a, a')$ and $r \in \mathrm{Hom}_{\mathcal{B}}(b, b')$ such that $u' \circ F(q)$ equals $G(r) \circ u$ in $\mathrm{Hom}_{\mathcal{C}}(F(a), G(b'))$. Composition is defined in the evident way. In particular, the arrow category of $\mathcal{C}$ is the comma category when $\mathcal{A}$ equals $\mathcal{B}$ equals $\mathcal{C}$ and each of F and G is the identity functor on $\mathcal{C}$. In general, there is a **domain functor** or **source functor**, $F \downarrow G \to \mathcal{A}$, associating to every object (a, b, u) the $\mathcal{A}$-object a and associating to every morphism (q, r) the $\mathcal{A}$-morphism q. There is also a **codomain functor** or **target functor**, $F \downarrow G \to \mathcal{B}$, associating to every object (a, b, u) the $\mathcal{B}$-object b and associating to every morphism (q, r) the $\mathcal{B}$-morphism r. Finally, there is an **arrow functor**, $F \downarrow G \to \mathcal{C}^{\to}$ associating to every object (a, b, u) the $\mathcal{C}^{\to}$-object $(F(a), G(b), u)$ and associating to every morphism (q, r) the $\mathcal{C}^{\to}$-morphism $(F(q), G(r))$.

Definition 4.4. For a category $\mathcal{A}$, a full subcategory is **skeletal** if every object of $\mathcal{A}$ is isomorphic to an object of the subcategory. If there exists a skeletal subcategory whose objects are indexed by a set, then $\mathcal{A}$ is a **small category**. If the objects of $\mathcal{A}$ form a set, then $\mathcal{A}$ is a **strictly small category**.

Example 4.5. Let **FinSets** be the full subcategory of **Sets** whose objects are the finite subsets. Let $\mathcal{B}$ be the full subcategory whose objects are the subsets $[1, n] = \{1, \dots, n\}$ of $\mathbb{Z}_{\geq 1}$ for every integer $n \geq 0$. Then $\mathcal{B}$ is a strictly small category that is a skeletal subcategory of **FinSets**, but **FinSets** is not a strictly small category.

Example 4.6. For every partially ordered set $(S, \leq)$ and for every partially ordered set $(T, \leq)$, a functor from the associated category of $(S, \leq)$ to the associated category of $(T, \leq)$ is equivalent to a order-preserving function from S to T. Such a functor is always faithful. It is full if and only if the function is **strict**, i.e., for every $(s, s') \in S \times S$, the image pair $(t, t') \in T \times T$ satisfies $t \leq t'$ if and only if $s \leq s'$. The functor is essentially surjective if and only if the set function is surjective.

Example 4.7. For every pair of categories $\mathcal{A}$ and $\mathcal{B}$, for every covariant functor F from $\mathcal{A}$ to $\mathcal{B}$, the **opposite functor** F^{opp} from the opposite category $\mathcal{A}^{\mathrm{opp}}$ to the opposite category $\mathcal{B}^{\mathrm{opp}}$ associates to every object a of $\mathcal{A}^{\mathrm{opp}}$ the object $F(a)$ of $\mathcal{B}^{\mathrm{opp}}$, and associates to every ordered pair (a, a') of objects of $\mathcal{A}^{\mathrm{opp}}$ the set function $F_{a',a}$ of (a', a). For a triple of categories $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, for covariant functors $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$, the functor $(G \circ F)^{\mathrm{opp}}$ is the composition $G^{\mathrm{opp}} \circ F^{\mathrm{opp}}$, and the opposite functor of the identity functor is the identity functor of the opposite category. The opposite functor is faithful, respectively full, essentially surjective if and only if the original functor is faithful, resp. full, essentially surjective. Finally, the opposite functor of F^{opp} is the original functor F.

Example 4.8. For every set a, denote by $\mathcal{P}(a)$ the power set of a, i.e., the set whose elements are all subsets of a. For every set map $f : a \to a'$, define $\mathcal{P}_{a,a'}(f)$ to be the set map from $\mathcal{P}(a)$ to $\mathcal{P}(a')$ associating to every subset b of a the image subset $f(b)$ of a'. Similarly, define $\mathcal{P}^{a',a}(f)$ to be the set map from $\mathcal{P}(a')$ to $\mathcal{P}(a)$ that associates to every subset b' of b the preimage subset $f^{\mathrm{pre}}(b')$

of a. This defines a covariant functor $\mathcal{P}_*$ from **Sets** to itself and a contravariant functor $\mathcal{P}^*$ from **Sets** to itself. These functors preserve the full subcategory **FinSets**, but they do not preserve the skeletal subcategory $\mathcal{B}$.

Example 4.9. There is a forgetful functor from **Groups** to **Sets** that forgets the group structure. Similarly, there is a forgetful functor from $R-\mathbf{mod}$ to **Groups** that remembers only the additive group structure on the R-module. Similarly, there is a forgetful functor from **Rings** to $\mathbb{Z}$-**mod** that remembers only the additive group structure. There is a forgetful functor from **UnitalRings** to **Rings**. All of these are faithful functors. The category **CommUnitalRings** is a full subcategory of **UnitalRings**.

Example 4.10. For every ordered pair of monoids, the covariant functors between the associated categories are naturally equivalent to the morphisms of monoids. For every commutative, unital ring R, for every ordered pair of central R-algebras, the covariant functors between the associated categories that are R-linear on sets of morphisms are naturally equivalent to the R-algebra homomorphisms between these central R-algebras.

Definition 4.11. For every category $\mathcal{A}$ and for every object a of $\mathcal{A}$, the **Yoneda covariant functor** of a is the covariant functor,

$$h^a : \mathcal{A} \to \mathbf{Sets}, \quad h^a(a') = \mathrm{Hom}_{\mathcal{A}}(a, a').$$

For every ordered pair of objects (a', a''), for every morphism $g \in \mathrm{Hom}_{\mathcal{A}}(a', a'')$, and for every element $f \in h^a(a')$, i.e., for every morphism $f \in \mathrm{Hom}_{\mathcal{A}}(a, a')$, composition defines an element $g \circ f$ in $h^a(a'')$. This defines a set function,

$$h^a_{a',a''} : \mathrm{Hom}_{\mathcal{A}}(a', a'') \to \mathrm{Hom}_{\mathbf{Sets}}(h^a(a'), h^a(a'')), \quad g \mapsto (f \mapsto g \circ f).$$

In particular, $h^a_{a',a'}$ sends the identity morphism of a' to the identity set function of $h^a(a')$. Also, since composition is associative, the set maps $h^a_{a',a''}$ respect composition. Altogether, this defines a covariant functor.

Similarly, the **Yoneda contravariant functor** of a'' is the contravariant functor,

$$h_{a''} : \mathcal{A} \to \mathbf{Sets}, \quad h_{a''}(a') = \mathrm{Hom}_{\mathcal{A}}(a', a'').$$

Each set map $h^{a,a'}_{a''}$ is defined by sending $g \in \mathrm{Hom}_{\mathcal{A}}(a, a')$ to the set map

$$h_{a''}(a') \to h_{a''}(a), \quad g \mapsto g \circ f.$$

Example 4.12. For every partially ordered set $(S, \leq)$, for the associated category, for every element $a \in S$, the Yoneda functor h_a associates to each element a' the empty set unless $a' \leq a$, in which case it associates a singleton set. Similarly, the Yoneda functor $h^{a'}$ associates to each element a the empty set unless $a' \leq a$.

5 Natural Transformations

Definition 5.1. For categories $\mathcal{A}$ and $\mathcal{B}$, for covariant functors F and G from $\mathcal{A}$ to $\mathcal{B}$, a **natural transformation** from F to G is a rule θ that associates to every object a of $\mathcal{A}$ an element $\theta_a \in \mathrm{Hom}_{\mathcal{B}}(F(a), G(a))$ such that for every ordered pair of objects (a, a') of $\mathcal{A}$, for every element $f \in \mathrm{Hom}_{\mathcal{A}}(a, a')$, the following compositions of morphisms in $\mathcal{B}$ are equal,

$$\theta_{a'} \circ F(f) = G(f) \circ \theta_a.$$

For covariant functors, F, G and H from $\mathcal{A}$ to $\mathcal{B}$, for natural transformations from F to G and from G to H, the (vertical) **composite natural transformation** from F to H is defined in the evident way. Also, for every functor F, the identity natural transformation from F to itself is defined in the evident way. An invertible natural transformation (with respect to composition of natural transformations and the identity natural transformations) is called a **natural equivalence** or **natural isomorphism**. This holds if and only if θ_a is an invertible morphism for every object a, in which case the inverse natural transformation associates to a the inverse of θ_a.

For every natural transformation θ between covariant functors $F, G : \mathcal{A} \to \mathcal{B}$, for every natural transformation θ' between covariant functors $F', G' : \mathcal{B} \to \mathcal{C}$, the **horizontal composition natural transformation**, or **Godement product**, is the natural transformation $\theta * \theta' : F' \circ F \to G' \circ G$ associating to every object a of $\mathcal{A}$ the $\mathcal{C}$-morphism,

$$\theta'_{G(a)} \circ_{\mathcal{C}} F'_{F(a),G(a)}(\theta_a) = (\theta * \theta')_a = G'_{F(a),G(a)}(\theta_a) \circ \theta'_{F(a)}.$$

This is associative in θ and θ'. For every covariant functor $I : \mathcal{B} \to \mathcal{C}$, the I-**pushforward natural transformation**, $I_*\theta = \theta * \mathrm{Id}_I$, is the natural transformation between the composition functors $I \circ F, I \circ G : \mathcal{A} \to \mathcal{C}$ associating to every object a of $\mathcal{A}$ the morphism $I_{F(a),G(a)}(\theta_a)$ in $\mathrm{Hom}_{\mathcal{C}}(I(F(a)), I(G(a)))$. Similarly, for every covariant functor $E : \mathcal{D} \to \mathcal{A}$, the E-**pullback natural transformation**, $E^*\theta = \mathrm{Id}_E * \theta$, is the natural transformation between the composition functors $F \circ E, G \circ E : \mathcal{D} \to \mathcal{B}$ that associates to every object d of $\mathcal{D}$ the morphism $\theta_{E(d)}$ in $\mathrm{Hom}_{\mathcal{B}}(F(E(d)), G(E(d)))$. Of course the Godement product can be expanded in terms of pushforward, pullback and vertical composition,

$$G^*\theta' \circ (F')_*\theta = \theta * \theta' = G'_*\theta \circ F^*\theta'.$$

In particular,

$$I_*(E^*(\theta)) = (\mathrm{Id}_E * \theta) * \mathrm{Id}_I = \mathrm{Id}_E * (\theta * \mathrm{Id}_I) = E^*(I_*(\theta)).$$

Example 5.2. For every partially ordered set $(S, \le)$, for every partially ordered set $(T, \le)$, for every pair of order-preserving functions,

$$F, G : (S, \le) \to (T, \le),$$

there exists a natural transformation from F to G if and only if $F \le G$, i.e., $F(s) \le G(s)$ for every $s \in S$. In this case, the natural transformation is unique. Notice that $F \le F$, and if both $F \le G$

and $G \leq H$ for order-preserving functions F, G, and H, then also $F \leq H$, reflecting composition of natural transformations. If $F \leq G$, then the natural transformation is a natural equivalence if and only if the set functions are equal. For order-preserving functions $I : (T, \leq) \to (U, \leq')$ and $E : (R, \leq') \to (S, \leq)$, if $F \leq G$, then also $I \circ F \leq' I \circ G$ and $F \circ E \leq G \circ E$, reflecting the I-pushforward and E-pullback of the natural transformation.

Example 5.3. For categories $\mathcal{A}$ and $\mathcal{B}$, for covariant functors F and G from $\mathcal{A}$ to $\mathcal{B}$, for every natural transformation θ from F to G, the **opposite natural transformation** θ^{opp} from G^{opp} to F^{opp} associates to every object a of $\mathcal{A}$ the element θ_a in $\text{Hom}_{\mathcal{B}}(F(a), G(a)) = \text{Hom}_{\mathcal{B}^{\text{opp}}}(G(a), F(a))$. The natural transformation θ is a natural equivalence if and only if θ^{opp} is a natural equivalence. The opposite natural transformation of θ^{opp} is the original natural transformation θ. The opposite natural transformation is compatible with vertical composition and Godement product.

Example 5.4. Let $F : \mathcal{A} \to \mathcal{C}$ and $G : \mathcal{B} \to \mathcal{C}$ be covariant functors, and let $F \downarrow G$ be the comma category with its domain functor, $s : F \downarrow G \to \mathcal{A}$, and its codomain functor $t : F \downarrow G \to \mathcal{B}$. For the composite functors, $F \circ s, G \circ t : F \downarrow G \to \mathcal{C}$, there is a natural transformation,

$$\theta : F \circ s \Rightarrow G \circ t, \quad \theta_{(a,b,u)} = u.$$

For every category $\mathcal{D}$, for every functor $E : \mathcal{D} \to F \downarrow G$, there is a triple (S, T, η) of functors,

$$S = s \circ E : \mathcal{D} \to \mathcal{A}, \quad T = t \circ E : \mathcal{D} \to \mathcal{B},$$

and a natural transformation $\eta = E^*\theta$ from $F \circ S$ to $G \circ T$. Conversely, for every natural transformation (S, T, η) as above, there is a unique functor $E : \mathcal{D} \to F \downarrow G$ such that $s \circ E$ (strictly) equals S, such that $t \circ E$ (strictly) equals T, and such that $E^*\theta$ equals η.

Example 5.5. As a special case of the preceding, for every category $\mathcal{A}$, for every category $\mathcal{D}$, a covariant functor to the arrow category,

$$E : \mathcal{D} \to \mathcal{A}^{\to},$$

is (strictly) equivalent to an ordered pair (S, T) of covariant functors,

$$S : \mathcal{D} \to \mathcal{A}, \quad T : \mathcal{D} \to \mathcal{A},$$

and a natural transformation $\eta : S \Rightarrow T$.

Example 5.6. For every set a, denote by $\theta_a : a \to \mathcal{P}(a)$ the set function that associates to every element $x \in a$ the singleton set of x. This defines a natural transformation from the identity functor of **Sets**, resp. **FinSets**, to the covariant functor $\mathcal{P}_*$.

Example 5.7. For every category $\mathcal{A}$, for every covariant functor $F : \mathcal{A} \to \textbf{Sets}$, for every object a of $\mathcal{A}$, for every element $t \in F(a)$, for every object a' of $\mathcal{A}$, for every element $f \in \text{Hom}_{\mathcal{A}}(a, a')$, denote by $f_*(t)$ the element of $F(a')$ that is the image of t under $F_{a,a'}(f)$. This defines a set function,

$$\widetilde{t}_{a'} : h^a(a') \to F(a'), \quad f \mapsto f_*(t).$$

This is a natural transformation $\widetilde{t}$ from the covariant functor h^a to F. Every natural transformation from h^a to F is of the form $\widetilde{t}$ for a unique element $t \in F(a)$.

Similarly, for every contravariant functor $G : \mathcal{A}^{\text{opp}} \to \mathbf{Sets}$, for every element $t \in G(a)$, for every object a' of $\mathcal{A}$, and for every element $f \in \text{Hom}_{\mathcal{A}}(a', a)$, denote by $f^*(t)$ the element of $G(a')$ that is the image of t under $G^{a',a}(f)$. This defines a set function,

$$\widetilde{t}^{a'} : h_a(a') \to G(a'), \quad f \mapsto f^*(t).$$

This is a natural transformation $\widetilde{t}$ from the contravariant functor h_a to G. Every natural transformation from h_a to G is of the form $\widetilde{t}$ for a unique element $t \in F(a)$.

Definition 5.8. For a category $\mathcal{A}$ and for a covariant functor $F : \mathcal{A} \to \mathbf{Sets}$, a **representation** of F is a pair (a, t) of an object a of $\mathcal{A}$ and an element $t \in F(a)$ such that the associated natural transformation $\widetilde{t} : h_a \Rightarrow F$ is a natural equivalence. If there exists a representation, then F is a **representable functor**. Similarly, a **representation** of a contravariant functor is a representation of the associated covariant functor from $\mathcal{A}^{\text{opp}}$ to **Sets**, and the contravariant functor is a **representable functor** if there exists a representation.

Example 5.9. For a covariant functor $F : \mathcal{A} \to \mathcal{B}$, for every object a of $\mathcal{A}$, let $\theta_a : F(a) \to G(a)$ be an isomorphism in $\mathcal{B}$. For every ordered pair (a, a') of objects of $\mathcal{A}$, denote by $G_{a,a'}$ the unique set map,

$$G_{a,a'} : \text{Hom}_{\mathcal{A}}(a, a') \to \text{Hom}_{\mathcal{B}}(G(a), G(a')),$$

such that for every $u \in \text{Hom}_{\mathcal{A}}(a, a')$, the composite $G_{a,a'}(u) \circ \theta_a$ equals $\theta_{a'} \circ F_{a,a'}(u)$. The rule associating to every object a of $\mathcal{A}$ the object $G(a)$ of $\mathcal{B}$ and associating to every ordered pair (a, a') of objects of $\mathcal{A}$ the set map $G_{a,a'}$ is a covariant functor $G : \mathcal{A} \to \mathcal{B}$, and the rule associating to every object a of $\mathcal{A}$ the isomorphism θ_a in $\mathcal{B}$ is a natural equivalence between F and G. In this sense, a rule that covariantly associates to every object of $\mathcal{A}$ an object of $\mathcal{B}$ only up to unique isomorphism in $\mathcal{B}$ defines a "natural equivalence class" of covariant functors.

Example 5.10. As an explicit example of the preceding, let $R : \mathcal{B} \to \mathcal{A}$ be a fully faithful, essentially surjective covariant functor, i.e., an equivalence of categories. Also assume that $\mathcal{A}$ is strictly small. For every object a of $\mathcal{A}$, since R is essentially surjective, there exists an object b of and an isomorphism, $a \to R(b)$. Using the Axiom of Choice, let $b = L(a)$ and $\theta_a : a \to R(L(a))$ be such a choice of object and isomorphism for every object a of $\mathcal{A}$. For every ordered pair (a, a') of objects, since R is fully faithful, there exists a unique bijection of sets,

$$L_{a,a'} : \text{Hom}_{\mathcal{A}}(a, a') \to \text{Hom}_{\mathcal{B}}(R(a), R(a')), \quad u \mapsto v = L_{a,a'}(u)$$

such that the composition $R(v) \circ \theta_a$ equals $\theta_{a'} \circ u$ for every u in $\text{Hom}_{\mathcal{A}}(a, a')$. This defines a covariant functor $L : \mathcal{A} \to \mathcal{B}$ and a natural equivalence $\theta : \text{Id}_{\mathcal{A}} \Rightarrow R \circ L$. Since R is fully faithful, also L is fully faithful.

Again using that R is fully faithful, there is a unique natural equivalence $\eta : L \circ R \Rightarrow \text{Id}_{\mathcal{B}}$ such that the R-pullback $R^*\eta$ equals the inverse natural isomorphism of the R-pushforward $R_*\theta$. In particular, L is essentially surjective. Thus, L is also an equivalence of categories. For a given equivalence R from a category $\mathcal{A}$ to a strictly small category $\mathcal{B}$, the extended datum of functors and natural transformations, (L, R, θ, η) as above, is unique up to unique natural equivalence in R.

6 Adjoint Pairs of Functors

Let $\mathcal{A}$ and $\mathcal{B}$ be categories.

Definition 6.1. An **adjoint pair** of (covariant) functors between $\mathcal{A}$ and $\mathcal{B}$ is a pair of (covariant) functors,

$$L : \mathcal{A} \to \mathcal{B}, \ \ R : \mathcal{B} \to \mathcal{A},$$

be (covariant) functors, and a pair of natural transformations of functors,

$$\theta : \mathrm{Id}_{\mathcal{A}} \Rightarrow RL, \ \ \theta(a) : a \to R(L(a)),$$

$$\eta : LR \Rightarrow \mathrm{Id}_{\mathcal{B}}, \ \ \eta(b) : L(R(b)) \to b,$$

such that the following compositions of natural transformations equal Id_R, resp. Id_L,

$$(*_R) : R \overset{\theta\circ R}{\Rightarrow} RLR \overset{R\circ\eta}{\Rightarrow} R,$$

$$(*_L) : L \overset{L\circ\theta}{\Rightarrow} LRL \overset{\eta\circ L}{\Rightarrow} R.$$

For every object a of $\mathcal{A}$ and for every object b of $\mathcal{B}$, define set maps,

$$H^L_R(a,b) : \mathrm{Hom}_{\mathcal{B}}(L(a), b) \to \mathrm{Hom}_{\mathcal{A}}(a, R(b)),$$

$$(L(a) \overset{\phi}{\to} b) \mapsto \left(a \xrightarrow{\theta(a)} R(L(a)) \xrightarrow{R(\phi)} R(b)\right),$$

and

$$H^R_L(a,b) : \mathrm{Hom}_{\mathcal{A}}(a, R(b)) \to \mathrm{Hom}_{\mathcal{B}}(L(a), b),$$

$$(a \overset{\psi}{\to} R(b)) \mapsto \left(L(a) \xrightarrow{L(\psi)} L(R(b)) \xrightarrow{\eta(b)} b\right).$$

Adjoint Pairs Exercise.

(i) For L, R, θ and η as above, the conditions $(*_R)$ and $(*_L)$ hold if and only if for every object a of $\mathcal{A}$ and every object b of $\mathcal{B}$, $H^L_R(a,b)$ and $H^R_L(a,b)$ are inverse bijections.

(ii) Prove that both $H^L_R(a,b)$ and $H^R_L(a,b)$ are binatural in a and b.

(iii) For functors L and R, and for binatural inverse bijections $H^L_R(a,b)$ and $H^R_L(a,b)$ between the bifunctors

$$\mathrm{Hom}_{\mathcal{B}}(L(a), b), \mathrm{Hom}_{\mathcal{A}}(a, R(b)) : \mathcal{A} \times \mathcal{B} \to \mathbf{Sets},$$

prove that there exist unique θ and η extending L and R to an adjoint pair such that H^L_R and H^R_L agree with the binatural inverse bijections defined above.

(iv) Let (L, R, θ, η) be an adjoint pair. Let a (covariant) functor

$$\widetilde{R} : \mathcal{B} \to \mathcal{A},$$

and natural transformations,

$$\widetilde{\theta} : \text{Id}_{\mathcal{A}} \Rightarrow \widetilde{R} \circ L, \widetilde{\eta} : L \circ \widetilde{R} \Rightarrow \text{Id}_{\mathcal{B}},$$

be natural transformations such that $(L, \widetilde{R}, \widetilde{\theta}, \widetilde{\eta})$ is also an adjoint pair. For every object b of B, define $I(b)$ in $\text{Hom}_{\mathcal{B}}(R(b), \widetilde{R}(b))$ to be the image of Id_b under the composition,

$$\text{Hom}_{\mathcal{B}}(b, b) \xrightarrow{\text{Hom}_{\mathcal{B}}(\theta(b), b)} \text{Hom}_{\mathcal{B}}(L(R(b)), b) \xrightarrow{H^{\widetilde{R}}_L(R(b), b)} \text{Hom}_{\mathcal{B}}(R(b), \widetilde{R}(b)).$$

Similarly, define $J(b)$ in $\text{Hom}_{\mathcal{B}}(\widetilde{R}(b), R(b))$, to be the image of Id_b under the composition,

$$\text{Hom}_{\mathcal{B}}(b, b) \xrightarrow{\text{Hom}_{\mathcal{B}}(\widetilde{\theta}(b), b)} \text{Hom}_{\mathcal{B}}(L(\widetilde{R}(b)), b) \xrightarrow{H^R_L(\widetilde{R}(b), b)} \text{Hom}_{\mathcal{B}}(\widetilde{R}(b), R(b)).$$

Prove that I and J are the unique natural transformations of functors,

$$I : R \Rightarrow \widetilde{R}, \ J : \widetilde{R} \Rightarrow R,$$

such that $\widetilde{\theta}$ equals $(I \circ L) \circ \theta$, θ equals $(J \circ L) \circ \widetilde{\theta}$, $\widetilde{\eta}$ equals $\eta \circ (L \circ I)$, and η equals $\widetilde{\eta} \circ (L \circ J)$. Moreover, prove that I and J are inverse natural isomorphisms. In this sense, every extension of a functor L to an adjoint pair (L, R, θ, η) is unique up to unique natural isomorphisms (I, J). Formulate and prove the symmetric statement for all extensions of a functor R to an adjoint pair (L, R, θ, η).

(v) For every adjoint pair (L, R, θ, η), prove that also $(R^{\text{opp}}, L^{\text{opp}}, \eta^{\text{opp}}, \theta^{\text{opp}})$ is an adjoint pair.

(vi) Formulate the corresponding notions of adjoint pairs when L and R are contravariant functors (just replace one of the categories by its opposite category).

Exercise on Composition of Adjoint Pairs. Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be categories. Let

$$L' : \mathcal{A} \to \mathcal{B}, R' : \mathcal{B} \to \mathcal{A},$$

be (covariant) functors, and let

$$\theta' : \text{Id}_{\mathcal{A}} \Rightarrow R'L', \ \eta' : L'R' \Rightarrow \text{Id}_{\mathcal{B}},$$

be natural transformations that are an adjoint pair of functors. Also let

$$L'' : \mathcal{B} \to \mathcal{C}, R'' : \mathcal{C} \to \mathcal{B},$$

be (covariant) functors, and let

$$\theta'' : \mathrm{Id}_{\mathcal{B}} \Rightarrow R''L'',\ \eta'' : L''R'' \Rightarrow \mathrm{Id}_{\mathcal{C}},$$

be natural transformations that are an adjoint pair of functors. Define functors

$$L : \mathcal{A} \to \mathcal{C},\ R : \mathcal{C} \to \mathcal{A}$$

by $L = L'' \circ L'$, $R = R' \circ R''$. Define the natural transformation,

$$\theta : \mathrm{Id}_{\mathcal{A}} \Rightarrow R \circ L,$$

to be the composition of natural transformations,

$$\mathrm{Id}_{\mathcal{A}} \overset{\theta'}{\Rightarrow} R' \circ L' \overset{R'\circ\theta''\circ L'}{\Rightarrow} R' \circ R'' \circ L'' \circ L'.$$

Similarly, define the natural transformation,

$$\eta : L \circ R \Rightarrow \mathrm{Id}_{\mathcal{C}},$$

to be the composition of natural transformations,

$$L'' \circ L' \circ R' \circ R'' \overset{L''\circ\eta'\circ R''}{\Rightarrow} L'' \circ R'' \overset{\eta''}{\Rightarrow} \mathrm{Id}_{\mathcal{C}}.$$

Prove that L, R, θ and η form an adjoint pair of functors. This is the **composition** of (L', R', θ', η') and $(L'', R'', \theta'', \eta'')$. If $\mathcal{A}$ equals $\mathcal{B}$, if L' and R' are the identity functors, and if θ' and η' are the identity natural transformations, prove that (L, R, θ, η) equals $(L'', R'', \theta'', \eta'')$. Similarly, if $\mathcal{B}$ equals $\mathcal{C}$, if L'' and R'' are the identity functors, and if θ'' and η'' are the identity natural transformations, prove that (L, R, θ, η) equals (L', R', θ', η'). Finally, prove that composition of three adjoint pairs is associative.

7 Adjoint Pairs of Partially Ordered Sets

Partially Ordered Sets Exercise. Let $(S, \leq)$ and $(T, \leq)$ be partially ordered sets, and consider the associated categories. For an order-preserving function,

$$L : (S, \leq) \to (T, \leq),$$

prove that there exists an order-preserving function,

$$R : (T, \leq) \to (S, \leq)$$

extending (uniquely) to an adjoint pair of functors (L, R, θ, η) if and only if for every element t of T, there exists an element s of S (necessarily unique) such that

$$L^{-1}\{\tau \in T | \tau \leq t\} = \{\sigma \in S | \sigma \leq s\}.$$

In particular, conclude that L is injective and strict, i.e., the associated functor is fully faithful. Formulate and prove a similar criterion for an order-preserving function R from $(T, \leq)$ to $(S, \leq)$ to admit a left adjoint.

8 Adjoint Pair between a Category and its Pointed Category

Definition 8.1. A **pointed set** is an ordered pair (S, s) of a set S and an element s of the set S. For pointed sets (S, s) and (S', s'), the **set of morphisms of pointed sets** is the subset of $\text{Hom}_{\mathbf{Sets}}(S, S')$ of set functions that map s to s'.

Notation 8.2. For every set S, denote by $\overline{S}$ the subset of the power set $\mathcal{P}(S)$ whose elements are $\{S\}$ and all singleton sets. Thus, $\overline{S}$ contains the image of the set function $\theta_S : S \to \mathcal{P}(S)$ from Example 5.6. For every set function $u : S \to S'$, define $\overline{u} : \overline{S} \to \overline{S'}$ to be the unique set function that maps $\{S\}$ to $\{S'\}$ and such that $\overline{u} \circ \theta_S$ equals $\theta_{S'} \circ u$. For every pointed set (S, s), define $\eta_{(S,s)} : (\overline{S}, \{S\}) \to (S, s)$ to be the unique function of pointed sets such that $\eta_{(S,s)} \circ \theta_S$ equals the identity function on S.

Pointed Sets Exercise.

(i) Prove that the rules above define a category **PtdSets** of pointed sets together with a faithful functor **PtdSets** → **Sets** associating to every pointed set (S, s) the set S and restricting to the inclusion from the set of morphisms of pointed sets from (S, s) to (S', s') inside the set of all set functions from S to s'. This is the **forgetful functor**.

(ii) Prove that the rule associating to every set S the ordered pair $(\overline{S}, \{S\})$ and associating to every set function $u : S \to S'$ the set function $\overline{u}$ defines a faithful functor from **Sets** to **PtdSets**.

(iii) Prove that the rule associating to every set S the set function $\theta_S : S \to \overline{S}$ defines a natural transformation from the identity functor on **Sets** to the composition of the above functors, **Sets** → **PtdSets** → **Sets**.

(iv) Prove that the rule associating to every pointed set (S, s) the set function $\eta_{(S,s)} : (\overline{S}, \{S\}) \to (S, s)$ is a natural transformation to the identity functor on **PtdSets** from the composition of the above functors **PtdSets** → **Sets** → **PtdSets**.

(v) Prove that these functors and natural transformations define an adjoint pair of functors.

Semigroups and Monoids Exercise. Modify the construction of the previous exercise to construct an adjoint pair of functors between **Semigroups** and **Monoids** whose right adjoint functor is the (faithful) forgetful functor from **Monoids** to **Semigroups** that "forgets" the specified identity element of the monoid (since identity elements in a monoid are unique, this functor is faithful).

Definition 8.3. A category is a **category with an initial object**, respectively a **category with a terminal object**, a **pointed category**, if it has an initial object, resp. if it has a terminal object, it has an object that is simultaneously an initial object and a terminal object, i.e., if it has a zero object. A functor between categories that both have an initial object, respectively a terminal object, a zero object, is a **initial preserving**, resp. **terminal preserving**, a **pointed functor**, if it maps each initial object to an initial object, resp. if it maps each terminal object to a terminal object, resp. if it maps each zero object to a zero object.

Definition 8.4. A **trivial category** is a pointed category such that every object is a zero object (i.e., there are objects, and every Hom set is a singleton set). A **terminal category** is a trivial category that has a unique object; every object of a trivial category gives a skeletal subcategory that is a terminal category.

Definition 8.5. For every category $\mathcal{C}$, for every set 0, the **associated category $\mathcal{C}_{0,\mathrm{init}}$ with initial object** 0 is the category whose objects consist of 0 together with ordered pairs $(A,0)$ for all objects A of $\mathcal{C}$. For every object of $\mathcal{C}_{\mathrm{init}}$, the set of morphisms from 0 to that object is a singleton set. For every pair of objects A and B of $\mathcal{C}$, the set of morphisms of $\mathcal{C}_{0,\mathrm{init}}$ from $(A,0)$ to $(B,0)$ is the set of morphisms of $\mathcal{C}$ from A to B. For every object A of $\mathcal{C}$, the set of morphisms in $\mathcal{C}_{0,\mathrm{init}}$ from $(A,0)$ to 0 is the empty set. There is a rule $F_{\mathcal{C},0}$ that associates to every object A of $\mathcal{C}$ the object $(A,0)$ of $\mathcal{C}_{0,\mathrm{init}}$ and, for every pair of objects A and B of $\mathcal{C}$, identifies the set of morphisms of $\mathcal{C}$ from A to B to the set of morphisms of $\mathcal{C}_{0,\mathrm{init}}$ from $(A,0)$ to $(B,0)$. There is a unique composition rule on $\mathcal{C}_{0,\mathrm{init}}$ that makes $\mathcal{C}_{0,\mathrm{init}}$ a category in such a way that $F_{\mathcal{C},0}$ is a fully faithful functor.

Adjointness property of the associated category with initial object. Show that the object 0 of $\mathcal{C}_{0,\mathrm{init}}$ is an initial object. Show that for every functor $G:\mathcal{C}\to\mathcal{B}$ to a category $\mathcal{B}$ and for every initial object b of $\mathcal{B}$, there exists a unique functor $G_{b,0,\mathrm{init}}:\mathcal{C}_{\mathrm{init}}\to\mathcal{B}$ that is initial preserving, that sends the initial object 0 of $\mathcal{C}_{0,\mathrm{init}}$ to b, and such that $G_{b,0,\mathrm{init}}\circ F$ equals G. Show that for every initial object b' of $\mathcal{B}$, there is a unique natural equivalence $G_{b',b,0,\mathrm{init}}:G_{b,0,\mathrm{init}}\Rightarrow G_{b',0,\mathrm{init}}$ such that $G_{b',b,0,\mathrm{init}}\circ F$ equals the identity natural equivalence of G to itself. In this sense, $(-)_{0,\mathrm{init}}$ is a 2-functor from the 2-category of categories to the 2-category of categories with initial objects with morphisms being natural equivalence classes of initial preserving functors, and $(-)_{\mathrm{init}}$ is "left adjoint" to the faithful (but not full) functor from the 2-category of categories with initial objects to the 2-category of categories (not necessarily having an initial object).

Associated category with a terminal object. For a category $\mathcal{C}$, define $\mathcal{C}_{0,\mathrm{term}}$ to be the opposite category of the associated category with initial object of the opposite category $\mathcal{C}^{\mathrm{opp}}$, i.e., $((\mathcal{C}^{\mathrm{opp}})_{0,\mathrm{init}})^{\mathrm{opp}}$. Formulate the analogues of the above for the associated functor $F_{0,\mathrm{term}}:\mathcal{C}\to\mathcal{C}_{0,\mathrm{term}}$.

Definition 8.6. For every category $\mathcal{C}$ that has a terminal object, for every terminal object 0, the **associated category $\mathcal{C}_0$ with final object** $(0,\mathrm{Id}_0)$ is the category whose objects are all ordered pairs (A,f) of an object A of $\mathcal{C}$ and a morphism $f:0\to A$ of $\mathcal{C}$. For every pair of such ordered pairs, (A,f) and (A',f'), the set of morphisms of $\mathcal{C}_0$ from (A,f) to (A',f') is the set of all morphisms of $\mathcal{C}$ $g:A\to A'$ such that $g\circ f$ equals f'. There is a rule $\Phi_{\mathcal{C},0}$ that associates to every object (A,f) of $\mathcal{C}_0$ the object A of $\mathcal{C}$ and that associates to every morphism of $\mathcal{C}_0$, $g:(A,f)\to(A',f')$, the morphism $g:A\to A'$ of $\mathcal{C}$. There is a unique composition rule on $\mathcal{C}_0$ that makes $\mathcal{C}_0$ a category in such a way that $\Phi_{\mathcal{C},0}$ is a faithful functor (usually not full).

Adjointness property of the associated category with zero object. Show that $(0,\mathrm{Id}_0)$ is a zero object of $\mathcal{C}_0$. Show that for every terminal-preserving functor $G:\mathcal{B}\to\mathcal{C}$ from a category with a zero object b to a category with a terminal object 0, there exists a unique zero-preserving functor $G_{0,b}:\mathcal{B}\to\mathcal{C}_0$ such that $\Phi_{\mathcal{C},0}\circ G_{0,b}$ equals G. In this sense, the rule associating to a category with a terminal object $\mathcal{C}$ the category with zero object $\mathcal{C}_0$ is right adjoint to the fully faithful 2-functor from the 2-category of categories with zero object and zero-preserving functors to the 2-category of categories with terminal object and terminal-preserving functors.

9 Adjoint Pairs of Free Objects

Definition 9.1. A **concrete category** is a category, $\mathcal{A}$, together with a faithful functor, $R : \mathcal{A} \to$ **Sets**, the **forgetful functor** of the concreted category. A left adjoint of R is a **free functor** for the specified concrete category. For concrete categories $(\mathcal{A}, R)$ and $(\mathcal{A}', R')$, a **functor of concrete categories** is a functor $F : \mathcal{A} \to \mathcal{A}'$ together with a natural equivalence $\theta : R \Rightarrow R' \circ F$, cf. the articles of Porst.

Remark 9.2. If there exists a free functor L for R, then the natural equivalence θ in a functor of concrete categories is uniquely determined by its value on the object $L(\{*\})$ for any singleton set $\{*\}$. for a given functor $F : \mathcal{A} \to \mathcal{A}'$, there is at most one natural equivalence θ such that (R, θ) is a functor of concrete categories. Thus, there is a unique concrete equivalence of the concrete category of sets extending the identity functor, but the extensions of the identity functor on the concrete category of groups has two elements (the identity extension and the extension given by group inversion).

Notation 9.3. For every nonnegative integer n, denote by $[1, n]$ the set $\{k \in \mathbb{Z}_{>0} | k \le n\}$, which has precisely n elements. For every ordered pair (n', n'') of nonnegative integers, denote by $q'_{n',n''}$ and $q''_{n',n''}$ the following set maps,

$$q'_{n',n''} : [n'] \to [n' + n''], \quad k \mapsto k,$$
$$q''_{n',n''} : [n''] \to [n' + n''], \quad k \mapsto n' + k.$$

For every set Σ and for every ordered pair of set functions,

$$f' : [n'] \to \Sigma, \quad f'' : [n''] \to \Sigma,$$

denote by $m_{\Sigma,n',n''}(f', f'')$ the unique set function

$$f : [n' + n''] \to \Sigma, \quad f \circ q'_{n',n''} = f', \quad f \circ q''_{n',n''} = f''.$$

Denote the unique set function $[0] \to \Sigma$ by 0_Σ. For every element $\sigma \in \Sigma$, denote by $\iota_{\Sigma,\sigma}$ the unique set function $[1] \to \Sigma$ with image $\{\sigma\}$.

Notation 9.4. For every set Σ, denote by $F(\Sigma)$ the set of all ordered pairs (n, f) of an integer $n \ge 0$ and a set map $f : [n] \to \Sigma$. For every set function $u : \Sigma \to \Pi$, denote by $F(u) : F(\Sigma) \to F(\Pi)$ the set function $(n, f) \mapsto (n, f \circ u)$. Denote by $\mathrm{pr}_{\Sigma,1} : F(\Sigma) \to \mathbb{Z}_{\ge 0}$ the set map that sends (n, f) to n. Denote by m_Σ the following binary operation,

$$m_\Sigma : F(\Sigma) \times F(\Sigma) \to F(\Sigma), \quad ((n', f'), (n'', f'')) \mapsto (n' + n'', m_{\Sigma,n',n''}(f', f'')).$$

Denote by ι_Σ the following set map,

$$\iota_\Sigma : \Sigma \to F(\Sigma), \quad \sigma \mapsto ([1], \iota_{\Sigma,\sigma}).$$

Free Monoids Exercise.

(i) Prove that the rule associating to every monoid (G, m, e) the set G and associating to every monoid morphism the same set map defines a faithful functor **Monoids** → **Sets**. This is the forgetful functor of the concrete category of monoids.

(ii) For every set Σ, prove that $(F(\Sigma), m_\Sigma, ([0], 0_\Sigma))$ is a monoid. For this monoid structure, for every set map $u : \Sigma \to \Pi$, prove that $F(u)$ is a monoid morphism. Prove that this defines a covariant functor **Sets** → **Monoids**.

(iii) Prove that the rule associating to every set Σ the set function ι_Σ is a natural transformation from the identity functor on **Sets** to the composition of the two functors above, **Sets** → **Monoids** → **Sets**.

(iv) For every monoid (G, m, e) and for every set function $j : \Sigma \to G$, use induction on the integer $n \geq 0$ to prove that there exists a unique morphism of monoids,

$$\widetilde{j} : F(\Sigma) \to G,$$

such that $\widetilde{j} \circ i_\Sigma$ equals j.

(v) For every monoid (G, m, e), for the identity set map $\mathrm{Id}_G : G \to G$, prove that the rule associating to (G, m, e) the monoid morphism $\widetilde{\mathrm{Id}}_G : F(G) \to G$ is a natural transformation to the identity functor on **Monoids** from the composition of the two functors above, **Monoids** → **Sets** → **Monoids**.

(vi) Check that these functors and natural transformations define an adjoint pair of functors. The monoid $F(\Sigma)$ is the **free monoid** on Σ.

(vii) Also check that for the functor **Sets** → **Monoids** that associates to every set the additive monoid $\mathbb{Z}_{\geq 0}$ and associates to every set function the identity morphism of $\mathbb{Z}_{\geq 0}$, the rule associating to every set Σ the monoid morphism $\mathrm{pr}_{1,\Sigma} : F(\Sigma) \to \mathbb{Z}_{\geq 0}$ is a natural transformation from the free monoid functor to this functor. Also, check that this equals the composition of the free monoid functor with the natural transformation from the identity functor on **Sets** to the "constant" functor from **Sets** to itself associating to every set the singleton $\{1\}$ and associating to every set function the identity set function on $\{1\}$ (since this singleton is a final object in **Sets**, there is a unique natural transformation from the identity functor to this constant functor).

Notation 9.5. For every set Σ, denote by $F_{>0}(\Sigma) \subset F(\Sigma)$ the inverse image under $\mathrm{pr}_{\Sigma,1}$ of the subset $\mathbb{Z}_{>0} \subset \mathbb{Z}_{\geq 0}$. For every set function $u : \Sigma \to \Pi$, define $F_{>0}(u)$ to be the restriction of $F(u)$ to $F_{>0}(\Sigma)$, which is a set function with image contained in $F_{>0}(\Pi)$.

Free Semigroups Exercise.

(i) Since $\mathbb{Z}_{>0}$ is a subsemigroup of $\mathbb{Z}_{\geq 0}$ (although not a submonoid), check that also $F_{>0}(\Sigma)$ is a subsemigroup of $F(\Sigma)$ for every set.

(ii) Also check that $F_{>0}(u)$ is a morphism of semigroups for every set function $u : \Sigma \to \Pi$.

(iii) Check that these rules define a functor from **Sets** to **Semigroups**. Check that the natural transformations of the previous exercise modify to define an adjoint pair of functors between **Sets** and **Semigroups** whose right adjoint functor is the forgetful functor.

(iv) Double-check that the composite of this adjoint pair with the adjoint pair between **Semigroups** and **Monoids** is naturally equivalent to the adjoint pair between **Sets** and **Monoids** from the previous exercise.

Notation 9.6. For every set Σ, denote the Cartesian product $\Sigma \times \{+1\}$, respectively $\Sigma \times \{-1\}$, by Σ_+, resp. Σ_-, with the corresponding bijections,

$$j_{\Sigma,+} : \Sigma \to \Sigma_+, \quad j_{\Sigma,-} : \Sigma \to \Sigma_-, \quad j_{\Sigma,+}(\sigma) = (\sigma, +1), \quad j_{\Sigma,-}(\sigma) = (\sigma, -1).$$

For every set function $u : \Sigma \to \Pi$, denote by $u_+ \sqcup u_-$ the unique set function from $\Sigma_+ \sqcup \Sigma_-$ to $\Pi_+ \sqcup \Pi_-$ whose composition with $j_{\Sigma,+}$, resp. with $j_{\Sigma,-}$, equals $j_{\Pi,+} \circ u$, resp. equals $j_{\Pi,-} \circ u$. Denote by $\Lambda_\Sigma \subset F(\Sigma_+ \sqcup \Sigma_-) \times F(\Sigma_+ \sqcup \Sigma_-)$, the subset whose elements are the following ordered pairs,

$$(f \cdot (i \circ j_{\Sigma,+})(\sigma) \cdot (i \circ j_{\Sigma,-})(\sigma) \cdot g, f \cdot (i \circ j_{\Sigma,-})(\sigma) \cdot (i \circ j_{\Sigma,+})(\sigma) \cdot g), \quad f, g \in F(\Sigma_+ \sqcup \Sigma_-), \quad \sigma \in \Sigma.$$

Denote by $\sim_\Sigma$ to be the weakest equivalent relation on $F(\Sigma_+ \sqcup \Sigma_-)$ generated by the relation Λ_Σ. Denote the quotient by this equivalence relation by

$$q_\Sigma : F(\Sigma_+ \sqcup \Sigma_-) \to F_{\mathbf{Groups}}(\Sigma).$$

Denote the composition $q_\Sigma \circ i \circ j_{\Sigma,+}$ by

$$i_{\mathbf{Groups},\Sigma} : \Sigma \to F_{\mathbf{Groups}}(\Sigma).$$

Free Groups Exercise.

(i) For an equivalence relation $\sim$ on a semigroup (G, m) with quotient $q : G \to H$, check that there exists a semigroup structure on H for which q is a morphism of semigroups if and only if there exists a left act of G on H for which q is a morphism of left G-acts if and only if there exists a right act of G on H for which q is a morphism of right acts if and only if $\sim$ satisfies the following: for every $g, g', g'' \in G$, if $g \sim g'$, then also $g \cdot g'' \sim g' \cdot g''$ and also $g'' \cdot g' \sim g'' \cdot g$.

(ii) For a monoid (G, m, e), check that every surjective morphism of semigroups $u : G \to G'$ is a morphism of monoids. Conclude that for an equivalence relation $\sim$ on G, the quotient is a morphism of monoids if and only if it is a morphism of semigroups.

(iii) Check that the rule associating to each set Σ the monoid $F(\Sigma_+ \sqcup \Sigma_-)$ and associating to each set function $u : \Sigma \to \Pi$ the monoid morphism $F(u_+ \sqcup u_-)$ is a functor from **Sets** to **Monoids**. Check that the functions $i \circ j_{\Sigma,+}$ and $i \circ j_{\Sigma,-}$ are natural transformations from the identity functor on **Sets** to the composite of this functor with the forgetful functor **Monoids** $\to$ **Sets**. Check that the rule associating to every set Σ the set $F(\Sigma_+ \sqcup \Sigma_-) \times F(\Sigma_+ \sqcup \Sigma_-)$ and associating to every set function $u : \Sigma \to \Pi$ the set function

$$F(u_+ \sqcup u_-) \times F(u_+ \sqcup u_-) : F(\Sigma_+ \sqcup \Sigma_-) \times F(\Sigma_+ \sqcup \Sigma_-) \to F(\Pi_+ \sqcup \Pi_-) \times F(\Pi_+ \sqcup \Pi_-)$$

is a functor from **Sets** to itself. Check that this function sends Λ_Σ to Λ_Π. Conclude that the rule associating to every set Σ the subset Λ_Σ and associating to every set function $u : \Sigma \to \Pi$ the restriction of $F(u_+ \sqcup u_-) \times F(u_+ \sqcup u_-)$ is a subfunctor of the previous functor. Conclude that the rule associating to every set Σ the equivalence relation $\sim_\Sigma$ is also a subfunctor.

(iv) For every set Σ, check that the equivalence relation $\sim_\Sigma$ satisfies the condition necessary for the quotient map to be a monoid morphism. Conclude that there is a unique pair of a functor **Sets** $\to$ **Monoids** and a natural transformation to this functor from the free monoid functor $F(\Sigma_+ \sqcup \Sigma_-)$ associating to every set Σ the monoid $F_{\mathbf{Groups}}(\Sigma)$ and the quotient monoid morphism q_Σ.

(v) Check that each of the monoid generators $i(j_{\Sigma,+}(\sigma))$ and $i(j_{\Sigma,-}(\sigma))$ of the free monoid $F(\Sigma_+ \sqcup \Sigma_-)$ map under q_Σ to an invertible element of $F_{\mathbf{Groups}}(\Sigma)$. Conclude that the functor $F_{\mathbf{Groups}}$ from **Sets** to **Monoids** factors through the full subcategory **Groups** of **Monoids**. Thus, $F_{\mathbf{Groups}}$ is a functor from **Sets** to **Groups**.

(vi) Check that the rule associating to every set Σ the set function $i_{\mathbf{Groups},\Sigma}$ is a natural transformation from the identity functor to the composition of the forgetful functor with the functor above, **Sets** $\to$ **Groups** $\to$ **Sets**. Similarly, modify the definition of η_Σ to obtain a natural transformation from the composition **Groups** $\to$ **Sets** $\to$ **Groups** to the identity functor on **Groups**. Prove that these functors and natural transformations define an adjoint pair whose right adjoint functor is the (faithful) forgetful functor **Groups** $\to$ **Sets**. The group $F_{\mathbf{Groups}}(\Sigma)$ is the **free group on the set** Σ.

(vii) For every monoid (G, m, e), denote by $N_{(G,m,e)}$ the fiber over e of the natural transformation,

$$\widetilde{\mathrm{Id}}_G : F(G) \to G.$$

Denote by $N_{\mathbf{Groups},(G,m,e)}$ the normal subgroup of $F_{\mathbf{Groups}}(G)$ generated by the image under $q \circ F(j_{G,+})$ of $N_{(G,m,e)}$. Check that this is functorial in (G, m, e) and that the quotient group $F_{\mathbf{Groups}}(G)/N_{\mathbf{Groups},(G,m,e)}$ define a left adjoint functor to the (fully faithful) forgetful functor from **Groups** to **Monoids**. This left adjoint functor is the **group completion functor**. Double-check that the composite of the group completion functor with the free monoids functor is naturally equivalent to $F_{\mathbf{Groups}}$.

(viii) For categories $\mathcal{B}$, $\mathcal{C}$, for functors

$$L'' : \mathcal{B} \to \mathcal{C}, \ R'' : \mathcal{C} \to \mathcal{B},$$

and for natural transformations

$$\theta'' : \mathrm{Id}_\mathcal{B} \Rightarrow R'' \circ L'', \ \eta'' : L'' \circ R'' \Rightarrow \mathrm{Id}_\mathcal{C},$$

such that $(L'', R'', \theta'', \eta'')$ is an adjoint pair, the adjoint pair is *reflective* if R'' is fully faithful. In this case, prove that there exists a unique binatural transformation

$$\widetilde{H}^{L''}_{R''}(b, b') : \mathrm{Hom}_\mathcal{C}(L''(R''(b)), b') \to \mathrm{Hom}_\mathcal{C}(b, b'),$$

such that the composition with R'',

$$\mathrm{Hom}_{\mathcal{C}}(L''(R''(b)),b') \xrightarrow{\widetilde{H}^{L''}_{R''}(b,b')} \mathrm{Hom}_{\mathcal{C}}(b,b') \xrightarrow{R''} \mathrm{Hom}_{\mathcal{B}}(R''(b),R''(b')),$$

equals $H^{L''}_{R''}(R(b),b')$. In particular, taking $b' = L''(R''(b))$, denote the image of $\mathrm{Id}_{b'}$ by

$$\widetilde{\eta}''_b : b \to L''(R''(b)).$$

Prove that $\widetilde{\eta}''_b$ is an inverse to $\eta''_b : L''(R''(b)) \to b$. Thus, for a reflective adjoint pair, η'' is a natural isomorphism. Conversely, if η'' is a natural isomorphism, prove that the adjoint pair is reflective, i.e., R'' is fully faithful. In particular, for the group completion, conclude that the group completion of the monoid underlying a group is naturally isomorphic to that group.

Free Abelian Groups Exercise. Denote by

$$\Phi : \mathbb{Z}-\mathrm{mod} \to \mathbf{Groups}$$

the full subcategory of **Groups** whose objects are Abelian groups. For every group $(G,\cdot,e)$, denote by $[G,G]$ the normal subgroup of G generated by all commutators

$$[g,h] = g\cdot h\cdot g^{-1}\cdot h^{-1}$$

for pairs $g,h\in G$. Denote by

$$\theta_G : G \to L(G),$$

the group quotient associated to the normal subgroup $[G,G]$ of G. Prove that $L(G)$ is an Abelian group. Moreover, for every Abelian group $(A,\cdot,e)$, prove that the set map

$$H^L_\Phi : \mathrm{Hom}_{\mathbb{Z}-\mathrm{mod}}(L(G),A) \to \mathrm{Hom}_{\mathbf{Groups}}(G,\Phi(A)),\ v\mapsto v\circ\theta_G,$$

is a bijection. In particular, for every group homomorphism,

$$u : G \to G',$$

the composition $\theta_{G'}\circ u : G \to L(G')$ is a group homomorphism, and thus there exists a unique group homomorphism,

$$L(u) : L(G) \to L(G'),$$

such that $H^L_\Phi(L(u))\circ\theta_G$ equals $\theta_{G'}\circ u$. Prove that the rule $G\mapsto L(G)$, $u\mapsto L(u)$ defines a functor,

$$L : \mathbf{Groups} \to \mathbb{Z}-\mathrm{mod}.$$

This functor is called *Abelianization*. Prove that $G\mapsto\theta_G$ is a natural transformation,

$$\theta : \mathrm{Id}_{\mathbf{Groups}} \Rightarrow \Phi\circ L.$$

For every Abelian group A, prove that $[A, A]$ is the identity subgroup, and thus the quotient homomorphism,

$$\theta_{\Phi(A)} : \Phi(A) \to \Phi(L(\Phi(A))),$$

is an isomorphism. Thus there exists a unique group homomorphism, just the inverse isomorphism of $\theta_{\Phi(A)}$,

$$\eta_A : L(\Phi(A)) \to A,$$

such that $\theta_{\Phi(A)} \circ \Phi(\eta_A)$ equals the $\text{Id}_{\Phi(A)}$. Prove that $A \mapsto \eta_A$ is a natural isomorphism,

$$\eta : L \circ \Phi \to \text{Id}_{\mathbb{Z}-\text{mod}}.$$

Prove that (L, Φ, θ, η) is an adjoint pair.

Factorization Exercise. Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be categories. Let

$$R' : \mathcal{B} \to \mathcal{A}, \ R'' : \mathcal{C} \to \mathcal{B},$$

be fully faithful functors. Denote the composition $R' \circ R''$ by

$$R : \mathcal{C} \to \mathcal{A}.$$

(i) If there exist extensions to reflective adjoint pairs (L', R', θ', η'), $(L'', R'', \theta'', \eta'')$, prove that there is also an extension to a reflective adjoint pair (L, R, θ, η).

(ii) If there exists an extension of R to a reflective adjoint pair (L, R, θ, η), prove that there exists an extension $(L'', R'', \theta'', \eta'')$. Give an example demonstrating that R' need not extend to a reflective adjoint pair (for instance, consider the full subcategory of Abelian groups in the full subcategory of solvable groups in the category of all groups).

(iii) A monoid $(G, \cdot, e)$ is called **left cancellative**, resp. **right cancellative**, if for every f, g, h in G, if $f \cdot g$ equals $f \cdot h$, resp. if $g \cdot f$ equals $h \cdot f$, then g equals h. A monoid is **cancellative** if it is both left cancellative and right cancellative. A monoid is **commutative** if for every $f, g \in G$, $f \cdot g$ equals $g \cdot f$. A commutative monoid is left cancellative if and only if it is right cancellative if and only if it is cancellative. Denote by

$$\textbf{LCanMonoids}, \textbf{RCanMonoids}, \textbf{CanMonoids}, \textbf{CommMonoids}, \textbf{CommCanMonoids} \subseteq \textbf{Monoids}$$

the full subcategories of the category of all monoids whose objects are left cancellative monoids, resp. right cancellative monoids, cancellative monoids, commutative monoids, commutative cancellative monoids. In each of these cases, prove that the fully faithful inclusion functor R extends to a reflective adjoint pair. Use (ii) to conclude that for every inclusion functor among the full subcategories listed above, there is an extension of the inclusion functor to a reflective adjoint pair.

(iv) In particular, prove that the group completion adjoint pair

$$(L : \textbf{Monoids} \to \textbf{Groups}, R : \textbf{Groups} \to \textbf{Monoids}, \theta, \eta)$$

factors as the composition of the reflective adjoint pair

$$(L' : \mathbf{Monoids} \to \mathbf{CanMonoids}, R' : \mathbf{CanMonoids} \to \mathbf{Monoids}, \theta', \eta'),$$

and the restriction to **CanMonoids** of the group completion adjoint pair

$$(L'' = L \circ R', R'', \theta'', \eta'').$$

Similarly, prove that the composition of the Abelianization functor and the group completion functor

$$(L : \mathbf{Monoids} \to \mathbb{Z} - \mathrm{mod}, R : \mathbb{Z} - \mathrm{mod} \to \mathbf{Monoids}, \theta, \eta),$$

factors through the reflection to the full subcategory of commutative, cancellative monoids,

$$(L' : \mathbf{Monoids} \to \mathbf{CommCanMonoids}, R' : \mathbf{CommCanMonoids} \to \mathbf{Monoids}, \theta', \eta').$$

Adjointness of Tensor and Hom Exercise. Let A and B be unital, associative rings, and let $\phi : A \to B$ be a morphism of unital, associative rings.

(i) For every left B-module,

$$(N, m_{B,N} : B \times N \to N),$$

prove that the composition

$$A \times N \xrightarrow{\phi \times \mathrm{Id}_N} B \times N \xrightarrow{m_{B,N}} N,$$

makes the datum

$$(N, m_{B,N} \circ (\phi \times \mathrm{Id}_N) : A \times N \to N),$$

an R-module. For every morphism of left B-modules,

$$u : (N, m_{B,N}) \to (N', m_{B,N'}),$$

prove that also

$$u : (N, m_{B,N} \circ (\phi \times \mathrm{Id}_N)) \to (N', m_{B,N'} \circ (\phi \times \mathrm{Id}_{N'}))$$

is a morphism of left A-modules. Altogether, prove that the association $(N, m_{B,N}) \mapsto (N, m_{B,N} \circ (\phi \times \mathrm{Id}_N))$ and $u \mapsto u$ is a faithful functor

$$R_\phi : B - \mathrm{mod} \to A - \mathrm{mod}.$$

In particular, in the usual manner, for every unital, associative ring C and for every $B - C$-bimodule N, prove that $R_\phi(N)$ is naturally an $A - C$-bimodule.

(ii) Formulate and prove the analogous results for right modules, giving a faithful functor

$$R^\phi : \mathrm{mod} - B \to \mathrm{mod} - A.$$

For every $C - B$-bimodule N, prove that $R^\phi(N)$ is naturally a $C - A$-bimodule. In particular for the $B - B$-bimodule $N = B$, $R^\phi(B)$ is naturally a $B - A$-bimodule.

For every left A-module M, denote $L_\phi(M) = R^\phi(B)\otimes_A M$. For every morphism of left A-modules,

$$u : M \to M',$$

denote by $L_\phi(u) = \mathrm{Id}_{R^\phi(B)} \otimes u$,

$$L_\phi(u) : L_\phi(M) \to L_\phi(M'),$$

the associated morphism of left B-modules. Prove that the associations $M \mapsto L_\phi(M)$ and $u \mapsto L_\phi(u)$ define a functor

$$L_\phi : A - \mathrm{mod} \to B - \mathrm{mod}.$$

(iv) Denote by 1_B the multiplicative unit in B. For every left A-module M, prove that the composition

$$M \xrightarrow{1_B \times \mathrm{Id}_M} B \times M \xrightarrow{\beta_{B,M}} B \otimes_A M,$$

is a morphism of left A-modules,

$$\theta_M : M \to R_\phi(L_\phi(M)),$$

i.e., for every $a \in A$ and for every $m \in M$,

$$\beta_{B,M}(1_B, a\cdot m) = \beta_{B,M}(1_B \cdot \phi(a), m) = \beta_{B,M}(\phi(a)\cdot 1_B, m).$$

Prove that the association $M \mapsto \theta_M$ defines a natural transformation

$$\theta : \mathrm{Id}_{A-\mathrm{mod}} \Rightarrow R_\phi \circ L_\phi.$$

(v) For every left B-module $(N, m_{B,N})$, for the induced right A-module structure on $R^\phi(B)$ and left A-module structure on N, prove that

$$m_{B,N} : B \times N \to N$$

is A-bilinear, i.e., for every $a \in A$, for every $b \in B$, and for every $n \in N$,

$$m_{B,N}(b, \phi(a)\cdot n) = m_{B,N}(b\cdot \phi(a), n).$$

Thus, by the universal property of tensor product, there exists a unique homomorphism of Abelian groups,

$$m_N : B \otimes_A N \to N,$$

such that $m_N \circ \beta_{B,N}$ equals $m_{B,N}$. Prove that m_N is a morphism of left B-modules, i.e., for every $b, b' \in B$ and for every $n \in N$,

$$m_N(b\cdot \beta_{B,N}(b', n)) = m_N(\beta_{B,N}(b\cdot b', n)) = m_{B,N}(b\cdot b', n) = m_{B,N}(b, m_{B,N}(b', n)).$$

Prove that the association $N \mapsto m_N$ defines a natural transformation

$$m : R_\phi \circ L_\phi \Rightarrow \mathrm{Id}_{B-\mathrm{mod}}.$$

(vi) Prove that $(L_\phi, R_\phi, \theta, m)$ is an adjoint pair of functors. In particular, even though R_ϕ is faithful, the natural transformation m is typically not a natural isomorphism. Conclude that one cannot weaken the definition of reflective adjoint pair from "fully faithful" to "faithful".

(vii) Prove the analogues of the above for right modules. Also, taking A to be $\mathbb{Z}$, and taking $\phi : \mathbb{Z} \to B$ to be the unique ring homomorphism, obtain an adjoint pair

$$(L'' : \mathbb{Z} - \text{mod} \to B - \text{mod}, R'' : B - \text{mod} \to \mathbb{Z} - \text{mod}, \theta'', \eta'')$$

whose composition with the adjoint pair

$$(L' : \textbf{CommCanMonoids} \to \mathbb{Z} - \text{mod}, R' : \mathbb{Z} - \text{mod} \to \textbf{CommCanMonoids}, \theta', \eta')$$

is an adjoint pair (L, R, θ, η) extending the forgetful functor

$$R : B - \text{mod} \to \textbf{CommCanMonoids}.$$

Composing this adjoint pair further with the other adjoint pairs above gives, in particular, an adjoint pair (F, Φ, i, η) extending the forgetful functor

$$\Phi : B - \text{mod} \to \textbf{Sets}.$$

The functor $F : \textbf{Set} \to B - \text{mod}$ and the natural transformation i is called the "free B-module". Use the usual functorial properties to conclude that F naturally maps to the category of $B - B$-bimodules.

Free Central A-algebras and Free Commutative Central A-algebras Exercise. Let A be an associative, unital ring that is commutative. Recall that a central A-**algebra** is a pair (B, ϕ) of an associative, unital ring B and a morphism of associative, unital rings, $\phi : A \to B$, such that for every $a \in A$ and every $b \in B$, $\phi(a) \cdot b$ equals $b \cdot \phi(a)$, i.e., $\phi(A)$ is contained in the center of B. In particular, the identity map

$$\text{Id}_B : R^\phi(B) \to R_\phi(B),$$

is an isomorphism of $A - A$-bimodules making B into a left-right A-module.

For central A-algebras (B, ϕ) and (B', ϕ'), a morphism of central A-algebras is a morphism of associative, unital rings, $\psi : B \to B'$, such that $\psi \circ \phi$ equals ϕ'. In particular, ψ is a morphism of left-right A-modules.

(i) Prove that the usual composition and the usual identity maps define a faithful (but not full!) subcategory

$$R : A - \textbf{algebra} \to A - \textbf{mod}$$

whose objects are central A-algebras and whose morphisms are morphisms of central A-algebras. The rest of this problem extends this to an adjoint pair that is a composition of two other (more elementary) adjoint pairs.

(ii) Let $n \geq 2$ be an integer. Let $M_1, \dots, M_n$ be (left-right) A-modules. For every A-module U, a map

$$\gamma : M_1 \times \cdots \times M_n \to U,$$

is an n-A**-multilinear** map if for every $i = 1, \dots, n$, for every choice of

$$\overline{m}_i = (m_1, \dots, m_{i-1}, m_{i+1}, \dots, m_n) \in M_1 \times \cdots \times M_{i-1} \times M_{i+1} \times \cdots \times M_n,$$

the induced map

$$\gamma_{\overline{m}_i} : M_i \to U, \ m_i \mapsto \gamma(m_1, \dots, m_{i-1}, m_i, m_{i+1}, \dots, m_n),$$

is a morphism of A-modules. Prove that there exists a pair $(T(M_1, \dots, M_n), \beta_{M_1,\dots,M_n})$ of an A-module $T(M_1, \dots, M_n)$ and an n-A-multilinear map

$$\beta_{M_1,\dots,M_N} : M_1 \times \cdots \times M_n \to T(M_1, \dots, M_n),$$

such that for every n-A-multilinear map γ as above, there exists a unique A-module homomorphism,

$$u : T(M_1, \dots, M_n) \to U,$$

such that $u \circ \beta_{M_1,\dots,M_n}$ equals γ. For $n = 3$, prove that β_{M_1,M_2,M_3} factors through

$$\beta_{M_1,M_2} \times \mathrm{Id}_{M_3} : M_1 \times M_2 \times M_3 \to (M_1 \otimes_A M_2) \times M_3.$$

Prove that the induced map

$$\beta_{M_1 \otimes M_2, M_3} : (M_1 \otimes_A M_2) \times M_3 \to T(M_1, M_2, M_3),$$

is A-bilinear. Conclude that there exists a unique A-module homomorphism,

$$u : (M_1 \otimes_A M_2) \otimes_A M_3 \to T(M_1, M_2, M_3).$$

Prove that this is an isomorphism of A-modules. Similarly, prove that there is a natural isomorphism of A-modules,

$$M_1 \otimes_A (M_2 \otimes_A M_3) \to T(M_1, M_2, M_3).$$

Conclude that there is a natural isomorphism of A-modules,

$$(M_1 \otimes_A M_2) \otimes_A M_3 \cong M_1 \otimes_A (M_2 \otimes_A M_3),$$

i.e., tensor product is associative for A-modules. Iterate this to conclude that there are natural isomorphisms between all the different interpretations of $M_1 \otimes_A \cdots \otimes_A M_n$, and each of these is naturally isomorphic to $T(M_1, \dots, M_n)$. (All of this is also true in the case of M_i that are $A_{i-1} - A_i$-bimodules with n-$(A_i)_i$-multilinearity defined appropriately.)

(iii) Let B be an A-algebra. A $\mathbb{Z}_+$-grading of B is a direct sum decomposition as an A-module,

$$B = \oplus_{n \geq 0} B_n,$$

such that for every pair of integers $n, p \geq 0$, the restriction to the summands B_n and B_p of the multiplication map,

$$m_B : B_n \times B_p \to B$$

factors through B_{n+p}. The induced A-bilinear map is denoted

$$m_{B,n,p} : B_n \times B_p \to B_{n+p}.$$

In particular, notice that this means that B_0 is an A-subalgebra of B, and every direct summand B_n is a $B_0 - B_0$-bimodule. Finally, for every triple of integers $n, p, r \geq 0$, the following diagram commutes,

$$\begin{array}{ccc} B_n \times B_p \times B_r & \xrightarrow{m_{B,n,p} \times \mathrm{Id}_{B_r}} & B_{n+p} \times B_r \\ {\scriptstyle \mathrm{Id}_{B_n} \times m_{B,p,r}} \downarrow & & \downarrow {\scriptstyle m_{B,n+p,r}} \\ B_n \times B_{p+r} & \xrightarrow[m_{B,n,p+r}]{} & B_{n+p+r} \end{array}$$

Prove that a $\mathbb{Z}_+$-graded A-algebra is equivalent to the data $((B_n)_{n \in \mathbb{Z}_+}, (m_{B,n,p})_{(n,p) \in \mathbb{Z}_+ \times \mathbb{Z}_+})$ satisfying the conditions above.

(iv) For $\mathbb{Z}_+$-graded A-algebras $((B_n)_{n \in \mathbb{Z}_+}, (m_{B,n,p})_{(n,p) \in \mathbb{Z}_+ \times \mathbb{Z}_+})$ and $((B'_n)_{n \in \mathbb{Z}_+}, (m_{B',n,p})_{(n,p) \in \mathbb{Z}_+ \times \mathbb{Z}_+})$, a morphism of $\mathbb{Z}_+$-graded A-algebras is a morphism of A-algebras,

$$\psi : B \to B',$$

such that for every integer $n \geq 0$, $\psi(B_n)$ is contained in B'_n. The induced A-linear map is denoted

$$\psi_n : B_n \to B'_n.$$

In particular, ψ_0 is a morphism of A-algebras. Relative to ψ_0, every map ψ_n is a morphism of $B_0 - B_0$-bimodules. Finally, for every pair of integers $n, p \geq 0$, the following diagram commutes,

$$\begin{array}{ccc} B_n \times B_p & \xrightarrow{\psi_n \times \psi_p} & B'_n \times B'_p \\ {\scriptstyle m_{B,n,p}} \downarrow & & \downarrow {\scriptstyle m_{B',n,p}} \\ B_{n+p} & \xrightarrow[\psi_{n+p}]{} & B'_{n+p} \end{array}$$

Prove that a morphism of $\mathbb{Z}_+$-graded A-algebras is equivalent to the data $(\psi_n)_{n \in \mathbb{Z}_+}$ satisfying the conditions above. Prove that composition of morphisms of $\mathbb{Z}_+$-graded A-algebras is a morphism of $\mathbb{Z}_+$-graded A-algebras. Prove that identity maps are morphisms of $\mathbb{Z}_+$-graded A-algebras. Conclude that there is a faithful (but not full!) subcategory,

$$L'' : \mathbb{Z}_+ - A - \textbf{algebra} \to A - \textbf{algebra},$$

whose objects are $\mathbb{Z}_+$=graded A-algebras and whose morphisms are morphisms of $\mathbb{Z}_+$-graded A-algebras. Prove that this extends to an adjoint pair $(L'', R'', \theta'', \eta'')$ where

$$R'' : A - \textbf{algebra} \to \mathbb{Z}_+ - A - \textbf{algebra},$$

associates to an associative, unital A-algebra (C, m_C) the $\mathbb{Z}_+$-graded A-algebra,

$$((C_n)_{n\in\mathbb{Z}_+}, (m_{n,p})_{(n,p)\in\mathbb{Z}_+\times\mathbb{Z}_+}) = ((C)_{n\in\mathbb{Z}_+}, (m)_{(n,p)}).$$

Thus C_0 equals C as an A-algebra, and the C_0-algebra $\oplus_n C_n$ is equivalent as a $\mathbb{Z}_+$-graded C-algebra to $C[t] = C\otimes_{\mathbb{Z}}\mathbb{Z}[t]$, where $\mathbb{Z}[t]$ is graded in the usual way.

(v) Let M be an A-module. For every integer $n\geq 1$, denote

$$T_A^n(M) = T(M_1,\dots,M_n) = M^{\otimes n} = M\otimes_A\cdots\otimes_A M,$$

with the universal n-A-multilinear map,

$$\beta_M^n : M^n \to T_A^n(M).$$

Similarly, denote $T_A^0(M) = A$. For every pair of integers $n,p\geq 0$, the composition,

$$M^n\times M^p \xrightarrow{=} M^{n+p} \xrightarrow{\beta^{n+p}} T_A^{n+p}(M),$$

is n-A-multilinear, resp. p-A-multilinear in the two arguments separately. Thus the composition factors as

$$M^n\times M^p \xrightarrow{\beta_M^n\times\beta_M^p} T_A^n(M)\times T_A^p(M) \xrightarrow{\mu_M^{n,p}} T_A^{n+p}(M),$$

where $\mu_M^{n,p}$ is A-bilinear. Finally, for every triple of integers $n,p,r\geq 0$, associativity of tensor products implies that the following diagram commutes,

$$\begin{array}{ccc} T_A^n(M)\times T_A^p(M)\times T_A^r(M) & \xrightarrow{\mu_M^{n,p}\times \mathrm{Id}_{T_A^r(M)}} & T_A^{n+p}(M)\times T_A^r(M) \\ {\scriptstyle \mathrm{Id}_{T_A^n(M)}\times\mu_M^{p,r}}\downarrow & & \downarrow{\scriptstyle \mu_M^{n+p,r}} \\ T_A^n(M)\times T_A^{p+r}(M) & \xrightarrow[\mu_M^{n,p+r}]{} & T_A^{n+p+r}(M) \end{array}.$$

Thus, the data $((T_A^n(M))_{m\in\mathbb{Z}_+}, (\mu_M^{n,p})_{(n,p)\in\mathbb{Z}_+\times\mathbb{Z}_+})$ defines a $\mathbb{Z}_+$-graded A-algebra, denoted $T_A(M)$ and called the *tensor algebra* associated to M. For every $\mathbb{Z}_+$-graded A-algebra

$$B = ((B_n)_{n\in\mathbb{Z}_+}, (m_{B,n,p})_{(n,p)\in\mathbb{Z}_+\times\mathbb{Z}_+}),$$

for every integer n, inductively define the A-module morphism

$$\eta'_{B,n} : T_A^n(B_1)\to B_n,$$

by $\eta'_{B,0} : A\to B_0$ is the usual A-algebra structure map ϕ, $\eta'_{B,1} : T_A^1(B_1)\to B_1$ is the usual identity morphism on B_1, and for every $n\geq 0$, assuming that $\eta'_{B,n}$ is defined,

$$\eta'_{B,n+1} : T_A^{n+1}(B_1) = B_1\otimes_A T_A^n(B)\to B_{n+1},$$

is the unique A-module homomorphism whose composition with the universal A-bilinear map,

$$\beta_M : B_1 \times T^n_A(B) \to B_A \otimes_A T^n_A(B),$$

equals the A-bilinear composition

$$B_1 \times T^n_A(B_1) \xrightarrow{\text{Id}_{B_1} \times \eta_{B,n}} B_1 \times B_n \xrightarrow{m_{B,1,n}} B_{n+1}.$$

Use associativity of tensor product (and induction) to prove that for every pair of integers $n, p \geq 0$, the following diagram commutes,

$$\begin{array}{ccc} T^n_A(B_1) \times T^p_A(B_1) & \xrightarrow{\eta'_{B,n} \times \eta'_{B,p}} & B_n \times B_p \\ {\scriptstyle \mu^{n,p}_{B_1}} \downarrow & & \downarrow {\scriptstyle m_{B,n,p}} \\ T^{n+p}_A(B_1) & \xrightarrow[\eta'_{B,n+p}]{} & B_{n+p} \end{array}$$

Conclude that $(\eta'_{B,n})_{n \in \mathbb{Z}_+}$ is a morphism of $\mathbb{Z}_+$-graded A-algebras,

$$\eta'_B : T_A(B_1) \to B.$$

(vi) Denote by

$$R' : \mathbb{Z}_+ - A - \text{algebra} \to A - \text{mod}$$

the functor that associates to a $\mathbb{Z}_+$-graded A-algebra $((B_n)_{n \in \mathbb{Z}_+}, (m_{B,n,p})_{(n,p) \in \mathbb{Z}_+ \times \mathbb{Z}_+})$ the A-module B_1 and that associates to a morphism $(\psi_n)_{n \in \mathbb{Z}_+}$ of $\mathbb{Z}_+$-graded A-algebras the A-module ψ_1. For every A-module M, denote by

$$\theta'_M : M \to R'(T_A(M))$$

the identity morphism $M \to T^1_A(M)$. Prove that this defines an adjoint pair $(T_A, R', \theta', \eta')$. Composing with the adjoint pair $(L'', R'', \theta'', \eta'')$ gives an adjoint pair $(L'' \circ T_A, R, \theta, \eta)$ extending the faithful (but not full!) forgetful functor

$$R : A - \textbf{algebra} \to A - \textbf{mod}, \ B \mapsto B.$$

10 Adjoint Pairs for Lawvere Theories

Definition 10.1. For a concrete category $\mathcal{A}$ with its forgetful functor $R : \mathcal{A} \to \textbf{Sets}$, for a category $\mathcal{B}$, an **$\mathcal{A}$-object** of $\mathcal{B}$ is a triple (b, F, θ) of an object b of $\mathcal{B}$, a contravariant functor $F : \mathcal{B}^{\text{opp}} \to \mathcal{A}$, and a natural equivalence of set-valued contravariant functors on $\mathcal{B}$, $\theta : h_a \Rightarrow R \circ F$. The contravariant functor F is the **Yoneda contravariant functor** associated to the $\mathcal{A}$-object of $\mathcal{B}$. For $\mathcal{A}$-objects of $\mathcal{B}$, (b, F, θ) and (b', F', θ'), a **morphism** of $\mathcal{A}$-objects of $\mathcal{B}$ from the first triple to the second triple is a pair $(u : b \to b', v : F' \Rightarrow F)$ of a $\mathcal{B}$-morphism u and a natural transformation of contravariant functors v such that $(F \circ v) \circ \theta'$ equals $\theta \circ h_u$ as natural transformations from $h_{b'}$ to $R \circ F$. Composition is defined in the evident way, and the identity of (b, F, θ) is $(\text{Id}_b, \text{Id}_F)$.

Remark 10.2. Because R is faithful, for every $\mathcal{B}$-morphism $u : b \to b'$, there is at most one morphism (u, v) from the $\mathcal{A}$-object (b, F, θ) to the morphism (b', F', θ'). Thus, the rule associating to each morphism (u, v) of $\mathcal{A}$-objects of $\mathcal{B}$ the $\mathcal{B}$-morphism u gives an identification of the morphisms (u, v) with a subset of the set of $\mathcal{B}$-morphisms; in particular, the morphisms (u, v) from (b, F, θ) to (b', F', θ') form a set. Using axioms on inaccessible cardinals or Grothendieck universes, one can also deal with the foundational issues around the objects. Altogether, this gives a category of $\mathcal{A}$-objects of $\mathcal{B}$, denoted $\mathcal{A} - \mathcal{B}$, together with a covariant, faithful functor, $L - \mathcal{B} : \mathcal{A} - \mathcal{B} \to \mathcal{B}$, sending (b, F, θ) to b and sending (u, v) to u.

The Yoneda Functor of an $\mathcal{A}$-Object. Formulate and prove the analogue of Problem for the Yoneda contravariant functors associated to $\mathcal{A}$-objects of $\mathcal{B}$.

Definition 10.3. Assume now that $\mathcal{A}$ has a terminal object and all finite products. A **Lawvere theory** for $\mathcal{A}$ is a category T with a terminal object and all finite products together with an $\mathcal{A}$-object (x_T, F_T, θ_T) in T such that for every category $\mathcal{B}$ having a terminal object and all finite products, every $\mathcal{A}$-object of $\mathcal{B}$ is equivalent to the $\mathcal{A}$-object of $\mathcal{B}$ associated to (b_T, F_T, θ_T) for a functor $G_{(b,F,\theta)} : T \to \mathcal{B}$ that is unique up to natural equivalence and satisfying the following minimality condition: every object of T equals the n-fold self product of x_T, x_T^n, for some nonnegative integer n.

Lawvere Theory for a Concrete Category with a Free Functor. If there exists a left adjoint $L : \mathbf{Sets} \to \mathcal{A}$ of R, then show that there is a Lawvere theory whose underlying category T equals the opposite category of the full subcategory of $\mathcal{A}$ obtained by evaluating L on the sets $[1, n]$ from Notation 9.3. In particular, conclude that there exists a Lawvere theory for monoids, for semigroups, for groups, for Abelian groups, for central A-algebras, and for commutative central A-algebras. When a Lawvere theory exists, use this to give another solution of the previous problem.

11 Adjoint Pairs of Limits and Colimits

Limits and Colimits Exercise. Mostly we use the special cases of products and coproducts. The notation here is meant to emphasize the connection with operations on presheaves and sheaves such as formation of global sections, stalks, pushforward and inverse image. Let τ be a small category. Let $\mathcal{C}$ be a category. A *τ-family* in $\mathcal{C}$ is a (covariant) functor,

$$\mathcal{F} : \tau \to \mathcal{C}.$$

Precisely, for every object U of τ, $\mathcal{F}(U)$ is a specified object of $\mathcal{C}$. For every morphism of objects of τ, $r : U \to V$, $\mathcal{F}(r) : \mathcal{F}(U) \to \mathcal{F}(V)$ is a morphism of $\mathcal{C}$. Also, $\mathcal{F}(\mathrm{Id}_U)$ equals $\mathrm{Id}_{\mathcal{F}(U)}$. Finally, for every pair of morphisms of τ, $r : U \to V$ and $s : V \to W$, $\mathcal{F}(s) \circ \mathcal{F}(r)$ equals $\mathcal{F}(s \circ r)$.

For every pair $\mathcal{F}$, $\mathcal{G}$ of τ-families in $\mathcal{C}$, a *morphism* of τ-families from $\mathcal{F}$ to $\mathcal{G}$ is a natural transformation of functors, $\phi : \mathcal{F} \Rightarrow \mathcal{G}$. For every object a of $\mathcal{C}$, denote by

$$\underline{a}_\tau : \tau \to \mathcal{C}$$

the functor that sends every object to a and that sends every morphism to Id_a. For every morphism in $\mathcal{C}$, $p : a \to b$, denote by
$$\underline{p}_\tau : \underline{a}_\tau \Rightarrow \underline{b}_\tau$$
the natural transformation that assigns to every object U of τ the morphism $p : a \to b$. Finally, for every object U of τ, denote
$$\Gamma(U, \mathcal{F}) = \mathcal{F}(U), \quad \Gamma(U, \theta) = \theta(U),$$
and for every morphism $r : U \to V$ of τ, denote
$$\Gamma(r, \mathcal{F}) = \mathcal{F}(r).$$

(i)(Functor Categories and Section Functors) For τ-families $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$, and for morphisms of τ-families, $\theta : \mathcal{F} \to \mathcal{G}$ and $\eta : \mathcal{G} \to \mathcal{H}$, define the composition of θ and η to be the composite natural transformation $\eta \circ \theta : \mathcal{F} \to \mathcal{H}$. **Prove** that with this notion, there is a category $\mathbf{Fun}(\tau, \mathcal{C})$ whose objects are τ-families $\mathcal{F}$ and whose morphisms are natural transformations. **Prove** that
$$\underline{*}_\tau : \mathcal{C} \to \mathbf{Fun}(\tau, \mathcal{C}), \ a \mapsto \underline{a}_\tau, \ p \mapsto \underline{p}_\tau,$$
is a functor that preserves monomorphisms, epimorphisms and isomorphisms. For every object U of τ, **prove** that
$$\Gamma(U, -) : \mathbf{Fun}(\tau, \mathcal{C}) \to \mathcal{C}, \ \mathcal{F} \mapsto \Gamma(U, \mathcal{F}), \ \theta \mapsto \Gamma(U, \theta),$$
is a functor. For every morphism $r : U \to V$ of τ, **prove** that $\Gamma(r, -)$ is a natural transformation $\Gamma(U, -) \Rightarrow \Gamma(V, -)$.

(ii)(Adjointness of Constant / Diagonal Functors and the Global Sections Functor) If $\mathcal{C}$ has an initial object X, **prove** that $(\underline{*}_\tau, \Gamma(X, -))$ extends to an adjoint pair of functors. More generally, a *limit* of a τ-family $\mathcal{F}$ (if it exists) is a natural transformation $\eta : \underline{a}_\tau \Rightarrow \mathcal{F}$ that is final among all such natural transformations, i.e., for every natural transformation $\theta : \underline{b}_\tau \Rightarrow \mathcal{F}$, there exists a unique morphism $t : b \to a$ in $\mathcal{C}$ such that θ equals $\eta \circ \underline{t}_\tau$. For a morphism $\phi : \mathcal{F} \to \mathcal{G}$, for limits $\eta : \underline{a}_\tau \Rightarrow \mathcal{F}$ and $\theta : \underline{b}_\tau \Rightarrow \mathcal{G}$, **prove** that there exists a unique morphism $f : a \to b$ such that $\theta \circ \underline{p}_\tau$ equals $\phi \circ \eta$. In particular, **prove** that if a limit of $\mathcal{F}$ exists, then it is unique up to unique isomorphism. In particular, for every object a of $\mathcal{C}$, **prove** that the identity transformation $\theta_a : \underline{a}_\tau \to \underline{a}_\tau$ is a limit of $\underline{a}_\tau$.

(iii)(Adjointness of Constant / Diagonal Functors and Limits) For this part, assume that every τ-family has a limit; a category $\mathcal{C}$ is said to *have all limits* if for every small category τ and for every τ-family $\mathcal{F}$, there is a limit. Assume further that there is a rule Γ_τ that assigns to every $\mathcal{F}$ an object $\Gamma_\tau(\mathcal{F})$ and a natural transformation $\eta_{\mathcal{F}} : \underline{\Gamma_\tau(\mathcal{F})}_\tau \to \mathcal{F}$ that is a limit. (Typically such a rule follows from the "construction" of limits, but such a rule also follows from some form of the Axiom of Choice.) **Prove** that this extends uniquely to a functor,
$$\Gamma_\tau : \mathbf{Fun}(\tau, \mathcal{C}) \to \mathcal{C},$$

and a natural transformation of functors

$$\eta : \underline{\star}_\tau \circ \Gamma_\tau \Rightarrow \mathrm{Id}_{\mathbf{Fun}(\tau,\mathcal{C})}.$$

Moreover, **prove** that the rule sending every object a of $\mathcal{C}$ to the identity natural transformation θ_a is a natural transformation $\theta : \mathrm{Id}_{\mathcal{C}} \Rightarrow \Gamma_\tau \circ \underline{\star}_\tau$. **Prove** that $(\underline{\star}_\tau, \Gamma, \theta, \eta)$ is an adjoint pair of functors. In particular, the limit functor Γ_τ preserves monomorphisms and sends injective objects of $\mathbf{Fun}(\tau, \mathcal{C})$ to injective objects of $\mathcal{C}$.

(iiii)(Adjointness of Colimits and Constant / Diagonal Functors) If $\mathcal{C}$ has a final object O, **prove** that $(\Gamma(O,-), \underline{\star}_\tau)$ extends to an adjoint pair of functors. More generally, a *colimit* of a τ-family $\mathcal{F}$ (if it exists) is a natural transformation $\theta : \mathcal{F} \Rightarrow \underline{a}_\tau$ that is final among all such natural transformations, i.e., for every natural transformation $\eta : \mathcal{F} \Rightarrow \underline{b}_\tau$, there exists a unique morphism $h : a \to b$ in $\mathcal{C}$ such that $\underline{h}_\tau \circ \theta$ equals η. For a morphism $\phi : \mathcal{F} \to \mathcal{G}$, for colimits $\theta : \mathcal{F} \Rightarrow \underline{a}_\tau$ and $\eta : \mathcal{G} \Rightarrow \underline{b}_\tau$, **prove** that there exists a unique morphism $f : a \to b$ such that $\underline{f}_\tau \circ \theta$ equals $\eta \circ \phi$. In particular, **prove** that if a colimit of $\mathcal{F}$ exists, then it is unique up to unique isomorphism. In particular, for every object a of $\mathcal{C}$, **prove** that the identity transformation $\theta_a : \underline{a}_\tau \to \underline{a}_\tau$ is a colimit of $\underline{a}_\tau$. Finally, **repeat** the previous part for colimits in place of limits. Deduce that colimits (if they exist) preserve epimorphisms and projective objects.

(v)(Functoriality in the Source) Let $x : \sigma \to \tau$ be a functor of small categories. For every τ-family $\mathcal{F}$, define $\mathcal{F}_x$ to be the composite functor $\mathcal{F} \circ x$, which is a σ-family. For every morphism of τ-families, $\phi : \mathcal{F} \to \mathcal{G}$, define $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$ to be $\phi \circ x$, which is a morphism of σ-families. **Prove** that this defines a functor

$$\star_x : \mathbf{Fun}(\tau, \mathcal{C}) \to \mathbf{Fun}(\sigma, \mathcal{C}).$$

For the identity functor $\mathrm{Id}_\tau : \tau \to \tau$, **prove** that $\star_{\mathrm{Id}_\tau}$ is the identity functor. For $y : \rho\ \sigma$ a functor of small categories, **prove** that $\star_{x \circ y}$ is the composite $\star_y \circ \star_x$. In this sense, deduce that $\star_x$ is a contravariant functor in x.

For two functors, $x, x_1 : \sigma \to \tau$ and for a natural transformation $n : x \Rightarrow x_1$, define $\mathcal{F}_n : \mathcal{F}_x \Rightarrow \mathcal{F}_{x_1}$ to be $\mathcal{F}(n(V)) : \mathcal{F}(x(V)) \to \mathcal{F}(x_1(V))$ for every object V of σ. **Prove** that $\mathcal{F}_n$ is a morphism of σ-families. For every morphism of τ-families, $\phi : \mathcal{F} \to \mathcal{G}$, **prove** that $\phi_{x_1} \circ \mathcal{F}_n$ equals $\mathcal{G}_n \circ \phi_x$. In this sense, conclude that $\star_n$ is a natural transformation $\star_x \Rightarrow \star_{x'}$. For the identity natural transformation $\mathrm{Id}_x : x \Rightarrow x$, **prove** that $\star_{\mathrm{Id}_x}$ is the identity natural transformation of $\star_x$. For a second natural transformation $m : x_1 \Rightarrow x_1$, **prove** that $\mathcal{F}_{m \circ n}$ equals $\mathcal{F}_m \circ \mathcal{F}_n$. In this sense, deduce that $\star_x$ is also compatible with natural transformations. In particular, if (x, y, θ, η) is an adjoint pair of functors, **prove** that $(\star_y, \star_x, \star_\theta, \star_\eta)$ is an adjoint pair of functors.

(vi)(Fiber Categories) The following notion of *fiber category* is a special case of the notion of *2-fiber product* of functors of categories. Let $x : \sigma \to \tau$ be a functor; this is also called a *category over* τ. For every object U of τ, a *$\sigma_{x,U}$-object* is a pair $(V, r : x(V) \to U)$ of an object V of σ and a τ-isomorphism $r : x(V) \to U$. For two objects $\sigma_{x,U}$-objects (V, r) and (V', r') of $\sigma_{x,U}$, a *$\sigma_{x,U}$-morphism* from (V, r) to (V', r') is a morphism of σ, $s : V \to V'$, such that $r' \circ x(s)$ equals r. **Prove** that Id_V is a $\sigma_{x,U}$-morphism from (V, r) to itself; more generally, the $\sigma_{x,U}$-morphisms from (V,r) to (V,r) are precisely the σ-morphisms $s:V\to V$ such that $x(s)$ equals $\text{Id}_{x(V)}$. For every pair of $\sigma_{x,U}$-morphisms, $s:(V,r)\to(V',r')$ and $s':(V',r')\to(V'',r'')$, **prove** that $s'\circ s$ is a $\sigma_{x,U}$-morphism from (V,r) to (V'',r''). Conclude that these rules form a category, denoted $\sigma_{x,U}$. **Prove** that the rule $(V,r)\mapsto V$ and $s\mapsto s$ defines a faithful functor,

$$\Phi_{x,U}:\sigma_{x,U}\to\sigma,$$

and $r:x(V)\to U$ defines a natural isomorphism $\theta_{x,U}:x\circ\Phi_{x,U}\Rightarrow \underline{U}_{\sigma_{x,U}}$. Finally, for every category σ', for every functor $\Phi':\sigma'\to\sigma$, and for every natural isomorphism $\theta':x\circ\Phi'\Rightarrow\underline{U}_{\sigma'}$, **prove** that there exists a unique functor $F:\sigma'\to\sigma_{x,U}$ such that Φ' equals $\Phi_{x,U}\circ F$ and θ' equals $\theta_{x,U}\circ F$. In this sense, $(\Phi_{x,U},\theta_{x,U})$ is final among pairs (Φ',θ') as above.

For every pair of functors $x,x_1:\sigma\to\tau$, and for every natural *isomorphism* $n:x\Rightarrow x_1$, for every $\sigma_{x_1,U}$-object $(V,r_1:x_1(V)\to U)$, **prove** that $(V,r_1\circ n_V:x(V)\to U)$ is an object of $\sigma_{x,U}$. For every morphism in $\sigma_{x_1,U}$, $s:(V,r_1)\to(V',r_1')$, **prove** that s is also a morphism $(V,r_1\circ n_V)\to(V',r_1'\circ n_{V'})$. Conclude that these rules define a functor,

$$\sigma_{n,U}:\sigma_{x_1,U}\to\sigma_{x,U}.$$

Prove that this functor is a *strict equivalence* of categories: it is a bijection on Hom sets (as for all equivalences), but it is also a bijection on objects (rather than merely being essentially surjective). **Prove** that $\sigma_{n,U}$ is functorial in n, i.e., for a second natural isomorphism $m:x_1\Rightarrow x_2$, prove that $\sigma_{m\circ n,U}$ equals $\sigma_{n,U}\circ\sigma_{m,U}$.

For every pair of functors, $x:\sigma\to\tau$ and $y:\rho\to\tau$, and for every functor $z:\sigma\to\rho$ such that x equals $y\circ z$ equals x, for every $\sigma_{x,U}$-object (V,r), **prove** that $(z(V),r)$ is a $\rho_{y,U}$-object. For every $\sigma_{x,U}$-morphism $s:(V,r)\to(V',r')$, **prove** that $z(s)$ is a $\rho_{y,U}$-morphism $(z(V),r)\to(z(V'),r')$. **Prove** that $z(\text{Id}_V)$ equals $\text{Id}_{z(V)}$, and **prove** that z preserves composition. Conclude that these rules define a functor,

$$z_U:\sigma_{x,U}\to\rho_{y,U}.$$

Prove that this is functorial in z: $(\text{Id}_\sigma)_U$ equals $\text{Id}_{\sigma_{x,U}}$, and for a third functor $w:\pi\to\tau$ and functor $z':\rho\to\pi$ such that y equals $w\circ z'$, then $(z'\circ z)_U$ equals $z'_U\circ z_U$. For an object (W,r_W) of $\rho_{y,U}$, for each object $((V,r_V),q:Z(V)\to W)$ of $(\sigma_{x,U})_{z,(W,r_W)}$, define the *associated* object of $\sigma_{z,W}$ to be (V,q). For an object $((V',r_{V'}),q':Z(V')\to W)$ of $(\sigma_{x,U})_{z,(W,r_W)}$, for every morphism $s:(V,r_V)\to(V',r_{V'})$ such that q equals $q'\circ z(s)$, define the *associated* morphism of $\sigma_{z,W}$ to be s. **Prove** that this defines a functor

$$\widetilde{z}_{U,(W,r_W)}:(\sigma_{x,U})_{z_U,(W,r_W)}\to\sigma_{z,W}.$$

Prove that this functor is a strict equivalence of categories. **Prove** that this equivalence is functorial in z. Finally, for two functors $z,z_1:\sigma\to\rho$ such that x equals both $y\circ z$ and $y\circ z_1$, and for a natural transformation $m:z\Rightarrow z_1$, for every object $(V,r:x(V)\to U)$ of $\sigma_{x,U}$, **prove** that m_V is a morphism in $\rho_{y,U}$ from $(z(V),r)$ to $(z_1(V),r)$. Moreover, for every morphism in $\sigma_{x,U}$, $s:(V,r)\to(V',r')$, **prove** that $m_{V'}\circ z(s)$ equals $z_1(s)\circ m_V$. Conclude that this rule is a natural transformation $m_U : z_U \Rightarrow (z_1)_U$. **Prove** that this is functorial in m. If m is a natural isomorphism, **prove** that also m_U is a natural isomorphism, and the strict equivalence $(m_U)_{(W,r_W)}$ is compatible with the strict equivalence m_W. Finally, **prove** that $m \mapsto m_U$ is compatible with precomposition and postcomposition of m with functors of categories over τ.

(vii)(Colimits and Limits along an Essentially Surjective Functor) Let $x : \sigma \to \tau$ be a functor of small categories. **Prove** that every fiber category $\sigma_{x,U}$ is small. Next, assume that x is *essentially surjective*, i.e., for every object U of τ, there exists a $\sigma_{x,U}$-object (V, r). Let $y : \tau \to \sigma$ be a functor, and let $\alpha : \mathrm{Id}_\sigma \Rightarrow y \circ x$ be a natural transformation. **Prove** that this extends to an adjoint pair of functors (x, y, α, β) if and only if for every object V of σ, the morphism $x(\alpha_V) : x(V) \to x(y(x(V)))$ is an isomorphism and $(y(x(V)), x(\alpha_V)^{-1})$ is a final object of the fiber category $\sigma_{x,x(V)}$. (Conversely, up to some form of the Axiom of Choice, there exists y and α extending to an adjoint pair if and only if every fiber category $\sigma_{x,U}$ has a final object.) For every adjoint pair (x, y, α, β), also $(\star_y, \star_x, \star_\alpha, \star_\beta)$ is an adjoint pair. More generally, no longer assume that there exists y and α, yet let L_x be a rule that assigns to every object $\mathcal{F}$ of $\mathbf{Fun}(\sigma, \mathcal{C})$ an object $L_x(\mathcal{F})$ of $\mathbf{Fun}(\tau, \mathcal{C})$ and a natural transformation,

$$\theta_{\mathcal{F}} : \mathcal{F} \to \star_x \circ L_x(\mathcal{F}),$$

of objects in $\mathbf{Fun}(\sigma, \mathcal{C})$. For every object U of τ, this defines a natural transformation

$$\theta_{\mathcal{F},x,U} : \mathcal{F} \circ \Phi_{x,U} \Rightarrow L_x(\mathcal{F}) \circ \underline{U}_{\sigma_{x,U}},$$

of objects in $\mathbf{Fun}(\sigma_{x,U}, \mathcal{C})$. Assume that each $(L_x(\mathcal{F})(U), \theta_{\mathcal{F},x,U})$ is a colimit of $\mathcal{F} \circ \Phi_{x,U}$. **Prove** that this extends uniquely to a functor,

$$L_x : \mathbf{Fun}(\sigma, \mathcal{C}) \to \mathbf{Fun}(\tau, \mathcal{C}),$$

and a natural transformation

$$\theta_x : \mathrm{Id}_{\mathbf{Fun}(\sigma,\mathcal{C})} \Rightarrow \star_x \circ L_x.$$

Moreover, for every $\mathcal{G}$ in $\mathbf{Fun}(\tau, \mathcal{C})$, the identity morphism,

$$\mathrm{Id}_{\mathcal{G}} : \mathcal{G} \circ x \circ \Phi_{x,U} \to \mathcal{G} \circ \underline{U}_{\sigma_{x,U}},$$

factors uniquely through a $\mathcal{C}$-morphism $L_x(\mathcal{G} \circ x)(U) \to \mathcal{G}(U)$. **Prove** that this defines a morphism $\eta_{\mathcal{G}} : L_x(\mathcal{G} \circ x) \to \mathcal{G}$ in $\mathbf{Fun}(\tau, \mathcal{C})$. **Prove** that is a natural transformation,

$$\eta : L_x \circ \star_x \Rightarrow \mathrm{Id}_{\mathbf{Fun}(\tau,\mathcal{C})}.$$

Prove that $(L_x, \star_x, \theta, \eta)$ is an adjoint pair of functors. (Using some version of the Axiom of Choice, if every $\mathcal{F} \circ \Phi_{x,U}$ admits a colimit, then there exists a Γ^x and θ as above.)

Next, as above, let $x : \sigma \to \tau$ be a functor of small catgories that is essentially surjective. Let $y : \tau \to sigma$ be a functor, and let $\beta : y \circ x \Rightarrow \mathrm{Id}_\sigma$ be a natural transformation. **Prove** that this extends to an adjoint pair of functors (x, y, α, β) if and only if for every object V of σ, the morphism $x(\beta_v) : x(y(x(V))) \to x(V)$ is an isomorphism and $(y(x(V)), x(\beta_v))$ is an initial object of the fiber category $\sigma_{x,x(V)}$. (Conversely, up to some form of the Axiom of Choice, there exists y and β extending to an adjoint pair if and only if every fiber category $\sigma_{x,U}$ has an initial object.) For every adjoint pair $(y, x, \alpha, beta)$ also $(*_x, *_y, *_\alpha, *_\beta)$ is an adjoint pair. More generally, no longer assume that there exists y and β, yet let R_x be a rule that assigns to every object $\mathcal{F}$ of $\mathbf{Fun}(\sigma, \mathcal{C})$ an object $R_x(\mathcal{F})$ of $\mathbf{Fun}(\tau, \mathcal{C})$ and a natural transformation,

$$\eta_{\mathcal{F}} : *_x \circ R_x(\mathcal{F}) \to \mathcal{F},$$

of objects in $\mathbf{Fun}(\sigma, \mathcal{C})$. For every object U of τ, this defines a natural transformation

$$\eta_{\mathcal{F},x,U} : R_x(\mathcal{F}) \circ \underline{U}_{\sigma_{x,U}} \Rightarrow \mathcal{F} \circ \Phi_{x,U},$$

of objects in $\mathbf{Fun}(\sigma_{x,U}, \mathcal{C})$. Assume that each $(R_x(\mathcal{F})(U), \eta_{\mathcal{F},x,U})$ is a limit of $\mathcal{F} \circ \Phi_{x,U}$. **Prove** that this extends uniquely to a functor,

$$R_x : \mathbf{Fun}(\sigma, \mathcal{C}) \to \mathbf{Fun}(\tau, \mathcal{C}),$$

and a natural transformation,

$$\eta : *_x \circ R_x \Rightarrow \mathrm{Id}_{\mathbf{Fun}(\sigma,\mathcal{C})}.$$

Moreover, for every $\mathcal{G}$ in $\mathbf{Fun}(\tau, \mathcal{C})$, the identity morphism,

$$\mathrm{Id}_{\mathcal{G}} : \mathcal{G} \circ \underline{U}_{\sigma_{x,U}} \Rightarrow \mathcal{G} \circ x \circ \Phi_{x,U},$$

factors uniquely through a $\mathcal{G}(U) \to \mathcal{C}$-morphism $R_x(\mathcal{G} \circ x)(U)$. **Prove** that this defines a morphism $\theta_{\mathcal{G}} : \mathcal{G} \to R_x(\mathcal{G} \circ x)$ in $\mathbf{Fun}(\tau, \mathcal{C})$. **Prove** that this is a natural transformation,

$$\theta : \mathrm{Id}_{\mathbf{Fun}(\tau,\mathcal{C})} \Rightarrow R_x \circ *_x.$$

Prove that $(*_x, R_x, \theta, \eta)$ is an adjoint pair of functors. (Using some version of the Axiom of Choice, if every $\mathcal{F} \circ \Phi_{x,U}$ admits a colimit, then there exists a R_x and η as above.)

(viii)(Adjoints Relative to a Full, Upper Subcategory) In a complementary direction to the previous case, let $x : \sigma \to \tau$ be an embedding of a full subcategory (thus, x is essentially surjective if and only if x is an equivalence of categories). In this case, the functor

$$*_x : \mathbf{Fun}(\tau, \mathcal{C}) \to \mathbf{Fun}(\sigma, \mathcal{C})$$

is called *restriction*. Assume further that σ is *upper* (a la the theory of partially ordered sets) in the sense that every morphism of τ whose source is an object of σ also has target an object of σ. Assume that $\mathcal{C}$ has an initial object, $\odot$. Let $\mathcal{G}$ be a σ-family of objects of $\mathcal{C}$. Also, let $\phi : \mathcal{G} \to \mathcal{H}$ be a morphism of σ-families. For every object U of τ, if U is an object of σ, then define ${}_x\mathcal{G}(U)$ to be $\mathcal{G}(U)$, and define ${}_x\phi(U)$ to be $\phi(U)$. For every object U of τ that is not an object of σ, define ${}_x\mathcal{G}(U)$ to be $\odot$, and define ${}_x\phi(U)$ to be $\mathrm{Id}_\odot$. For every morphism $r : U \to V$, if U is an object of σ, then r is a morphism of σ. In this case, define ${}_x\mathcal{G}(r)$ to be $\mathcal{G}(r)$. On the other hand, if U is not an object of σ, then $\mathcal{G}(U)$ is the initial object $\odot$. In this case, define ${}_x\mathcal{G}(r)$ to be the unique morphism ${}_x\mathcal{G}(U) \to {}_x\mathcal{G}(V)$. **Prove** that ${}_x\mathcal{G}$ is a τ-family of objects, i.e., the definitions above are compatible with composition of morphisms in τ and with identity morphisms. Also **prove** that ${}_x\phi$ is a morphism of τ-families. **Prove** that ${}_x\,\mathrm{Id}_{\mathcal{G}}$ equals $\mathrm{Id}_{{}_x\mathcal{G}}$. Also, for a second morphism of σ-families, $\psi : \mathcal{H} \to \mathcal{I}$, **prove** that ${}_x(\psi \circ \phi)$ equals ${}_x\psi \circ {}_x\phi$. Conclude that these rules form a functor,

$$ {}_x* : \mathbf{Fun}(\sigma, \mathcal{C}) \to \mathbf{Fun}(\tau, \mathcal{C}). $$

Prove that $({}_x*, *_x)$ extends to an adjoint pair of functors. In particular, conclude that $*_x$ preserves epimorphisms and ${}_x*$ preserves monomorphisms.

Next assume that $\mathcal{C}$ is an Abelian category that satisfies (AB3). For every τ-family $\mathcal{F}$, for every object U of τ, define $\theta_{\mathcal{F}}(U) : \mathcal{F}(U) \to {}^x\mathcal{F}(U)$ to be the cokernel of $\mathcal{F}(U)$ by the direct sum of the images of

$$ \mathcal{F}(s) : \mathcal{F}(T) \to \mathcal{F}(U), $$

for all morphisms $s : T \to U$ with V *not* in σ (possibly empty, in which case $\theta_{\mathcal{F}}(U)$ is the identity on $\mathcal{F}(U)$). In particular, if U is not in σ, then ${}^x\mathcal{F}(U)$ is zero. For every morphism $r : U \to V$ in τ, **prove** that the composition $\theta_{\mathcal{F}}(V) \circ \mathcal{F}(r)$ equals ${}^x\mathcal{F}(r) \circ \theta_{\mathcal{F}}(U)$ for a unique morphism

$$ {}^x\mathcal{F}(r) : {}^x\mathcal{F}(U) \to {}^x\mathcal{F}(V). $$

Prove that ${}^x\,\mathcal{F}(\mathrm{Id}_U)$ is the identity morphism of ${}^x\,\mathcal{F}(U)$. **Prove** that $r \mapsto {}^x\,\mathcal{F}(r)$ is compatible with composition in τ. Conclude that ${}^x\mathcal{F}$ is a τ-family, and $\theta_{\mathcal{F}}$ is a morphism of τ-families. For every morphism $\phi : \mathcal{F} \to \mathcal{E}$ of τ-families, for every object U of τ, **prove** that $\theta_{\mathcal{E}}(U) \circ \phi(U)$ equals ${}^x\phi(U) \circ \theta_{\mathcal{F}}(U)$ for a unique morphism

$$ {}^x\phi(U) : {}^x\mathcal{F}(U) \to {}^x\mathcal{E}(U). $$

Prove that the rule $U \mapsto {}^x\,\phi(U)$ is a morphism of τ-families. **Prove** that ${}^x\,\mathrm{Id}_{\mathcal{F}}$ is the identity on ${}^x\,\mathcal{F}$. Also **prove** that $\phi \mapsto {}^x\,\phi$ is compatible with composition. Conclude that these rules define a functor

$$ {}^x* : \mathbf{Fun}(\tau, \mathcal{C}) \to \mathbf{Fun}(\tau, \mathcal{C}). $$

Prove that the rule $\mathcal{F} \mapsto \theta_{\mathcal{F}}$ is a natural transformation $\mathrm{Id}_{\mathbf{Fun}(\tau,\mathcal{C})} \Rightarrow {}^x*$. **Prove** that the natural morphism of τ-families,

$$ {}^x\mathcal{F} \to {}_x(({}^x\mathcal{F})_x), $$

is an isomorphism. Conclude that there exists a unique functor,

$$ *^x : \mathbf{Fun}(\tau, \mathcal{C}) \to \mathbf{Fun}(\sigma, \mathcal{C}), $$

and a natural isomorphism ${}^x* \Rightarrow {}_x(*^x)$. **Prove** that $(*^x, {}_x*, \theta)$ extends to an adjoint pair of functors. In particular, conclude that ${}_x*$ preserves epimorphisms and $*^x$ preserves monomorphisms.

Finally, drop the assumption that $\mathcal{C}$ has an initial object, but assume that σ is upper, assume that σ has an initial object, W_σ, and assume that there is a functor

$$ y : \tau \to \sigma $$

and a natural transformation $\theta : \mathrm{Id}_\tau \Rightarrow x \circ y$, such that for every object U of τ, the unique morphism $W_\sigma \to y(U)$ and the morphism $\theta_U : U \to y(U)$ make $y(U)$ into a coproduct of W_σ and U in τ. For simplicity, for every object U of σ, assume that $\theta_U : U \to y(U)$ is the identity Id_U (rather than merely being an isomorphism), and for every morphism $r : U \to V$ in σ, assume that $y(r)$ equals r. Thus, for every object V of σ, the identity morphism $y(V) \to V$ defines a natural transformation $\eta : y \circ x \Rightarrow \mathrm{Id}_\sigma$. **Prove** that (y, x, θ, η) is an adjoint pair of functors. Conclude that $(\star_x, \star_y, \star_\theta, \star_\eta)$ is an adjoint pair of functors. In particular, conclude that $\star_x$ preserves monomorphisms and $\star_y$ preserves epimorphisms.

(ix)(Compatibility of Limits and Colimits with Functors) Denote by 0 the "singleton category" 0 with a single object and a single morphism. **Prove** that $\Gamma(0, -)$ is an equivalence of categories. For an arbitrary category τ, for the unique natural transformation $\hat{\tau} : \tau \to 0$, **prove** that $\star_{\hat{\tau}}$ equals the composite $\underline{\star}_\tau \circ \Gamma(0, -)$ so that $\underline{\star}_\tau$ is an example of this construction. In particular, for every functor $x : \sigma \to \tau$, **prove** that $(\underline{a}_\tau)_x$ equals $\underline{a}_\sigma$. If $\eta : \underline{a}_\tau \Rightarrow \mathcal{F}$ is a limit of a τ-family $\mathcal{F}$, and if $\theta : \underline{b}_\sigma \Rightarrow \mathcal{F}_x$ is a limit of the associated σ-family $\mathcal{F}_x$, then **prove** that there is a unique morphism $h : a \to b$ in $\mathcal{C}$ such that η_x equals $\theta \circ \underline{p}_\sigma$. If there are right adjoints Γ_τ of $\underline{\star}_\tau$ and Γ_σ of $\underline{\star}_\sigma$, conclude that there exists a unique natural transformation

$$\Gamma_x : \Gamma_\tau \Rightarrow \Gamma_\sigma \circ \star_x$$

so that $\eta_{\mathcal{F}_x} \circ \underline{\Gamma_x(\mathcal{F})}_\sigma$ equals $(\eta_{\mathcal{F}})_x$. **Repeat** this construction for colimits.

(x)(Limits / Colimits of a Concrete Category) Let σ be a small category in which the only morphisms are identity morphisms: identify σ with the underlying set of objects. Let $\mathcal{C}$ be the category **Sets**. For every σ-family $\mathcal{F}$, **prove** that the rule

$$\Gamma_\sigma(\mathcal{F}) := \prod_{U \in \Sigma} \Gamma(U, \mathcal{F})$$

together with the morphism

$$\eta_{\mathcal{F}} : \underline{\Gamma_\sigma(\mathcal{F})}_\sigma \Rightarrow \mathcal{F},$$
$$\eta_{\mathcal{F}}(V) = \mathrm{pr}_V : \prod_{U \in \Sigma} \Gamma(U, \mathcal{F}) \to \Gamma(V, \mathcal{F}),$$

is a limit of $\mathcal{F}$. Next, for every small category τ, define σ to be the category with the same objects as τ, but with the only morphisms being identity morphisms. Define $x : \sigma \to \tau$ to be the unique functor that sends every object to itself. Define $\Gamma_\tau(\mathcal{F})$ to be the subobject of $\Gamma_\sigma(\mathcal{F}_x)$ of data $(f_U)_{U \in \Sigma}$ such that for every morphism $r : U \to V$, $\mathcal{F}(r)$ maps f_U to f_V. **Prove** that with this definition, there exists a unique natural transformation $\eta_{\mathcal{F}} : \underline{\Gamma_\tau(\mathcal{F})}_\tau \Rightarrow \mathcal{F}$ such that the natural transformation $\underline{\Gamma_\tau(\mathcal{F})}_\sigma \Rightarrow \underline{\Gamma_\sigma(\mathcal{F}_x)} \Rightarrow \mathcal{F}_x$ equals $(\eta_{\mathcal{F}})_x$. **Prove** that $\eta_{\mathcal{F}}$ is a limit of $\mathcal{F}$. Conclude that **Sets** has all small limits. Similarly, for associative, unital rings R and S, **prove** that the forgetful functor

$$\Phi : R - S - \mathrm{mod} \to \mathbf{Sets}$$

sends products to products. Let $\mathcal{F}$ be a τ-family of $R - S$-modules. **Prove** that the defining relations for $\Gamma_\tau(\Phi \circ \mathcal{F})$ as a subset of $\Gamma_\sigma(\Phi \circ \mathcal{F})$ are the simultaneous kernels of $R - S$-module homomorphisms. Conclude that there is a natural $R-S$-module structure on $\Gamma_\tau(\Phi \circ \mathcal{F})$, and use this to **prove** that $R-S$-mod has all limits.

(xi)(Functoriality in the Target) For every functor of categories,

$$H : \mathcal{C} \to \mathcal{D},$$

for every τ-family $\mathcal{F}$ in $\mathcal{C}$, **prove** that $H \circ \mathcal{F}$ is a τ-family in $\mathcal{D}$. For every morphism of τ-families in $\mathcal{C}$, $\phi : \mathcal{F} \Rightarrow \mathcal{G}$, **prove** that $H \circ \phi$ is a morphism of τ-families in $\mathcal{D}$. **Prove** that this defines a functor

$$H_\tau : \mathbf{Fun}(\tau, \mathcal{C}) \to \mathbf{Fun}(\tau, \mathcal{D}).$$

For the identity functor $\text{Id}_{\mathcal{C}}$, **prove** that $(\text{Id}_{\mathcal{C}})_\tau$ is the identity functor. For $I : \mathcal{D} \to \mathcal{E}$ a functor of categories, **prove** that $(I \circ H)_\tau$ is the composite $I_\tau \circ H_\tau$. In this sense, deduce that H_τ is functorial in H.

For two functors, $H, I : \mathcal{C} \to \mathcal{D}$, and for a natural transformation $N : H \Rightarrow I$, for every τ-family $\mathcal{F}$ in $\mathcal{C}$, define $N_\tau(\mathcal{F})$ to be

$$N \circ \mathcal{F} : H \circ \mathcal{F} \Rightarrow I \circ \mathcal{F}.$$

Prove that $N_\tau(\mathcal{F})$ is a morphism of τ-families in $\mathcal{D}$. For every morphism of τ-families in $\mathcal{C}$, $\phi : \mathcal{F} \to \mathcal{G}$, **prove** that $N_\tau(\mathcal{G}) \circ H_\tau(\phi)$ equals $I_\tau(\phi) \circ N_\tau(\mathcal{F})$. In this sense, conclude that N_τ is a natural transformation $H_\tau \Rightarrow I_\tau$. For the identity natural transformation $\text{Id}_H : H \Rightarrow H$, **prove** that $(\text{Id}_H)_\tau$ is the identity natural transformation of H_τ. For a second natural transformation $M : I \Rightarrow J$, **prove** that $(M \circ N)_\tau$ equals $M_\tau \circ N_\tau$. In this sense, deduce that $(-)_\tau$ is also compatible with natural transformations.

(xii)(Reductions of Limits to Finite Systems for Concrete Categories) A category is *cofiltering* if for every pair of objects U and V there exists a pair of morphisms, $r : W \to U$ and $s : W \to V$, and for every pair of morphisms, $r, s : V \to U$, there exists a morphism $t : W \to V$ such that $r \circ t$ equals $s \circ t$ (both of these are automatic if the category has an initial object X). Assume that the category $\mathcal{C}$ has limits for all categories τ with finitely many objects, and also for all small cofiltering categories. For an arbitrary small category τ, define $\widehat{\tau}$ to be the small category whose objects are finite full subcategories σ of τ, and whose morphisms are inclusions of subcategories, $\rho \subset \sigma$, of τ. **Prove** that $\widehat{\tau}$ is cofiltering. Let $\mathcal{F}$ be a τ-family in $\mathcal{C}$. For every finite full subcategory $\sigma \subset \tau$, denote by $\mathcal{F}_\sigma$ the restriction as in (f) above. By hypothesis, there is a limit $\eta_\sigma : \underline{\widehat{\mathcal{F}}(\sigma)}_\sigma \Rightarrow \mathcal{F}_\sigma$. Moreover, by (g), for every inclusion of full subcategories $\rho \subset \sigma$, there is a natural morphism in $\mathcal{C}$, $\widehat{\mathcal{F}}(\rho) \to \widehat{\mathcal{F}}(\sigma)$, and this is functorial. Conclude that $\widehat{\mathcal{F}}$ is a $\widehat{\tau}$-family in $\mathcal{C}$. Since $\widehat{\tau}$ is filtering, there is a limit

$$\eta_{\widehat{\mathcal{F}}} : \underline{a}_{\widehat{\tau}} \Rightarrow \widehat{\mathcal{F}}.$$

Prove that this defines a limit $\eta_{\mathcal{F}} \underline{a}_\tau \Rightarrow \mathcal{F}$.

Finally, use this to **prove** that limits exist in each of the following categories: the category of (not necessarily Abelian) groups, the category of Abelian groups, the category of associative, unital (not necessarily commutative) rings, the category of commutative rings, and the category of $R-S$-bimodules (where R and S are associative, unital rings).

(xiii)(bis, Colimits) Repeat the steps above for colimits in place of limits. Use this to **prove** that colimits exist in each of the following categories: the category of (not necessarily Abelian) groups, the category of Abelian groups, the category of associative, unital (not necessarily commutative) rings, the category of commutative rings, and the category of $R-S$-bimodules (where R and S are associative, unital rings).

Practice with Limits and Colimits Exercise. In each of the following cases, say whether the given category (a) has an initial object, (b) has a final object, (c) has a zero object, (d) has finite products, (e) has finite coproducts, (f) has arbitrary products, (g) has arbitrary coproducts, (h) has arbitrary limits (sometimes called *inverse limits*), (i) has arbitrary colimits (sometimes called *direct limits*), (j) coproducts / filtering colimits preserve monomorphisms, (k) products / cofiltering limits preserve epimorphisms.

(i) The category **Sets** whose objects are sets, whose morphisms are set maps, whose composition is usual composition, and whose identity morphisms are usual identity maps.

(ii) The opposite category $\mathbf{Sets}^{\text{opp}}$.

(iii) For a given set S, the category whose objects are elements of the set, and where the only morphisms are the identity morphisms from an element to that same element. What if the set is the empty set? What if the set is a singleton set?

(iv) For a partially ordered set $(S, \leq)$, the category whose objects are elements of S, and where the Hom set between two elements x, y of S is a singleton set if $x \leq y$ and empty otherwise. What if the partially ordered set $(S, \leq)$ is a **lattice**, i.e., every finite subset (resp. arbitrary subset) has a least upper bound and has a greatest lower bound?

(v) For a monoid $(M, \cdot, 1)$, the category with only one object whose Hom set, with its natural composition and identity, is $(M, \cdot, 1)$. What is M equals $\{1\}$?

(vi) For a monoid $(M, \cdot, 1)$ and an action of that monoid on a set, $\rho : M \times S \to S$, the category whose objects are the elements of S, and where the Hom set from x to y is the subset $M_{x,y} = \{m \in M | m \cdot x = y\}$. What if the action is both transitive and faithful, i.e., S equals M with its left regular representation?

(vii) The category **PtdSets** whose objects are pairs (S, s_0) of a set S and a specified element s_0 of S, i.e., *pointed sets*, whose morphisms are set maps that send the specified point of the domain to the specified point of the target, whose composition is usual composition, and whose identity morphisms are usual identity maps.

(viii) The category **Monoids** whose objects are monoids, whose morphisms are homomorphisms of monoids, whose composition is sual composition, and whose identity morphisms are usual identity maps.

(ix) For a specified monoid $(M, \cdot, 1)$, the category whose objects are pairs (S, ρ) of a set S and an action $\rho : M \times S \to S$ of M on S, whose morphisms are set maps compatible with the action, whose composition is usual composition, and whose identity morphisms are usual identity maps.

(x) The full subcategory **Groups** of **Monoids** whose objects are groups. Does the inclusion functor preserve coproducts, resp. products? Does the inclusion functor preserve monomorphisms, resp. epimorphisms?

(xi) The full subcategory $\mathbb{Z}-\text{mod}$ of **Groups** whose objects are Abelian groups. Does the inclusion functor preserve coproducts, resp. products? Does the inclusion functor preserve monomorphisms, resp. epimorphisms?

(xii) The full subcategory **FiniteGroups** of **Groups** whose objects are finite groups. Are coproducts, resp. products, in the subcategory also coproducts, resp. products, in the larger category **Groups**? Does the inclusion functor preserve monomorphisms, resp. epimorphisms?

(xiii) The full subcategory $\mathbb{Z}-\text{mod}_{\text{tor}}$ of $\mathbb{Z}-\text{mod}$ consisting of torsion Abelian groups, i.e., every element has finite order (allowed to vary from element to element). Are coproducts, resp. products, preserved by the inclusion functor? Are monomorphisms, resp. epimorphisms preserved?

(xiv) The category **Rings** whose objects are associative, unital rings, whose morphisms are homomorphisms of rings (preserving the multiplicative identity), whose composition is the usual composition, and whose identity morphisms are the usual identity maps. **Hint.** For the coproduct of two associative, unital rings $(R', +, 0, \cdot', 1')$ and $(R'', +, 0, \cdot'', 1'')$, first form the coproduct $R' \oplus R''$ of $(R', +, 0)$ and $(R'', +, 0)$ as a $\mathbb{Z}$-module, then form the total tensor product ring $T^\bullet_{\mathbb{Z}}(R' \oplus R'')$ as in the previous problem set. For the two natural maps $q' : R' \hookrightarrow T^1_{\mathbb{Z}}(R' \oplus R'')$ and $q'' : R'' \hookrightarrow T^1_{\mathbb{Z}}(R' \oplus R'')$ form the left-right ideal $I \subset T^\bullet_{\mathbb{Z}}(R' \oplus R'')$ generated by $q'(1') - 1$, $q''(1'') - 1$, $q'(r' \cdot' s') - q'(r') \cdot q'(s')$, and $q''(r'' \cdot'' s'') - q''(r'') \cdot q''(s'')$ for all elements $r', s' \in R'$ and $r'', s'' \in R''$. Define

$$p : T^1_{\mathbb{Z}}(R' \oplus R'') \to R,$$

to be the quotient by I. Prove that $p \circ q' : R' \to R$ and $p \circ q'' : R'' \to R$ are ring homomorphisms that make R into a coproduct of R' and R''.

(xv) The full subcategory **CommRings** of **Rings** whose objects are commutative, unital rings. Does the inclusion functor preserve coproducts, resp. products? Does the inclusion functor preserve monomorphisms, resp. epimorphisms?

(xvi) The full subcategory **NilCommRings** of **CommRings** whose objects are commutative, unital rings such that every noninvertible element is nilpotent. Does the inclusion functor preserve coproducts, resp. products? (Be careful about products!) Does the inclusion functor preserve monomorphisms, resp. epimorphisms?

(xvii) Let R and S be associative, unital rings. Let $R-\text{mod}$, resp. $\text{mod}-S$, $R-S-\text{mod}$, be the category of left R-modules, resp. right S-modules, $R-S$-bimodules. Does the inclusion functor from $R-S-\text{mod}$ to $R-\text{mod}$, resp. to $\text{mod}-S$, preserve coproduct, products, monomorphisms and epimorphisms?

(xviii) Let $(I,\le)$ be a partially ordered set. Let $\mathcal{C}$ be a category. An $(I,\le)$-*system* in $\mathcal{C}$ is a datum

$$c = ((c_i)_{i\in I}, (f_{i,j})_{(i,j)\in I\times I, i\le j})$$

where every c_i is an object of $\mathcal{C}$, where for every pair $(i,j)\in I\times I$ with $i\le j$, $c_{i,j}$ is an element of $\text{Hom}_{\mathcal{C}}(c_i,c_j)$, and satisfying the following conditions: (a) for every $i\in I$, $c_{i,i}$ equals Id_{c_i}, and (b) for every triple $(i,j,k)\in I$ with $i\le j$ and $j\le k$, $c_{j,k}\circ c_{i,j}$ equals $c_{i,k}$. For every pair of $(I,\le)$-systems in $\mathcal{C}$, $c=((c_i)_{i\in I},(c_{i,j})_{i\le j})$ and $c'=((c'_i)_{i\in I},(c'_{i,j})_{i\le j})$, a morphism $g:c\to c'$ is defined to be a datum $(g_i)_{i\in I}$ of morphisms $g_i\in\text{Hom}_{\mathcal{C}}(c_i,c'_i)$ such that for every $(i,j)\in I\times I$ with $i\le j$, $g_j\circ c_{i,j}$ equals $c'_{i,j}\circ g_i$. Composition of morphisms g and g' is componentwise $g'_i\circ g_i$, and identities are $\text{Id}_c=(\text{Id}_{c_i})_{i\in I}$. This category is $\text{Fun}((I,\le),\mathcal{C})$, and is sometimes referred to as the category of $(I,\le)$-*presheaves*. Assuming $\mathcal{C}$ has finite coproducts, resp. finite products, arbitrary coproducts, arbitrary products, a zero object, kernels, cokernels, etc., what can you say about $\text{Fun}((I,\le),\mathcal{C})$?

(xix) Let $\mathcal{C}$ be a category that has arbitrary products. Let $(I,\le)$ be a partially ordered set whose associated category as in (iv) has finite coproducts and has arbitrary products. The main example is when $I=\mathfrak{U}$ is the collection of all open subsets U of a topology on a set X, and where $U\le V$ if $U\supseteq V$. Then coproduct is intersection and product is union. Motivated by this case, an *covering* of an element i of I is a collection $\underline{j}=(j_\alpha)_{\alpha\in A}$ of elements j_α of I such that for every α, $i\le j_\alpha$, and such that i is the product of $(j_\alpha)_{\alpha\in A}$ in the sense of (iv). In this case, for every $(\alpha,\beta)\in A\times A$, define $j_{\alpha,\beta}$ to be the element of I such that $j_\alpha\le j_{\alpha,\beta}$, such that $j_\beta\le j_{\alpha,\beta}$, and such that $j_{\alpha,\beta}$ is a coprodcut of (j_α,j_β). An $(I,\le)$-presheaf $c=((c_i)_{i\in I},(c_{i,j})_{i\le j})$ is an $(I,\le)$-*sheaf* if for every element i of I and for every covering $\underline{j}=(j_\alpha)_{\alpha\in A}$, the following diagram in $\mathcal{C}$ is *exact* in a sense to be made precise,

$$c_i \xrightarrow{q} \prod_{\alpha\in A} c_{j_\alpha} \overset{p'}{\rightrightarrows} p'' \prod_{(\alpha,\beta)\in A\times A} c_{j_{\alpha,\beta}}.$$

For every $\alpha\in A$, the factor of q,

$$\text{pr}_\alpha\circ q : c_i\to c_{j_\alpha},$$

is defined to be c_{i,j_α}. For every $(\alpha,\beta)\in A\times A$, the factor of p',

$$\text{pr}_{\alpha,\beta}\circ p' : \prod_{\gamma\in A} c_{j_\gamma}\to c_{j_{\alpha,\beta}},$$

is defined to be $c_{j_\alpha,j_{\alpha,\beta}}\circ\text{pr}_\alpha$. Similarly, $\text{pr}_{\alpha,\beta}\circ p''$ is defined to be $c_{j_\beta,j_{\alpha,\beta}}\circ\text{pr}_\beta$. The diagram above is *exact* in the sense that q is a monomorphism in $\mathcal{C}$ and q is a fiber product in $\mathcal{C}$ of the pair of morphisms (p',p''). The category of $(I,\le)$ is the full subcategoryof the category of $(I,\le)$-presheaves whose objects are $(I,\le)$-sheaves. Does this subcategory have coproducts, products, etc.? Does the inclusion functor preserve coproducts, resp. products, monomorphisms, epimorphisms? Before considering the general case, it is probably best to first consider the case that $\mathcal{C}$ is $\mathbb{Z}-\text{mod}$, and then consider the case that $\mathcal{C}$ is **Sets**.

12 Adjoint Pairs and Yoneda Functors

Adjoint Pairs and Representable Functors. Let $\mathcal{A}$ be a category, and let $\mathcal{B}$ be a strictly small category. Let $L : \mathcal{A} \to \mathcal{B}$ be a covariant functor. For every object b of $\mathcal{B}$, assume that the following contravariant functor from $\mathcal{A}$ to **Sets** is representable,

$$\mathrm{Hom}_{\mathcal{B}}(L(-), b) : \mathcal{A}^{\mathrm{opp}} \to \mathbf{Sets}.$$

Prove that there exists an adjoint pair (L, R, θ, η). Using the opposite adjoint pair $(R^{\mathrm{opp}}, L^{\mathrm{opp}}, \eta^{\mathrm{opp}}, \theta^{\mathrm{opp}})$, formulate and prove the analogous result for a contravariant functor R from a category $\mathcal{A}$ to a strictly small category $\mathcal{B}$.

The Yoneda Functor as an Adjoint Functor. Let $\mathcal{A}$ be a strictly small category, so that there is a well-defined category $\mathbf{Sets}^{\mathcal{A}}$ of set-valued covariant functors from $\mathcal{A}$ with natural transformations as morphisms (independent of axioms on inaccessible cardinals or Grothendieck universes). As in Example , for every ordered pair (a, a') of objects of $\mathcal{A}$, composition in $\mathcal{A}$ enriches the set $H^a_{a'} := \mathrm{Hom}_{\mathcal{A}}(a, a')$ with an $H^{a'}_{a'} - H^a_a$-action. For every set S together with a right H^a_a-action, define $H^{S,a}_{a'}$ to be the set of right H^a_a-equivariant maps from S to $H^a_{a'}$,

$$H^{S,a}_{a'} = \mathrm{Hom}_{\mathbf{Set}-H^a_a}(S, H^a_{a'}).$$

This is compatible with postcomposition by $\mathcal{A}$-morphisms in $H^{a'}_{a''}$. Altogether, this defines a covariant, set-valued functor,

$$h^{S,a} : \mathcal{A} \to \mathbf{Sets}, \quad h^{S,a}(a') = H^{S,a}_{a'},$$

the **Yoneda functor** of a and S. Prove that the rule that associates to a set with right H^a_a-action the covariant functor $h^{S,a}$ is itself a functor,

$$h^{-,a} : \mathbf{Sets} - H^a_a \to \mathbf{Sets}^{\mathcal{A}}.$$

Conversely, for every set-valued functor F on $\mathcal{A}$, the set $F(a)$ is enriched with a right H^a_a-action. Prove that the rule associating to each set-valued functor F on $\mathcal{A}$ the set $F(a)$ with its right H^a_a-action is itself a functor,

$$-(a) : \mathbf{Sets}^{\mathcal{A}} \to \mathbf{Sets} - H^a_a.$$

Prove that these two functors are adjoint, i.e., there is a binatural bijection

$$\mathbf{Hom}_{\mathbf{Sets}-H^a_a}(S, F(a)) \cong \mathbf{Hom}_{\mathbf{Sets}^{\mathcal{A}}}(h^{S,a}, F).$$

In particular, when S equals H^a_a with its right regular action this gives the usual Yoneda bijection,

$$F(a) \cong \mathbf{Hom}_{\mathbf{Sets}^{\mathcal{A}}}(h^a, F).$$

Specializing further, when F equals the Yoneda functor $h^{a'}$, this gives a binatural bijection,

$$H^a_{a'} \cong \mathrm{Hom}_{\mathbf{Sets}^{\mathcal{A}}}(h^a, h^{a'}).$$

Deduce that the rule,

$$h : \mathcal{A} \to \mathbf{Sets}^{\mathcal{A}}, \quad a \mapsto h^a,$$

is an equivalence of the category $\mathcal{A}$ with a full subcategory of the functor category $\mathbf{Sets}^{\mathcal{A}}$. Formulate and prove the analogous result for the contravariant Yoneda functors. Finally, if you know the axioms about inaccessible cardinals or the notion of Grothendieck universes, formulate a version of this for categories that are not necessarily strictly small.

13 Preservation of Exactness by Adjoint Additive Functors

Exactness and adjoint pairs. Let $\mathcal{A}$ and $\mathcal{B}$ be Abelian categories. Let (L, R, θ, η) be an adjoint pair of additive functors

$$L : \mathcal{A} \to \mathcal{B}, \ R : \mathcal{B} \to \mathcal{A}.$$

(a) For every short exact sequence in $\mathcal{A}$,

$$\Sigma : \ 0 \longrightarrow A' \xrightarrow{q_A} A \xrightarrow{p_A} A'' \longrightarrow 0,$$

for every object B in $\mathcal{B}$, prove that the induced morphism of Abelian groups,

$$\mathrm{Hom}_{\mathcal{A}}(p_A, R(B)) : \mathrm{Hom}_{\mathcal{A}}(A'', R(B)) \to \mathrm{Hom}_{\mathcal{A}}(A, R(B)),$$

is a monomorphism. Conclude that also the associated morphism of Abelian groups,

$$\mathrm{Hom}_{\mathcal{B}}(L(p_A), B) : \mathrm{Hom}_{\mathcal{B}}(L(A''), B) \to \mathrm{Hom}_{\mathcal{B}}(L(A), B),$$

is a monomorphism. In the special case that B equals $\mathrm{Coker}(L(p_A))$, use this to conclude that B must be a zero object. Conclude that R preserves epimorphisms.

(b) Prove that the following induced diagram of Abelian groups is exact,

$$\mathrm{Hom}_{\mathcal{A}}(A'', R(B)) \xrightarrow{p_A^*} \mathrm{Hom}_{\mathcal{A}}(A, R(B)) \xrightarrow{q_A^*} \mathrm{Hom}_{\mathcal{A}}(A', R(B)).$$

Conclude that also the following associated diagram of Abelian groups is exact,

$$\mathrm{Hom}_{\mathcal{B}}(L(A''), B) \xrightarrow{p_A^*} \mathrm{Hom}_{\mathcal{B}}(L(A), B) \xrightarrow{q_A^*} \mathrm{Hom}_{\mathcal{B}}(L(A'), B).$$

In the special case that B equals $\mathrm{Coker}(L(q_A))$, conclude that the induced epimorphism $B \to L(A'')$ is split. Conclude that L is half-exact, hence right exact.

(c) Use similar arguments, or opposite categories, to conclude that also R is left exact.

(d) In case R is exact (not just left exact), prove that for every projective object P of $\mathcal{A}$, also $L(P)$ is a projective object of $\mathcal{B}$. Similarly, if L is exact (not just right exact), prove that for every injective object I of $\mathcal{A}$, also $R(I)$ is an injective object of $\mathcal{A}$.

14 Derived Functors as Adjoint Pairs

Problem 0.(The Cochain Functor of an Additive Functor) Let $\mathcal{A}$ and $\mathcal{B}$ be Abelian categories. Denote by $\text{Ch}(\mathcal{A})$, respectively $\text{Ch}(\mathcal{B})$, the associated Abelian category of cochain complexes of objects of $\mathcal{A}$, resp. of objects of $\mathcal{B}$.

Let

$$F : \mathcal{A} \to \mathcal{B}$$

be an additive functor. There is an induced additive functor,

$$\text{Ch}(F) : \text{Ch}(\mathcal{A}) \to \text{Ch}(\mathcal{B})$$

that associates to a cochain complex

$$A^\bullet = ((A^n)_{n\in\mathbb{Z}}, (d_A^n : A^n \to A^{n+1})_{n\in\mathbb{Z}}),$$

in $\mathcal{A}$ the cochain complex

$$F(A^\bullet) = ((F(A^n))_{n\in\mathbb{Z}}, (F(d_A^n) : F(A^n) \to F(A^{n+1}))_{n\in\mathbb{Z}}).$$

(a) Prove that F is half-exact, resp. left exact, right exact, exact, if and only if $\text{Ch}(F)$ is half-exact, resp. left exact, right exact, exact.

(b) Prove that the functor $\text{Ch}(F)$ induces natural transformations,

$$\theta^n_{B,F} : B^n \circ \text{Ch}(F) \Rightarrow F \circ B^n,\ \ \theta^n_{F,Z} : F \circ Z^n \Rightarrow Z^n \circ \text{Ch}(F).$$

Thus, for the functor $\overline{A}^n = A^n/B^n(A^\bullet)$, there is also an induced natural transformation,

$$\theta_{\overline{\cdot},F} : \overline{\cdot}^n \circ \text{Ch}(F) \Rightarrow F \circ \overline{\cdot}^n.$$

(c) Assume now that F is right exact (half-exact and preserves epimorphisms). Denote by

$$p^n : Z^n \Rightarrow H^n,$$

the usual natural transformation of functors $\text{Ch}(\mathcal{A}) \to \mathcal{A}$. Conclude the existence of a unique natural transformation

$$\theta^n_{F,H} : F \circ H^n \Rightarrow H^n \circ \text{Ch}(F),$$

such that for every $A^\bullet$ in $\text{Ch}(\mathcal{A})$, the following diagram commutes,

$$\begin{array}{ccc} F(Z^n(A^\bullet)) & \xrightarrow{F(p^n)} & F(H^n(A^\bullet)) \\ \Big\downarrow{\scriptstyle \theta^n_{F,Z}(A^\bullet)} & & \Big\downarrow{\scriptstyle \theta^n_{F,H}(A^\bullet)} \\ Z^n(\text{Ch}(F)(A^\bullet)) & \xrightarrow[p^n]{} & H^n(\text{Ch}(F)(A^\bullet)) \end{array}.$$

Finally, for every short exact sequence in $\text{Ch}(\mathcal{A})$,

$$\Sigma : \ 0 \longrightarrow K^\bullet \xrightarrow{u^\bullet} A^\bullet \xrightarrow{v^\bullet} 0,$$

such that also $F(\Sigma)$ is a short exact sequence in $\text{Ch}(\mathcal{B})$ (this holds, for instance, if Σ is term-by-term split), prove that the following diagram commutes,

$$\begin{array}{ccc} F(H^n(Q^\bullet)) & \xrightarrow{F(\delta^n_\Sigma)} & F(H^{n+1}(K^\bullet)) \\ {\scriptstyle \theta^n_{F,H}(Q^\bullet)} \big\downarrow & & \big\downarrow {\scriptstyle \theta^{n+1}_{F,H}(K^\bullet)} \\ H^n(F(Q^\bullet)) & \xrightarrow[\delta^n_{F(\Sigma)}]{} & > H^{n+1}(F(K^\bullet)) \end{array}.$$

(d) Assume now that F is left exact (half-exact and preserves monomorphisms). Denote by

$$q^n : H^n(A^\bullet) \Rightarrow \overline{A}^n = A^n/B^n(A^\bullet),$$

the usual natural transformation of functors $\text{Ch}(\mathcal{A}) \to \mathcal{A}$. Conclude the existence of a unique natural transformation

$$\theta^n_{H,F} : H^n \circ \text{Ch}(F) \Rightarrow F \circ H^n,$$

such that for every $A^\bullet$ in $\text{Ch}(\mathcal{A})$, the following diagram commutes,

$$\begin{array}{ccc} H^n(\text{Ch}(F)(A^\bullet)) & \xrightarrow{q^n} & \overline{\text{Ch}(F)(A^\bullet)}^n \\ {\scriptstyle \theta^n_{H,F}(A^\bullet)} \big\downarrow & & \big\downarrow {\scriptstyle \theta^n_{\cdot,F}(A^\bullet)} \\ \overline{\text{Ch}(F)(A^\bullet)}^n & \xrightarrow[F(q^n)]{} & F(\overline{A}^n) \end{array}.$$

Finally, for every short exact sequence in $\text{Ch}(\mathcal{A})$,

$$\Sigma : \ 0 \longrightarrow K^\bullet \xrightarrow{u^\bullet} A^\bullet \xrightarrow{v^\bullet} 0,$$

such that also $F(\Sigma)$ is a short exact sequence in $\text{Ch}(\mathcal{B})$ (this holds, for instance, if Σ is term-by-term split), prove that the following diagram commutes,

$$\begin{array}{ccc} H^n(F(Q^\bullet)) & \xrightarrow{\delta^n_{F(\Sigma)}} & H^{n+1}(F(K^\bullet)) \\ {\scriptstyle \theta^n_{H,F}(Q^\bullet)} \big\downarrow & & \big\downarrow {\scriptstyle \theta^{n+1}_{H,F}(K^\bullet)} \\ F(H^n(Q^\bullet)) & \xrightarrow[F(\delta^n_\Sigma)]{} & F(H^{n+1}(K^\bullet)) \end{array}$$

Preservation of Direct Sums Exercise. Let $\mathcal{A}$ be an additive category. Let A_1 and A_2 be objects of $\mathcal{A}$. Let $(q_1 : A_1 \to A, q_2 : A_2 \to A)$ be a coproduct (direct sum) in $\mathcal{A}$. Define $p_1 : A \to A_1$

to be the unique morphism in $\mathcal{A}$ such that $p_1 \circ q_1$ equals Id_{A_1} and $p_1 \circ q_2$ is zero. Similarly define $p_2 : A \to A_2$ to be the unique morphism in $\mathcal{A}$ such that $p_2 \circ q_1$ is zero and $p_2 \circ q_2$ equals Id_{A_2}. Prove that $q_1 \circ p_1 + q_2 \circ p_2$ equals Id_A both compose with q_i to equal q_i, and thus both are equal. Conclude that $(p_1 : A \to A_1, p_2 : A \to A_2)$ is a product in $\mathcal{A}$.

Now let $\mathcal{B}$ be a second additive category, and let

$$F : \mathcal{A} \to \mathcal{B}$$

be an additive functor. Define $B_i = F(A_i)$ and $B = F(A)$. Prove that $F(p_i) \circ F(q_j)$ equals Id_{B_i} if $j = i$ and equals 0 otherwise. Also prove that Id_B equals $F(q_1) \circ F(p_1) + F(q_2) \circ F(p_2)$. Conclude that both $(F(q_1) : B_1 \to B, F(q_2) : B_2 \to B)$ is a coproduct in $\mathcal{B}$ and $(F(p_1) : B \to B_1, F(p_2) : B \to B_2)$ is a product in $\mathcal{B}$. Hence, additive functors preserve direct sums. In particular, additive functors send split exact sequences to split exact sequences.

Preservation of Homotopies Exercise. Let $\mathcal{A}$ be an Abelian category. Let $A^\bullet$ and $C^\bullet$ be cochain complexes in $\mathrm{Ch}(\mathcal{A})$. Let $f^\bullet : A^\bullet \to C^\bullet$ be a cochain morphism. A *homotopy* from $f^\bullet$ to 0 is a sequence $(s^n : A^n \to C^{n-1})_{n \in \mathbb{Z}}$ such that for every $n \in \mathbb{Z}$,

$$f^n = d_C^{n-1} \circ s^n + s^{n+1} \circ d_A^n.$$

In this case, $f^\bullet$ is called *homotopic* to 0 or *null homotopic*. Cochain morphisms $g^\bullet, h^\bullet : A^\bullet \to C^\bullet$ are *homotopic* if $f^\bullet = g^\bullet - h^\bullet$ is homotopic to 0.

(a) Prove that the null homotopic cochain morphisms form an Abelian subgroup of $\mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(A^\bullet, C^\bullet)$. Moreover, prove that the precomposition or postcomposition of a null homotopic cochain morphism with an arbitrary cochain morphism is again null homotopic (the null homotopic cochain morphisms form a "left-right ideal" with respect to composition).

(b) If $f^\bullet$ is homotopic to 0, prove that for every $n \in \mathbb{Z}$, the induced morphism,

$$H^n(f^\bullet) : H^n(A^\bullet) \to H^n(C^\bullet),$$

is the zero morphism. In particular, if $\mathrm{Id}_{A^\bullet}$ is homotopic to 0, conclude that every $H^n(A^\bullet)$ is a zero object.

(c) For a short exact sequence in $\mathcal{A}$

$$\Sigma : \; 0 \longrightarrow K \xrightarrow{q} A \xrightarrow{p} Q \longrightarrow 0,$$

considered as a cochain complex $A^\bullet$ in $\mathcal{A}$ concentrated in degrees -1, 0, 1, prove that a homotopy from $\mathrm{Id}_{A^\bullet}$ to 0 is the same thing as a splitting of the short exact sequence.

(d) Let $\mathcal{B}$ be an Abelian category. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor. This induces an additive functor

$$\mathrm{Ch}(F) : \mathrm{Ch}(\mathcal{A}) \to \mathrm{Ch}(\mathcal{B}).$$

If F is half-exact, resp. left exact, right exact, exact, prove that also $\mathrm{Ch}(F)$ is half-exact, resp. left exact, right exact, exact. Prove that $\mathrm{Ch}(F)$ preserves homotopies. In particular, if $g^\bullet$ and $h^\bullet$ are homotopic in $\mathrm{Ch}(\mathcal{A})$, then for every integer $n \in \mathbb{Z}$, $H^n(\mathrm{Ch}(F)(g^\bullet))$ equals $H^n(\mathrm{Ch}(F)(h^\bullet))$.

Preservation of Translation Exercise. Let $\mathcal{A}$ be an Abelian category. For every integer m, for every cochain complex $A^\bullet$ in $\mathrm{Ch}(\mathcal{A})$, define $T^m(A^\bullet) = A^\bullet[m]$ to be the cochain complex such that $T^m(A^\bullet)^n = A^{m+n}$, and with differential

$$d^n_{T^m(A^\bullet)} : T^m(A^\bullet)^n \to T^m(A^\bullet)^{n+1}$$

equal to $(-1)^m d^{m+n}_{A^\bullet}$. For every cochain morphism

$$f^\bullet : A^\bullet \to C^\bullet,$$

define

$$T^m(f^\bullet)^n : T^m(A^\bullet)^n \to T^m(C^\bullet)^n$$

to be f^{m+n}. Finally, for every homotopy $s^\bullet$ from $g^\bullet - h^\bullet$ to 0, define

$$T^m(s^\bullet)^n = (-1)^m s^{m+n}.$$

(a) Prove that $T^m : \mathrm{Ch}(\mathcal{A}) \to \mathrm{Ch}(\mathcal{A})$ is an additive functor that is exact. Prove that T^0 is the identity functor. Also prove that $T^m \circ T^\ell$ equals $T^{m+\ell}$. Prove that not only are T^m and T^{-m} inverse functors, but also (T^m, T^{-m}) is an adjoint pair of functors (which implies that also (T^{-m}, T^m) is an adjoint pair). Finally, if $s^\bullet$ is a homotopy from $g^\bullet - h^\bullet$ to 0, prove that $T^m(s^\bullet)$ is a homotopy from $T^m(g^\bullet) - T^m(h^\bullet)$ to 0.

(b) Via the identification $T^m(A^\bullet)^n = A^{m+n}$, prove that the subfunctor $Z^n(T^m(A^\bullet))$ is naturally identified with $Z^{m+n}(A^\bullet)$. Similarly, prove that the subfunctor $B^n(T^m(A^\bullet))$ is naturally identified with $B^{m+n}(A^\bullet)$. Thus, show that the epimorphism $(T^m(A^\bullet))^n \to \overline{T^m(A^\bullet)}^n$ is identified with the epimorphism $A^{m+n} \to \overline{A}^{m+n}$. Finally, use these natural equivalences to deduce a natural equivalence of half-exact, additive functors $\mathrm{Ch}(\mathcal{A}) \to \mathcal{A}$,

$$\iota^{m,n} : H^{m+n} \Rightarrow H^n \circ T^m.$$

(c) For a short exact sequence in $\mathrm{Ch}(\mathcal{A})$,

$$\Sigma : \ K^\bullet \xrightarrow{q^\bullet} A^\bullet \xrightarrow{p^\bullet} Q^\bullet \longrightarrow 0,$$

for the associated short exact sequence,

$$\Sigma[+1] = T(\Sigma) : \ T(K^\bullet) \xrightarrow{T(q^\bullet)} T(A^\bullet) \xrightarrow{T(p^\bullet)} T(Q^\bullet) \longrightarrow 0,$$

prove that the following diagram commutes,

$$\begin{array}{ccc} H^{n+1}(Q^\bullet) & \xrightarrow{-\delta^{n+1}_{\Sigma}} & H^{n+1}(K^\bullet) \\ \downarrow{\scriptstyle \iota^n(Q^\bullet)} & & \downarrow{\scriptstyle \iota^{n+1}(K^\bullet)} \\ H^n(T(Q^\bullet)) & \xrightarrow[\delta^n_{T(\Sigma)}]{} & H^{n+1}(T(K^\bullet)) \end{array}$$

Iterate this to prove that for every $m \in \mathbb{Z}$, $\delta^n_{\Sigma[m]}$ is identified with $(-1)^m \delta^{n+m}_{\Sigma}$.

(d) For every integer m, define

$$e_{\geq m} : \mathrm{Ch}^{\geq m}(\mathcal{A}) \to \mathrm{Ch}(\mathcal{A})$$

to be the full additive subcategory whose objects are complexes $A^\bullet$ such that for every $n < m$, A^n is a zero object. (From here on, writing $A = 0$ for an object A means "A is a zero object".) Check that $\mathrm{Ch}^{\geq m}(\mathcal{A})$ is an Abelian category, and $e_{\geq m}$ is an exact functor. For every integer m, define the "brutal truncation"

$$\sigma_{\geq m} : \mathrm{Ch}(\mathcal{A}) \to \mathrm{Ch}^{\geq m}(\mathcal{A}),$$

to be the additive functor such that for every object $A^\bullet$

$$(\sigma_{\geq m} A^\bullet)^n = \begin{cases} A^n, & n \geq m \\ 0, & n < m \end{cases}$$

and for every morphism $u^\bullet : A^\bullet \to C^\bullet$,

$$(\sigma_{\geq m} f^\bullet)^n = \begin{cases} f^n, & n \geq m, \\ 0, & n < m \end{cases}$$

Check that $\sigma_{\geq m}$ is exact and is right adjoint to $e_{\geq m}$. For the natural transformation,

$$\eta_{\geq m} : e_{\geq m} \circ \sigma_{\geq m} \Rightarrow \mathrm{Id}_{\mathrm{Ch}(\mathcal{A})},$$

check that the induced natural transformation,

$$\overline{\eta_{\geq m}(A^\bullet)^n} : \overline{(\sigma_{\geq m}(A))^n} \to \overline{A^n},$$

is zero for $n < m$, is the identity for $n > m$, and for $n = m$ it is the epimorphism,

$$A^m \twoheadrightarrow \overline{A^m}.$$

Check that the induced natural transformation

$$Z^n(\eta_{\geq m}(A^\bullet)) : Z^n(\sigma_{\geq m}(A^\bullet)) \to Z^n(A^\bullet),$$

is zero for $n < m$, and it is the identity for $n \geq m$. Check that the induced natural transformation,

$$B^n(\eta_{\geq m}(A^\bullet)) : B^n(\sigma_{\geq m}(A^\bullet)) \to B^n(A^\bullet),$$

is zero for $n \le m$, and it is the identity for $n > m$. Check that the induced natural transformation,

$$H^n(\eta_{\ge m}(A^\bullet)) : H^n(\sigma_{\ge m}(A^\bullet)) \to H^n(A^\bullet),$$

is zero for $n < m$, is the identity for $n > m$, and for $n = m$ it is the epimorphism,

$$Z^m(A^\bullet) \twoheadrightarrow H^n(A^\bullet).$$

Check that for every integer ℓ, there is a unique (exact) equivalence of categories,

$$T^\ell_m : \mathrm{Ch}^{\ge m}(\mathcal{A}) \to \mathrm{Ch}^{\ge \ell+m}(\mathcal{A}),$$

such that $T^\ell_m \circ \sigma_{\ge m}$ equals $\sigma_{\ge \ell+m} \circ T^\ell$, and T^ℓ_m. Check that $(T^\ell_m, T^{-\ell}_{\ell+m})$ is an adjoint pair of functors, so that also $(T^{-\ell}_{\ell+m}, T^\ell_m)$ is an adjoint pair of functors.

(d)bis Similarly, define the "good truncation"

$$\tau_{\ge m} : \mathrm{Ch}(\mathcal{A}) \to \mathrm{Ch}^{\ge m}(\mathcal{A}),$$

to be the additive functor such that for every object $A^\bullet$

$$(\tau_{\ge m} A^\bullet)^n = \begin{cases} A^n, & n > m, \\ \overline{A^m}, & n = m, \\ 0, & n < m \end{cases}$$

and for every morphism $u^\bullet : A^\bullet \to C^\bullet$,

$$(\tau_{\ge m} f^\bullet)^n = \begin{cases} f^n, & n > m, \\ \overline{f^m}, & n = m, \\ 0, & n < m \end{cases}$$

Check that τ_m is right exact and is left adjoint to $e_{\ge m}$. For the natural transformation

$$\theta_m : \mathrm{Id}_{\mathrm{Ch}(\mathcal{A})} \Rightarrow e_m \circ \tau_{\ge m},$$

check that the induced morphism,

$$Z^n(\theta_{A^\bullet}) : Z^n(A^\bullet) \to Z^n(\tau_{\ge m}(A^\bullet)),$$

is zero for $n < m$, is the identity for $n > m$, and for $n = m$ it is the epimorphism,

$$Z^n(A^\bullet) \to H^n(A^\bullet).$$

Check that the induced natural transformation,

$$B^n(\theta_{A^\bullet}) : B^n(A^\bullet) \to B^n(\tau_{\ge m}(A^\bullet)),$$

is zero for $n \leq m$, and it is the identity for $n > m$. Check that the induced natural transformation,

$$\overline{\theta_{A^\bullet}}^n : \overline{A}^n \to \overline{\tau_{\geq m}(A^\bullet)}^n$$

is zero for $n < m$, and it is the identity for $n \geq m$. Check that the induced natural transformation,

$$H^n(\theta_{A^\bullet}) : H^n(A^\bullet) \to H^n(\tau_{\geq m}(A^\bullet)),$$

is zero for $n < m$, and it is the identity for $n \geq m$.

Finally, e.g., using the opposite category, formulate and prove the corresponding results for the full embedding,

$$e_{\leq m} : \mathrm{Ch}^{\leq m}(\mathcal{A}) \to \mathrm{Ch}(\mathcal{A}),$$

whose objects are complexes $A^\bullet$ such that A^n is a zero object for all $n > m$. In particular, note that although the sequence of brutal truncations,

$$0 \longrightarrow \sigma_{\geq m}(A^\bullet) \xrightarrow{\eta_{\geq m}(A^\bullet)} A^\bullet \xrightarrow{\theta_{\leq m-1}(A^\bullet)} \sigma_{\leq m-1}(A^\bullet) \longrightarrow 0$$

is exact, the corresponding morphisms of good truncations,

$$\mathrm{Ker}(\theta_{\geq m}(A^\bullet)) \hookrightarrow \tau_{\leq m}(A^\bullet), \quad \tau_{\geq m}(A^\bullet) \twoheadrightarrow \mathrm{Coker}(\eta_{\leq m}(A^\bullet)),$$

are not isomorphisms; in the first case the cokernel is $H^m(A^\bullet)[m]$, and in the second case the kernel is $H^m(A^\bullet)[m]$. However, check that the natural morphisms,

$$\tau_{\leq m-1}(A^\bullet) \xrightarrow{\eta_{\leq m-1}} \mathrm{Ker}(\theta_{\geq m}(A^\bullet),$$

$$\mathrm{Coker}(\eta_{\leq m-1}(A^\bullet)) \xrightarrow{\theta_{\geq m}} \tau_{\geq m}(A^\bullet),$$

are quasi-isomorphisms. (One reference slightly misstates this, claiming that the morphisms are isomorphisms, which is "morally" correct after passing to the derived category.)

(e) Beginning with the cohomological δ-functor (in all degrees) $\mathrm{Ch}(\mathcal{A}) \to \mathcal{A}$,

$$H^\bullet = ((H^n)_{n\in\mathbb{Z}}, (\delta^n)_{n\in\mathbb{Z}}),$$

the associated cohomological δ-functor,

$$H^\bullet \circ T^\ell = ((H^n \circ T^\ell)_{n\in}, (\delta^n \circ T^\ell)_{n\in\mathbb{Z}}),$$

the cohomological δ-functor

$$H^{\bullet+\ell} = ((H^{n+\ell})_{n\in\mathbb{Z}}, (\delta^{n+\ell})_{n\in\mathbb{Z}}),$$

and the equivalence,

$$\iota^{\ell,0} : H^\ell \Rightarrow H^0 \circ T^\ell,$$

prove that there exists a unique natural transformation of cohomological δ-functors,

$$\theta_\ell : H^{\bullet+\ell} \Rightarrow H^\bullet \circ T^\ell, \quad (\theta_\ell^n : H^{n+\ell} \Rightarrow H^n \circ T^\ell)_{n\in\mathbb{Z}},$$

and that $\theta_\ell^n = (-1)^{n\ell}\iota^{\ell,n}$.

(e)bis The truncation $\tau_{\geq m}H^\bullet$ in degrees $\geq m$ is obtained by replacing H^m by the subfunctor Z^m. Check that θ_ℓ restricts to a natural transformation $\tau_{\geq \ell+m}H^{\bullet+\ell} \to \tau_{\geq m}H^\bullet \circ T^\ell$. Assuming that $\tau_{\geq m}H^\bullet$ is a universal cohomological δ-functor in degrees $\geq m$, conclude that also $\tau_{\geq \ell+m}H^\bullet$ is a universal cohomological δ-functor in degrees $\geq \ell + m$. Also, formulate and prove the corresponding result for the universal δ-functors $\tau_{\leq 0}H^\bullet$ and $\tau_{\leq m}H^\bullet$.

(f) Let $\mathcal{B}$ be an Abelian category. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor. This induces an additive functor

$$\mathrm{Ch}(F) : \mathrm{Ch}(\mathcal{A}) \to \mathrm{Ch}(\mathcal{B}).$$

Prove that $\mathrm{Ch}(F) \circ T_{\mathcal{A}}$ equals $T_{\mathcal{B}} \circ \mathrm{Ch}(F)$.

Compatibility with Automorphisms Exercise. Let $\mathcal{A}$ be an Abelian category. Let

$$\Sigma : \ 0 \longrightarrow K^\bullet \xrightarrow{q^\bullet} A^\bullet \xrightarrow{p^\bullet} Q^\bullet \longrightarrow 0$$

be a short exact sequence in $\mathrm{Ch}(\mathcal{A})$. Let

$$u^\bullet : K^\bullet \to K^\bullet, \quad v^\bullet : Q^\bullet \to Q^\bullet$$

be isomorphisms in $\mathrm{Ch}(\mathcal{A})$.

(a) Prove that the following sequence is a short exact sequence,

$$\Sigma_{u^\bullet,v^\bullet} : \ 0 \longrightarrow K^\bullet \xrightarrow{q^\bullet\circ u^\bullet} A^\bullet \xrightarrow{v^\bullet\circ p^\bullet} Q^\bullet \longrightarrow 0\cdot$$

(b) Prove that the following diagrams are commutative diagrams.

$$\begin{array}{cccccccccccc}
\Sigma_{u^\bullet,\mathrm{Id}_Q} : & 0 & \longrightarrow & K^\bullet & \xrightarrow{q^\bullet\circ u^\bullet} & A^\bullet & \xrightarrow{p^\bullet} & Q^\bullet & \longrightarrow & 0 \\
\widetilde{u}\downarrow & & & u^\bullet\downarrow & & \downarrow \mathrm{Id}_A & & \downarrow \mathrm{Id}_Q & & & , \\
\Sigma_{\mathrm{Id}_K,\mathrm{Id}_Q} : & 0 & \longrightarrow & K^\bullet & \xrightarrow{q^\bullet} & A^\bullet & \xrightarrow{p^\bullet} & Q^\bullet & \longrightarrow & 0
\end{array}$$

$$\begin{array}{cccccccccccc}
\Sigma_{u^\bullet,\mathrm{Id}_Q} : & 0 & \longrightarrow & K^\bullet & \xrightarrow{q^\bullet\circ u^\bullet} & A^\bullet & \xrightarrow{p^\bullet} & Q^\bullet & \longrightarrow & 0 \\
\widetilde{v}\downarrow & & & \mathrm{Id}_K\downarrow & & \downarrow \mathrm{Id}_A & & \downarrow v^\bullet & & & . \\
\Sigma_{u^\bullet,v^\bullet} : & 0 & \longrightarrow & K^\bullet & \xrightarrow{q^\bullet\circ u^\bullet} & A^\bullet & \xrightarrow{v^\bullet\circ p^\bullet} & Q^\bullet & \longrightarrow & 0
\end{array}$$

(c) Use the commutative diagram of long exact sequences associated to a commutative diagrams of short exact sequences to prove that

$$\delta^n_\Sigma = H^{n+1}(u^\bullet) \circ \delta^n_{\Sigma_{u^\bullet, v^\bullet}} \circ H^n(v^\bullet),$$

for every integer n.

Compatibility with Natural Transformations of Additive Functors. Let $\mathcal{A}$ and $\mathcal{B}$ be Abelian categories.

(a) For additive functors,

$$F, G : \mathcal{A} \to \mathcal{B},$$

let

$$\alpha : F \Rightarrow G,$$

be a natural transformation. For every cochain complex $A^\bullet$ in $\text{Ch}(\mathcal{A})$, prove that

$$(\alpha_{A^n} : F(A^n) \to G(A^n))_{n\in\mathbb{Z}}$$

is a morphism of cochain complexes in $\text{Ch}(\mathcal{B})$,

$$\text{Ch}(\alpha)(A^\bullet) : \text{Ch}(F)(A^\bullet) \to \text{Ch}(G)(A^\bullet).$$

(b) Prove that the rule $A^\bullet \mapsto \text{Ch}(\alpha)(A^\bullet)$ is a natural transformation

$$\text{Ch}(\alpha) : \text{Ch}(F) \Rightarrow \text{Ch}(G).$$

Moreover, for every morphism $u^\bullet : C^\bullet \to A^\bullet$ in $\text{Ch}(\mathcal{A})$, and for every homotopy $(s^n : C^n \to A^{n-1})_{n\in\mathbb{Z}}$ from $u^\bullet$ to 0, prove that also $\text{Ch}(\alpha)(A^\bullet) \circ \text{Ch}(F)(s^\bullet)$ equals $\text{Ch}(G)(s^\bullet) \circ \text{Ch}(\alpha)(C^\bullet)$.

(c) For the identity natural transformation $\text{Id}_F : F \Rightarrow F$, prove that $\text{Ch}(\text{Id}_F)$ is the identity natural transformation $\text{Ch}(F) \Rightarrow \text{Ch}(F)$. Also, for every pair of natural transformations of additive functors $\mathcal{A} \to \mathcal{B}$,

$$\alpha : F \Rightarrow G, \quad \beta : E \Rightarrow F,$$

for the composite natural transformation $\alpha \circ \beta$, prove that $\text{Ch}(\alpha \circ \beta)$ equals $\text{Ch}(\alpha) \circ \text{Ch}(\beta)$. In this sense, Ch is a "functor" from the "2-category" of Abelian categories to the "2-category" of Abelian categories.

Derived Functors as Adjoint Pairs Exercise. Let $\mathcal{A}$ and $\mathcal{B}$ be Abelian categories. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor. Assume that $\mathcal{A}$ has enough injective objects. Thus, every object A admits an injective resolution in $\text{Ch}(\mathcal{A})$,

$$\begin{array}{ccccccccccc}
A[0]: & \dots & \longrightarrow & 0 & \longrightarrow & A & \longrightarrow & 0 & \longrightarrow & \dots \\
{\scriptstyle \epsilon_A}\downarrow & & & \downarrow & & {\scriptstyle \epsilon}\downarrow & & \downarrow & & \\
I^\bullet_A: & \dots & \longrightarrow & 0 & \longrightarrow & I^0 & \xrightarrow{d^0_I} & I^1 & \xrightarrow{d^1_I} & \dots
\end{array},$$

which is functorial up to null homotopies (in particular, any two injective resolutions are homotopy equivalent). Moreover, for every short exact sequence in $\mathcal{A}$,

$$\Sigma:\ 0 \longrightarrow K \xrightarrow{q} A \xrightarrow{p} Q \longrightarrow 0,$$

there exists a diagram of injective resolutions with rows being short exact sequences in $\mathrm{Ch}(\mathcal{A})$,

$$\begin{array}{ccccccccccc} \Sigma[0]: & 0 & \longrightarrow & K[0] & \xrightarrow{q[0]} & A[0] & \xrightarrow{p[0]} & Q[0] & \longrightarrow & 0 \\ {\scriptstyle \epsilon_\Sigma}\downarrow & & & {\scriptstyle \epsilon_K}\downarrow & & \downarrow{\scriptstyle \epsilon_A} & & \downarrow{\scriptstyle \epsilon_Q} & & \\ I_\Sigma: & 0 & \longrightarrow & I_K^\bullet & \xrightarrow{q^\bullet} & I_A^\bullet & \xrightarrow{p^\bullet} & I_Q^\bullet & \longrightarrow & 0 \end{array}$$

whose associated short exact sequences in $\mathcal{A}$,

$$I_\Sigma^n:\ 0 \longrightarrow I_K^n \xrightarrow{q^n} I_A^n \xrightarrow{p^n} I_Q^n \longrightarrow 0,$$

are automatically split. Moreover, this diagram of injective resolutions is functorial up to homotopy, i.e., for every commutative diagram of short exact sequences in $\mathcal{A}$,

$$\begin{array}{ccccccccccc} \Sigma: & 0 & \longrightarrow & K & \xrightarrow{q} & A & \xrightarrow{p} & Q & \longrightarrow & 0 \\ {\scriptstyle u}\downarrow & & & {\scriptstyle u_K}\downarrow & & \downarrow{\scriptstyle u_A} & & \downarrow{\scriptstyle u_Q} & & \\ \widetilde{\Sigma}: & 0 & \longrightarrow & \widetilde{K} & \xrightarrow{\widetilde{q}} & \widetilde{A} & \xrightarrow{\widetilde{p}} & \widetilde{Q} & \longrightarrow & 0 \end{array},$$

there exists a commutative diagram in $\mathrm{Ch}(\mathcal{A})$,

$$\begin{array}{ccccccccccc} I_\Sigma: & 0 & \longrightarrow & I_K & \xrightarrow{q^\bullet} & I_A & \xrightarrow{p^\bullet} & I_Q & \longrightarrow & 0 \\ {\scriptstyle u^\bullet}\downarrow & & & {\scriptstyle u_K^\bullet}\downarrow & & \downarrow{\scriptstyle u_A^\bullet} & & \downarrow{\scriptstyle u_Q^\bullet} & & \\ I_{\widetilde{\Sigma}}: & 0 & \longrightarrow & I_{\widetilde{K}} & \xrightarrow{\widetilde{q}^\bullet} & I_{\widetilde{A}} & \xrightarrow{\widetilde{p}^\bullet} & I_{\widetilde{Q}} & \longrightarrow & 0 \end{array}$$

compatible with the morphisms ϵ_-, and the cochain morphisms $u^\bullet$ making all diagrams commute are unique up to homotopy.

As proved in lecture, there is an associated cohomological δ-functor in degrees ≥ 0, $R^\bullet F$, with

$$R^nF : \mathcal{A} \to \mathcal{B}, \quad R^nF(A) = H^n(\mathrm{Ch}(F)(A^\bullet)).$$

For every short exact sequence in $\mathcal{A}$,

$$\Sigma:\ 0 \longrightarrow K \xrightarrow{q} A \xrightarrow{p} Q \longrightarrow 0,$$

the corresponding complex in $\mathcal{B}$, $\mathrm{Ch}(\mathcal{B})$,

$$\mathrm{Ch}(F)(I_\Sigma):\ 0 \longrightarrow \mathrm{Ch}(F)(I_K^\bullet) \xrightarrow{\mathrm{Ch}(F)(q^\bullet)} \mathrm{Ch}(F)(I_A^\bullet) \xrightarrow{\mathrm{Ch}(F)(p^\bullet)} \mathrm{Ch}(F)(I_Q^\bullet) \longrightarrow 0,$$

has associated complexes in $\mathcal{B}$,

$$\mathrm{Ch}(F)(I_\Sigma)^n: \ 0 \longrightarrow F(I_K^n) \xrightarrow{F(q^n)} F(I_A^n) \xrightarrow{F(p^n)} F(I_Q^n) \longrightarrow 0,$$

being split exact sequences (since the additive functor F preserves split exact sequences), and hence $\mathrm{Ch}(F)(I_\Sigma)$ is a short exact sequence in $\mathcal{B}$. The maps $\delta^n_{R^\bullet F,\Sigma}$ are the connecting maps determined by the Snake Lemma for this short exact sequence,

$$\delta^n_{\mathrm{Ch}(F)(I_\Sigma)} : H^n(\mathrm{Ch}(F)(I_Q^\bullet)) \to H^{n+1}(\mathrm{Ch}(F)(I_K^\bullet)).$$

Associated to ϵ, there are morphisms in $\mathcal{B}$

$$F(\epsilon_A) : F(A) \to R^0F(A).$$

(a) Let $G : \mathcal{A} \to \mathcal{B}$ be an additive functor. Let

$$\alpha : F \Rightarrow G,$$

be a natural transformation. For every object A of $\mathcal{A}$ and for every injective resolution $\epsilon : A[0] \to I_A^\bullet$, there is an induced morphism in $\mathrm{Ch}(\mathcal{B})$,

$$\mathrm{Ch}(\alpha)(I_A^\bullet) : \mathrm{Ch}(F)(I_A^\bullet) \to \mathrm{Ch}(G)(I_A^\bullet).$$

This induces morphisms,

$$R^n\alpha(A) : R^nF(A) \to R^nG(A),$$

given by,

$$H^n(\mathrm{Ch}(\alpha)(I_A^\bullet)) : H^n(\mathrm{Ch}(F)(I_A^\bullet)) \to H^n(\mathrm{Ch}(G)(I_A^\bullet)).$$

For every n, prove that $A \mapsto R^n\alpha(A)$ defines a natural transformation

$$R^n\alpha : R^nF \Rightarrow R^nG.$$

Moreover, prove that this natural transformation is a morphism of δ-functors, i.e., for every short exact sequence,

$$\Sigma: \ 0 \longrightarrow K \xrightarrow{q} A \xrightarrow{p} Q \longrightarrow 0,$$

for every integer n, the following diagram commutes,

$$\begin{array}{ccc} R^nF(Q) & \xrightarrow{\delta^n_{R^\bullet F,\Sigma}} & R^{n+1}F(K) \\ \downarrow{\scriptstyle R^n\alpha(Q)} & & \downarrow{\scriptstyle R^{n+1}\alpha(K)} \\ R^nG(Q) & \xrightarrow[\delta^n_{R^\bullet G,\Sigma}]{} & R^{n+1}G(K) \end{array}$$

(b) Prove that the morphisms $F(\epsilon_A)$ form a natural transformation, $\rho_F : F \to R^0F$.

(c) Prove that R^0F is a left-exact functor. Assuming that F is left-exact, prove that ρ_F is a natural equivalence of funcors. In particular, conclude that $\rho_{R^0F} : R^0F \to R^0(R^0F)$ is a natural equivalence of functors.

(d) For every half-exact functor,

$$G : \mathcal{A} \to \mathcal{B},$$

and for every natural transformation,

$$\gamma : F \Rightarrow G,$$

prove that the two natural transformations,

$$R^0\gamma \circ \rho_F, \rho_G \circ \gamma : F \Rightarrow R^0G,$$

are equal. In particular, if G is left-exact, so that ρ_G is a natural equivalence, conclude that there exists a unique natural transformation,

$$\widetilde{\gamma} : R^0F \Rightarrow G,$$

such that γ equals $\widetilde{\gamma} \circ \rho_F$.

(e) Now assume that $\mathcal{A}$ and $\mathcal{B}$ are small Abelian categories. Thus, functors from $\mathcal{A}$ to $\mathcal{B}$ are well-defined in the usual axiomatization of set theory. Let $\text{Fun}(\mathcal{A}, \mathcal{B})$ be the category whose objects are functors from $\mathcal{A}$ to $\mathcal{B}$ and whose morphisms are natural transformations of functors. Let $\text{AddFun}(\mathcal{A}, \mathcal{B})$ be the full subcategory of additive functors. Let

$$e : \text{LExactFun}(\mathcal{A}, \mathcal{B}) \to \text{AddFun}(\mathcal{A}, \mathcal{B}),$$

be the full subcategory whose objects are left-exact additive functors from $\mathcal{A}$ to $\mathcal{B}$. Prove that the rule associating to F the left-exact functor R^0F and associating to every natural transformation $\alpha : F \Rightarrow G$ the natural transformation $R^0\alpha : R^0F \Rightarrow R^0G$ is a left adjoint to e.

(f) With the same hypotheses as above, denote by $\text{Fun}_\delta^{\ge 0}(\mathcal{A}, \mathcal{B})$ the category whose objects are cohomological δ-functors from $\mathcal{A}$ to $\mathcal{B}$ concentrated in degrees ≥ 0,

$$T^\bullet = ((T^n : \mathcal{A} \to \mathcal{B})_{n \in \mathbb{Z}}, (\delta_T^n)_{n \in \mathbb{Z}}),$$

and whose morphisms are natural transformations of δ-functors,

$$\alpha^\bullet : S^\bullet \to T^\bullet, \quad (\alpha^n : S^n \Rightarrow T^n)_{n \in \mathbb{Z}}.$$

Denote by

$$(-)^0 : \text{Fun}_\delta^{\ge 0}(\mathcal{A}, \mathcal{B}) \to \text{LExactFun}(\mathcal{A}, \mathcal{B}),$$

the functor that associates to every cohomological δ-functor, $T^\bullet$, the functor, T^0, and that associates to every natural transformation of cohomological δ-functors, $u^\bullet : S^\bullet \to T^\bullet$, the natural transformation $u^0 : S^0 \to T^0$. Denote by

$$R : \text{LExactFun}(\mathcal{A}, \mathcal{B}) \to \text{Fun}_\delta^{\ge 0}(\mathcal{A}, \mathcal{B}),$$

the functor that associates to every left-exact functor, F, the cohomological δ-functor, $R^\bullet F$, and that associates to the natural transformation, $\alpha : F \Rightarrow G$, the natural transformation of cohomological δ-functors, $R^\bullet\alpha : R^\bullet F \Rightarrow R^\bullet G$. Prove that R is left adjoint to $(-)^0$.

(g) In particular, for $n > 0$, prove that $R^0(R^nF)$ is the zero functor. Thus, for every $m \geq n$, $R^m(R^nF)$ is the zero functor.

Right Derived Functors and Filtering Colimits Exercise. Let $\mathcal{B}$ be a cocomplete Abelian category satisfying Grothendieck's condition (AB5). Let I be a small filtering category. Let $C^\bullet : I \to \mathrm{Ch}^\bullet(\mathcal{B})$ be a functor.

(a) For every $n \in \mathbb{Z}$, prove that the natural $\mathcal{B}$-morphism,

$$\operatorname*{colim}_{i\in I} H^n(C^\bullet(i)) \to H^n(\operatorname*{colim}_{i\in I} C^\bullet(i)),$$

is an isomorphism. **Prove** that this extends to a natural isomorphism of cohomological δ-functors. This is "commutation of cohomology with filtered colimits".

(b) Let $\mathcal{A}$ be an Abelian category with enough injective objects. Let $F : I \times \mathcal{A} \to \mathcal{B}$ be a bifunctor such that for every object i of I, the functor $F_i : \mathcal{A} \to \mathcal{B}$ is additive and left-exact. Prove that $F_\infty(-) := \mathrm{colim}_{i\in I} F_i(-)$ also forms an additive functor that is left-exact. Also prove that the natural map

$$\operatorname*{colim}_{i\in I} R^n(F_i) \to R^n(F_\infty)$$

is an isomorphism. This is "commutation of right derived functors with filtered colimits".

15 Constructing Injectives via Adjoint Pairs

Projective / Injective Objects and Adjoint Pairs Exercise. Recall that for a category $\mathcal{C}$, for every object X of $\mathcal{C}$, there is a covariant Yoneda functor,

$$h^X : \mathcal{C} \to \mathbf{Sets}, \quad B \mapsto \mathrm{Hom}_{\mathcal{C}}(X, B),$$

and for every object Y of $\mathcal{C}$, there is a contravariant Yoneda functor,

$$h_Y : \mathcal{C}^{\mathrm{opp}} \to \mathbf{Sets}, \quad A \mapsto \mathrm{Hom}_{\mathcal{C}}(A, Y).$$

An object X of $\mathcal{C}$ is **projective** if the Yoneda functor h^X sends epimorphisms to epimorphisms. An object Y of $\mathcal{C}$ is **injective** if the Yoneda functor h_Y sends monomorphisms to epimorphisms. The category has **enough projectives** if for every object B there exists a projective object X and an epimorphism $X \to B$. The category has **enough injectives** if for every object A there exists an injective object Y and a monomorphism from A to Y.

(a) Check that this notion agrees with the usual definition of projective and injective for objects in an Abelian category.

(b) For the category **Sets**, assuming the Axioms of Choice, prove that every object is both projective and injective. Deduce the same for the opposite category, $\mathbf{Sets}^{\text{opp}}$.

(c) Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Let (L, R, θ, η) be an adjoint pair of covariant functors,

$$L : \mathcal{C} \to \mathcal{D}, \quad R : \mathcal{D} \to \mathcal{C}.$$

For every object d of $\mathcal{D}$, prove that

$$\eta(d) : L(R(d)) \to d,$$

is an epimorphism. For every object c of $\mathcal{C}$, prove that

$$\theta : c \to R(L(c)),$$

is a monomorphism. Thus, if every $L(R(d))$ is a projective object, then $\mathcal{C}$ has enough projective objects. Similarly, if every $R(L(c))$ is an injective object, then $\mathcal{C}$ has enough injective objects.

(d) Assuming that R sends epimorphisms to epimorphisms, prove that L sends projective objects of $\mathcal{C}$ to projective objects of $\mathcal{D}$. Thus, if every object of $\mathcal{C}$ is projective, conclude that $\mathcal{D}$ has enough projective objects. More generally, assume further that R is **faithful**, i.e., R sends distinct morphisms to distinct morphisms. Then conclude for every epimorphism $X \to R(D)$ in $\mathcal{C}$, the associated morphism $L(X) \to D$ in $\mathcal{D}$ is an epimorphism. Thus, if $\mathcal{C}$ has enough projective objects, also $\mathcal{D}$ has enough projective objects.

Similarly, assuming that L sends monomorphisms to monomorphisms, prove that R sends injective objects of $\mathcal{D}$ to injective objects of $\mathcal{C}$. Thus, if every object of $\mathcal{D}$ is injective, conclude that there are enough injective objects of $\mathcal{C}$. More generally, assume further that L is faithful. Then conclude for every monomorphism $L(C) \to Y$ in $\mathcal{D}$, the associated morphism $C \to R(Y)$ in $\mathcal{C}$ is a monomorphism. Thus, if $\mathcal{D}$ has enough injective objects, also $\mathcal{C}$ has enough injective objects.

(e) Let S and T be associative, unital algebras. Let $\mathcal{C}$ be the category **Sets**. Let $\mathcal{D}$ be the category $S - T - \text{mod}$ of $S - T$-bimodules. Let

$$R : S - T - \text{mod} \to \mathbf{Sets}$$

be the forgetful functor that sends every $S - T$-bimodule to the underlying set of elements of the bimodule. Prove that R sends epimorphisms to epimorphisms and R is faithful. Prove that there exists a left adjoint functor,

$$L : \mathbf{Sets} \to S - T - \text{mod},$$

that sends every set Σ to the corresponding $S - T$-bimodule, $L(\Sigma)$ of functions $f : \Sigma \to S \otimes_{\mathbb{Z}} T$ that are zero except on finitely many elements of Σ. Since **Sets** has enough projective objects (in fact every object is projective), conclude that $S - T - \text{mod}$ has enough projective objects.

(e) Let S, T and U be associative, unital rings. Let B be a $T - U$-bimodule. Let $\mathcal{C}$ be the Abelian category of $S - T$-bimodules, let $\mathcal{D}$ be the Abelian category of $S - U$-bimodules, let L be the exact, additive functor,

$$L : S - T - \text{mod} \to S - U - \text{mod}, \quad L(A) = A \otimes_T B,$$

and let R be the right adjoint functor,

$$R : S - U - \text{mod} \to S - T - \text{mod}, \quad R(C) = \text{Hom}_{\text{mod}-U}(B, C).$$

Prove that if B is a flat (left) T-module, resp. a faithfully flat (left) T-module, then L sends monomorphisms to monomorphisms, resp. L sends monomorphism to monomorphisms and is faithful. Conclude, then, that R sends injective objects of $S - U - \text{mod}$ to injective objects of $S - T - \text{mod}$, resp. if $S - U - \text{mod}$ has enough injective objects then also $S - T - \text{mod}$ has enough injective objects.

(f) Continuing as above, for every ring homomorphism $U \to T$, prove that the induced $T-U$-module structure on T is faithfully flat as a left T-module. Thus, given rings Λ and Π, define $S = \Lambda$, define $T = \Pi$, and define U to be $\mathbb{Z}$ with its unique ring homomorphism to T. Conclude that if there exist enough injective objects in $\Lambda - \text{mod}$, then there exist enough injective objects in $\Lambda - \Pi - \text{mod}$.

(g) For the next step, define T and U to be Λ, define B to be Λ as a left-right T-module, and define S to be $\mathbb{Z}$. Conclude that if there are enough injective $\mathbb{Z}$-modules, then there are enough injective Λ-modules, and hence there are enough injective $\Lambda - \Pi$-bimodules. Thus, to prove that there are enough $\Lambda - \Pi$-bimodules, it is enough to prove that there are enough $\mathbb{Z}$-modules.

Enough Injective Modules Exercise. Let $\mathcal{A}$ be an Abelian category that has all small products. An object Y of $\mathcal{A}$ is an **injective cogenerator** if Y is injective and for every pair of distinct morphisms,

$$u, v : A' \to A,$$

in $\mathcal{A}$, there exists a morphism $w : A \to Y$ such that $w \circ u$ and $w \circ v$ are also distinct.

(a) Let $\mathcal{C}$ be the category $\textbf{Sets}^{\text{opp}}$. For an object Y of $\mathcal{A}$, define L to be the Yoneda functor

$$h_Y : \mathcal{A} \to \textbf{Sets}^{\text{opp}}, \quad h_Y(A) = \text{Hom}_{\mathcal{A}}(A, Y).$$

Similarly, define the functor,

$$R : \textbf{Sets}^{\text{opp}} \to \mathcal{A}, \quad L(\Sigma) = \text{"}\text{Hom}_{\textbf{Sets}}(\Sigma, Y)\text{"},$$

that sends every set Σ to the object $R(\Sigma)$ in $\mathcal{A}$ that is the direct product of copies of Y indexed by elements of Σ. Prove that L and R are an adjoint pair of functors.

(b) Assuming that $\mathcal{A}$ has an injective cogenerator Y, prove that L sends monomorphisms to monomorphisms, and prove that L is faithful. Conclude that $\mathcal{A}$ has enough injective objects.

(c) Now let S be an associative, unital ring (it suffices to consider the special case that S is $\mathbb{Z}$). Let $\mathcal{A}$ be $\text{mod} - S$. Use the Axiom of Choice to prove Baer's criterion: a right S-module Y is injective if and only if for every right ideal J of S, the induced set map

$$\text{Hom}_{\text{mod}-S}(S, Y) \to \text{Hom}_{\text{mod}-S}(J, Y)$$

is surjective. In particular, if S is a principal ideal domain, conclude that Y is injective if and only if Y is divisible.

(d) Finally, defining S to be $\mathbb{Z}$, conclude that $Y = \mathbb{Q}/\mathbb{Z}$ is injective, since it is divisible. Finally, for every Abelian group A and for every nonzero element a of A, conclude that there is a nonzero $\mathbb{Z}$-module homomorphism $\mathbb{Z} \cdot a \to \mathbb{Q}/\mathbb{Z}$. Thus, for every pair of elements $a', a'' \in A$ such that $a = a' - a''$ is nonzero, conclude that there exists a $\mathbb{Z}$-module homomorphisms $w : A \to \mathbb{Q}/\mathbb{Z}$ such that $w(a') - w(a'')$ is nonzero. Conclude that $\mathbb{Q}/\mathbb{Z}$ is an injective cogenerator of $\mathbb{Z}$. Thus $\text{mod} - \mathbb{Z}$ has enough injective objects. Thus, for every pair of associative, unital rings Λ, Π, the Abelian category $\Lambda - \Pi - \text{mod}$ has enough injective objects.

Enough Injectives / Projectives in the Cochain Category Exercise. Let S be an associative, unital ring. Prove that $\text{Ch}^{\geq 0}(S)$ has enough injective objects, and prove that $\text{Ch}^{\leq 0}(S)$ has enough projective objects.

16 The Koszul Complex via Adjoint Pairs

Exterior Algebra CDGA as an Adjoint Pair Exercise. Let R be a commutative, unital ring. An *associative, unital, graded commutative R-algebra* (with homological indexing) is a triple

$$A_\bullet = ((A_n)_{n\in\mathbb{Z}}, (m_{p,q} : A_p \times A_q \to A_{p+q})_{p,q\in\mathbb{Z}}, (\epsilon : R \to A_0))$$

of a sequence $(A_n)_{n\in\mathbb{Z}}$ of R-modules, of a sequence $(m_{p,q})_{p,q\in\mathbb{Z}}$ of R-bilinear maps, and an R-module morphism ϵ such that the following hold.

(i) For the associated R-module $A = \bigoplus_{n\in\mathbb{Z}} A_n$ and the induced morphism $m : A \times A \to A$ whose restriction to each $A_p \times A_q$ equals $m_{p,q}$, $(A, m, \epsilon(1))$ is an associative, unital, R-algebra.

(ii) For every $p, q \in \mathbb{Z}$, for every $a_p \in A_p$ and for every $a_q \in A_q$, $m_{q,p}(a_q, a_p)$ equals $(-1)^{pq} m_{p,q}(a_p, a_q)$.

(a) Prove that the R-submodules of A,

$$A_{\geq 0} = \bigoplus_{n\geq 0} A_n, \quad A_{\leq 0} = \bigoplus_{n\leq 0} A_n,$$

are both associative, unital R-subalgebras. Moreover, prove that the R-submodule,

$$A_{>0} = \bigoplus_{n>0} A_n, \text{ resp. } A_{<0} = \bigoplus_{n<0} A_n,$$

is a left-right ideal in $A_{\geq 0}$, resp. in $A_{\leq 0}$.

(b) For associative, unital, graded commutative R-algebras $A_\bullet$ and $B_\bullet$, a graded homomorphism of R-algebras is a collection

$$f_\bullet = (f_n : A_n \to B_n)_{n\geq 0}$$

such that for the unique R-module homomorphism $f : A \to B$ whose restriction to every A_n equals f_n, f is an R-algebra homomorphism. Prove that such $f_\bullet$ is uniquely reconstructed from the homomorphism f. Prove that Id_A comes from a unique graded homomorphism $\text{Id}_{A_\bullet}$. Prove that for a graded homomorphism of R-algebras, $g_\bullet : B_\bullet \to C_\bullet$, the composition $g \circ f$ arises from a unique graded homomorphism of R-algebras, $A_\bullet \to C_\bullet$. Using this to define composition of homomorphisms of graded R-algebras, prove that composition is associative and the identity morphisms abe are left-right identities for composition. Conclude that these notions form a category $R-\text{GrComm}$ of associative, unital, graded commutative R-algebras. Prove that the rule $A_\bullet \mapsto A$, $f_\bullet \mapsto f$ defines a faithful functor

$$R - \text{GrComm} \to R - \text{Algebra}.$$

Give an example showing that this functor is not typically full.

(c) Let $A_\bullet$ be an associative, unital, graded commutative R-algebra. Prove that R is commutative (in the usual sense) if and only if A_n is a zero module for every even integer n. Denote by $R-\text{Comm}$ the category of associative, unital R-algebras S that are commutative. Denote by $\mathbb{Z} - R - \text{Comm}$ the faithful (but not full) subcategory whose objects are triples,

$$S_\bullet = ((S_n)_{n\in\mathbb{Z}}, (m_{p,q} : S_p \times S_q \to S_{p+q})_{p,q\in\mathbb{Z}}, (\epsilon : R \to S_0))$$

as above, but such that the multiplication is commutative rather than graded commutative, i.e., $m_{q,p}(s_q, s_p) = m_{p,q}(s_p, s_q)$. Prove that there is a functor,

$$v_{\text{even}} : R - \text{GrComm} \to \mathbb{Z} - R - \text{Comm},$$

$$((A_n)_{n\in\mathbb{Z}}, (m_{p,q} : A_p \times A_q \to A_{p+q})_{p,q\in\mathbb{Z}}, (\epsilon : R \to A_0)) \mapsto ((A_{2n})_{n\in\mathbb{Z}}, (m_{2p,2q} : A_{2p} \times A_{2q} \to A_{2(p+q)})_{p,q\in\mathbb{Z}}, (\epsilon : R \to A_0 =$$

and $f_\bullet : A_\bullet \to B_\bullet$ maps to $v_{\text{ev}}(f) = (f_{2n})_{n\in\mathbb{Z}}$. Also prove that there is a left adjoint to v_{even},

$$w_{\text{even}} : \mathbb{Z} - R - \text{Comm} \to R - \text{GrComm},$$

where $w_{\text{even}}(S_\bullet)_{2n}$ equals S_n, where $w_{\text{even}}(S_\bullet)_p$ is the zero module for every odd p, where

$$A_{2p} \times A_{2q} \to A_{2(p+q)}$$

is $m_{p,q}$ for $S_\bullet$, and where $R \to A_0$ is $\epsilon : R \to S_0$. For a morphism $f_\bullet : S_\bullet \to T_\bullet$ in $\mathbb{Z} - R - \text{Comm}$, $w_{\text{even}}(f_\bullet)$ is the unique morphism whose component in degree $2n$ equals f_n for every $n \in \mathbb{Z}$.

(d) Let e be an odd integer. For every associative, unital, graded commutative R-algebra $A_\bullet$ define $v_e(A_\bullet)$ to be the collection

$$((A_{ne})_{n\in\mathbb{Z}}, (m_{pe,qe} : A_{pe} \times A_{qe} \to A_{(p+q)e})_{p,q\in\mathbb{Z}}, \epsilon : R \to A_0 = A_{0e}).$$

Prove that $v_e(A_\bullet)$ is again an associative, unital, graded commutative R-algebra. For every morphism of associative, unital, graded commutative R-algebras, $f_\bullet : A_\bullet \to B_\bullet$, the collection $v_e(f_\bullet) = (f_{ne})_{n\in\mathbb{Z}}$ is a morphism of associative, unival, graded commutative R-algebras, $v_e(A_\bullet) \to v_e(B_\bullet)$. Prove that this defines a functor,

$$v_e : R - \text{GrComm} \to R - \text{GrComm}.$$

This is sometimes called the *Veronese functor* (it is closely related to the Veronese morphism of projective spaces). If e is positive, prove that the induced morphism $v_e(A_{\geq 0}) \to v_e(A_\bullet)$, resp. $v_e(A_{\leq 0}) \to v_e(A_\bullet)$, is an isomorphism to $(v_e(A_\bullet))_{\geq 0}$, resp. to $(v_e(A_\bullet))_{\leq 0}$. Similarly, if e is negative (e.g., if e equals -1), this defines an isomorphism to $(v_e(A_\bullet))_{\leq 0}$, resp. to $(v_e(A_\bullet))_{\geq 0}$. Prove that v_1 is the identity functor. For odd integers d and e, construct a natural isomorphism of functors,

$$v_{d,e} : v_d \circ v_e \Rightarrow v_{de},$$

prove that $v_{d,1}$ and $v_{1,e}$ are identity natural transformations, and prove that these natural isomorphisms are associative: $v_{de,f} \circ (v_{d,e} \circ v_f)$ equals $v_{d,ef} \circ (v_d \circ v_{e,f})$ for all odd integers d, e and f.

(e) For every associative, unital, graded commutative R-algebra $A_\bullet$, for every odd integer e, define

$$w_e : R - \text{GrComm} \to R - \text{GrComm},$$

where $w_e(A_\bullet)_{ne}$ equals A_n for every integer n, and where $w_e(A_\bullet)_m$ is a zero module if e does not m. For every morphism $f_\bullet : A_\bullet \to B_\bullet$, define $w_e(f_\bullet)$ to the be the unique morphism whose component in degree en equals f_n for every $n \in \mathbb{Z}$. Prove that w_e is a functor. For the natural isomorphism,

$$\theta_e(A_\bullet) : A_\bullet \to v_e(w_e(A_\bullet)), (A_n \xrightarrow{=} A_n)_{n\in\mathbb{Z}}$$

and the natural monomorphisms

$$\eta_e(B_\bullet) : w_e(v_e(B_\bullet)) \to B_\bullet, (B_{ne} \xrightarrow{=} B_{ne})_{n\in\mathbb{Z}},$$

prove that $(w_e, v_e, \theta_e, \eta_e)$ is an adjoint pair.

(f) For every integer $n \geq 0$, recall from Problem 5(iv) of Problem Set 1, that there is a functor,

$$\bigwedge_R^n : R - \text{mod} \to R - \text{mod},\ M \mapsto \bigwedge_R^n(M).$$

In particular, there is a natural isomorphism

$$\epsilon(M) : R \to \bigwedge_R^0(M),$$

and there is a natural isomorphism,

$$\theta(M) : M \to \bigwedge_R^1(M).$$

By convention, for every integer $n < 0$, define $\bigwedge_R^n(M)$ to be the zero module. For every pair of integers $q, r \geq 0$, prove that the natural R-bilinear map

$$\otimes : M^{\otimes q} \times M^{\otimes r} \to M^{\otimes(q+r)},\ ((m_1 \otimes \cdots \otimes m_q), (m_1' \otimes \cdots \otimes m_r')) \mapsto m_1 \otimes \ldots m_q \otimes m_1' \otimes \cdots \otimes m_r',$$

factors uniquely through an R-bilinear map,

$$\wedge : \bigwedge_R^q(M) \times \bigwedge_R^r(M) \to \bigwedge_R^{q+r}(M).$$

Prove that $\bigwedge_R^\bullet(M)$ is an associative, unital, graded commutative R-algebra. For every R-module homomorphism $\phi : M \to N$, prove that the associated R-module homomorphisms,

$$\bigwedge_R^n(\phi) : \bigwedge_R^n(M) \to \bigwedge_R^n(N),$$

define a morpism of associative, unital, graded commutative R-algebras,

$$\bigwedge_R^\bullet(\phi) : \bigwedge_R^\bullet(M) \to \bigwedge_R^\bullet(N).$$

Prove that for every R-module homomorphism $\psi : N \to P$, $\bigwedge_R^\bullet(\psi \circ \phi)$ equals $\bigwedge_R^\bullet(\psi) \circ \bigwedge_R^\bullet(\phi)$. Also prove that $\bigwedge_R^\bullet(\mathrm{Id}_M)$ is the identity morphism of $\bigwedge_R^\bullet(M)$.

(g) An associative, unital, graded commutative R-algebra $A_\bullet$ is (strictly) 0-*connected*, resp. *weakly* 0-*connected*, if the inclusion $A_{\geq 0} \to A$ is an isomorphism and the R-module homorphism ϵ is an isomorphism, resp. an epimorphism. If R is a field, prove that every weakly 0-connected algebra is strictly 0-connected. Denote by

$$R-\mathrm{GrComm}_{\geq 0}, \text{ resp. } R-\mathrm{GrComm}'_{\geq 0}$$

the full subcategory of $R-\mathrm{GrComm}$ whose objects are the 0-connected algebras, resp. the weakly 0-connected algebras. Prove that v_{even} restricts to a functor,

$$R-\mathrm{GrComm}_{\geq 0} \to \mathbb{Z}_+ - R - \mathrm{Comm},$$

where $\mathbb{Z}_+ - R - \mathrm{Comm}$ is the full subcategory of $\mathbb{Z} - R - \mathrm{Comm}$ of algebras graded in nonnegative degrees such that $R \to S_0$ is an isomorphism. For e an odd positive integer, prove that v_e and w_e restrict to an adjoint pair of functors,

$$v_e : R-\mathrm{GrComm}_{\geq 0} \to R-\mathrm{GrComm}_{\geq 0},$$

$$w_e : R-\mathrm{GrComm}_{\geq 0} \to R-\mathrm{GrComm}_{\geq 0}.$$

For every odd positive integer e, define a functor

$$\Phi_e : R-\mathrm{GrComm}_{\geq 0} \to R-\mathrm{mod},$$

that sends $A_\bullet$ to A_e and sends $f_\bullet$ to f_e. Of course, for every odd positive integer d, $\Phi_e \circ v_d$ is naturally isomorphic to Φ_{de} and $\Phi_{de} \circ w_d$ is Φ_e. By the previous part, there is a functor

$$\bigwedge_R^\bullet : R-\mathrm{mod} \to R-\mathrm{GrComm}_{\geq 0}$$

that sends every module M to the 0-connected, associative, unital, graded commutative R-algebra $(\bigwedge_R^n(M))_{n\geq 0}$. Moreover, there is a natural transformation,

$$\theta : \mathrm{Id}_{R-\mathrm{mod}} \Rightarrow \Phi_1 \circ \bigwedge_R^{\bullet}.$$

Prove that this extends uniquely to an adjoint pair of functors

$$(\bigwedge_R^{\bullet}, \Phi_1, \theta, \eta).$$

Using the natural isomorphisms $\Phi_e \circ v_d = \Phi_{de}$ and $\Phi_{de} \circ w_d = \Phi_e$, prove that there is also an adjoint pair of functors

$$(w_e \circ \bigwedge_R^{\bullet}, \Phi_e, \theta, \eta_e).$$

The Koszul Complex CDGA as an Adjoint Pair. Let R be a commutative, unital ring. A (homological, unital, associative, graded commutative) *differential graded R-algebra* is a pair

$$((C_n)_{n\in\mathbb{Z}}, (\wedge : C_p \times C_q \to C_{p+q})_{p,q\in\mathbb{Z}}, (\epsilon : R \to C_0), (d_n : C_n \to C_{n-1})_{n\in\mathbb{Z}}),$$

of an associative, unital, graded commutative R-algebra $C_\bullet$ together with R-linear morphisms $(d_n)_{n\in\mathbb{Z}}$ such that $d_{n-1} \circ d_n$ equals 0 for every $n \in \mathbb{Z}$, and that satisfies the graded Leibniz identity,

$$d_{p+q}(c_p \wedge c_q) = d_p(c_p) \wedge c_q + (-1)^p c_p \wedge d_q(c_q),$$

for every $p, q \in \mathbb{Z}$, for every $c_p \in C_p$, and for every $c_q \in C_q$. A *morphism* of differential graded R-algebras,

$$\phi_\bullet : C_\bullet \to A_\bullet,$$

is a morphism $\phi_\bullet = (\phi_n)_{n\in\mathbb{Z}}$ that is simultaneously a morphism of chain complexes of R-modules and a morphism of associative, unital, graded commutative R-algebras.

(a) For morphisms of differential graded R-algebras, $\phi_\bullet : C_\bullet \to A_\bullet$, $\psi_\bullet : D_\bullet \to C_\bullet$, prove that the composition of $\psi_\bullet \circ \phi_\bullet$ of graded R-modules is both a morphism of chain complexes of R-modules and a morphism of associative, unital, graded commutative R-algebras. Thus, it is a composition of morphisms of differential graded R-algebras. With this composition, prove that this defines a category $R - \mathrm{CDGA}$ of differential graded R-algebras.

(b) For every associative, unital, graded commutative R-algebra $A_\bullet$, for every integer n, define $d_{E(A),n} : A_n \to A_{n-1}$ to be the zero morphism. Prove that this gives a differential graded R-algebra, denoted $E(A_\bullet)$. For every morphism $f_\bullet : A_\bullet \to B_\bullet$ of associative, unital, graded commutative R-algebras, prove that $f_\bullet : E(A_\bullet) \to E(B_\bullet)$ is a morphism of differential graded R-algebras, denoted $E(f_\bullet)$. Prove that this defines a functor

$$E : R - \mathrm{GrComm} \to R - \mathrm{CDGA}.$$

For every differential graded R-algebra $C_\bullet$, prove that the subcomplex $Z_\bullet(C_\bullet)$ is a differential graded R-subalgebra with zero differential, and the inclusion,

$$\eta(C_\bullet) : E(Z_\bullet(C_\bullet)) \to C_\bullet,$$

is a morphism of differential graded R-algebras. Also, for every morphism $\phi_\bullet : C_\bullet \to D_\bullet$ of differential graded R-algebras, prove that the induced morphism $Z_\bullet(f_\bullet) : Z_\bullet(C_\bullet) \to Z_\bullet(D_\bullet)$ is a morphism of associative, unital, graded commutative R-algebras. Prove that this defines a functor

$$Z_\bullet : R-\text{CDGA} \to R-\text{GrComm}.$$

For every associative, unital, graded commutative R-algebra $A_\bullet$, the inclusion $Z_\bullet(E(A_\bullet)) \to E(A_\bullet)$ is just the identity map, whose inverse,

$$\theta(A_\bullet) : A_\bullet \to Z_\bullet(E(A_\bullet)),$$

is an isomorphism. Prove that $(E, Z_\bullet, \theta, \eta)$ is an adjoint pair of functors. Finally, prove that the subcomplex $B_\bullet(C_\bullet) \subset Z_\bullet(C_\bullet)$ is a left-right ideal in the associative, unital, graded commutative R-algebra $Z_\bullet(C_\bullet)$. Conclude that there is a unique structure of associative, unital, graded commutative R-algebra on the cokernel $H_\bullet(C_\bullet)$ such that the quotient morphism $Z_\bullet(C_\bullet) \to H_\bullet(C_\bullet)$ is a morphism of differential graded R-algebras. Prove that altogether this defines a functor,

$$H : R-\text{CDGA} \to R-\text{GrComm}.$$

(c) A differential graded R-algebra $C_\bullet$ is (strictly) 0-*connected*, resp. *weakly* 0-*connected*, if the underlying associative, unital, graded commutative R-algebra is 0-connected, resp. weakly 0-connected. Denote by $R-\text{CDGA}_{\geq 0}$, resp. $R-\text{CDGA}'_{\geq 0}$, the full subcategory of $R-\text{CDGA}$ whose objects are the 0-connected differential graded R-algebras, resp. those that are weakly 0-connected. Prove that the functors above restrict to functors,

$$E : R-\text{GrComm}_{\geq 0} \to R-\text{CDGA}_{\geq 0},$$

$$Z_\bullet : R-\text{CDGA}_{\geq 0} \to R-\text{GrComm}_{\geq 0},$$

such that (E, Z, θ, η) is still an adjoint pair. Similarly, show that H restricts to a functor

$$H : R-\text{CDGA}_{\geq 0} \to R-\text{GrComm}'_{\geq 0}.$$

(d) Denote by $R-\text{CDGA}_{[0,1]}$ the full subcategory of $R-\text{CDGA}_{\geq 0}$ whose objects are 0-connected differential graded R-algebras $C_\bullet$ such that C_n is a zero object for $n > 1$. Prove that every such object is uniquely determined by the data of an R-module C_1 and an R-module homomorphism $d_{C,1} : C_1 \to C_0 = R$, and conversely such data uniquely determine an object of $R-\text{CDGA}_{[0,1]}$. Prove that for such algebras $C_\bullet$ and $D_\bullet$, every morphism $\phi_\bullet : C_\bullet \to D_\bullet$ of differential graded R-algebras is uniquely determined by an R-module homomorphism $\phi_1 : C_1 \to D_1$ such that $d_{D,1} \circ \phi_1$ equals $d_{C,1}$, and conversely, such an R-module homomorphism uniquely determines a morphism of differential graded R-algebras. Conclude that there is a functor

$$\sigma_{[0,1]} : R-\text{CDGA}_{\geq 0} \to R-\text{CDGA}_{[0,1]},$$

that associates to every 0-connected differential graded R-algebra $C_\bullet$ the algebra $\sigma_{[0,1]}(C_\bullet)$ uniquely determined by the R-module homomorphism $d_{C,1} : C_1 \to C_0 = R$ and that associates to every morphism $\phi_\bullet : C_\bullet \to D_\bullet$ of 0-connected differential graded R-algebras the morphism,

$$\sigma_{[0,1]}(\phi_\bullet) : \sigma_{[0,1]}(C_\bullet) \to \sigma_{[0,1]}(D_\bullet),$$

uniquely determined by the morphism $\phi_1 : C_1 \to D_1$.

(e) For every R-module M and for every R-module homomorphism $\phi : M \to R$, prove that there exists a unique sequence of R-module homomorphisms,

$$(d_{M,\phi,n} : \bigwedge_R^n(M) \to \bigwedge_R^{n-1}(M))_{n>0},$$

such that d_1 equals ϕ and such that $(\bigwedge_R^\bullet(M), d_{M,\phi})$ is a 0-connected differential graded R-algebra. It may be convenient to begin with the case of a free R-module P and a morphism $\psi : P \to R$, in which case every $\bigwedge_R^n(P)$ is also free and the R-module homomorphisms d_n is uniquely determined by its restriction to a convenient basis. Given a presentation $M = P/K$ such that ψ factors uniquely through $\phi : M \to R$, prove that the associative, unital, graded commutative R-algebra $\bigwedge_R^\bullet(M)$ is the quotient of $\bigwedge_R^\bullet(P)$ by the left-right ideal generated by $K \subset P = \bigwedge_R^1(P)$. Also prove that $d_{P,\psi}$ maps this ideal to itself, i.e., the ideal is differentially-closed. Conclude that there is a unique structure of differential graded algebra on the quotient $\bigwedge_R^\bullet(M)$ such that the quotient map is a morphism of differential graded R-algebras.

(f) Prove that the construction of the previous part defines a functor,

$$\bigwedge_R^\bullet : R-\text{CDGA}_{[0,1]} \to R-\text{CDGA}_{\geq 0}.$$

Prove that for every object $(\phi : M \to R)$ of $R-\text{CDGA}_{[0,1]}$, the morphism

$$\theta(M,\phi) : M \xrightarrow{=} \bigwedge_R^1(M)$$

is a natural isomorphism

$$\theta : \text{Id}_{R-\text{CDGA}_{[0,1]}} \Rightarrow \sigma_{[0,1]} \circ \bigwedge_R^\bullet.$$

Similarly, for every object 0-connected differential graded R-algebra $C_\bullet$, prove that the natural transformation from Problem 10(g),

$$\eta(C_\bullet) : \bigwedge_R^\bullet(C_1) \to C_\bullet,$$

is compatible with the differential on $\bigwedge_R^\bullet(C_1)$ induced by $d_{C,1} : C_1 \to C_0 = R$, i.e., $\eta(C_\bullet)$ is a natural transformation,

$$\eta : \bigwedge_R^\bullet \circ \sigma_{[0,1]} \to \mathrm{Id}_{R-\mathrm{CDGA}_{\geq 0}}.$$

Conclude that $(\bigwedge_R^\bullet, \sigma_{[0,1]}, \theta, \eta)$ is an adjoint pair of functors. For every $\phi : M \to R$ in $R-\mathrm{CDGA}_{[0,1]}$, the associated 0-complete differential graded R-algebra structure on $\bigwedge_R^\bullet(M)$ is called the *Koszul algebra* of $\phi : M \to R$ and denoted $K_\bullet(M,\phi)$.

(g) For every R-module M, and for every R-submodule M' of M, denote by $F^1 \subset \bigwedge_R^\bullet(M)$ the left-right ideal generated by $M' \subset M = \bigwedge_R^1(M)$. For every integer $n \leq 0$, denote by $F^n \subset \bigwedge_R^\bullet(M)$ the entire algebra. For every integer $n \geq 1$, denote by F^n the left-right ideal of $\bigwedge_R^\bullet(M)$ generated by the n-fold self-product $F^1 \cdot \cdots \cdot F^1$. For every pair of nonnegative integers p, q, prove that the ideal $F^p \cdot F^q$ equals F^{p+q}. In particular, prove that there is a natural epimorphism,

$$\bigwedge_R^p (F_1^1) \otimes_R \bigwedge_R^q (M) \to F_{p+q}^p.$$

Denote the quotient M/M' by M'', and denote by Σ the short exact sequence,

$$\Sigma :\ 0 \longrightarrow M' \xrightarrow{u} M \xrightarrow{v} M'' \longrightarrow 0.$$

For every nonnegative integer q, prove that the R-module morphism,

$$\bigwedge_R^q (v) : \bigwedge_R^q (M) \to \bigwedge_R^q (M''),$$

is an epimorphism whose kernel equals F_q^1. Conclude that the composite epimorphism

$$\bigwedge_R^p (M') \otimes_R \bigwedge_R^q (M) \to F_{p+q}^p \to F_{p+q}^p / F_{p+q}^{p+1}$$

factors uniquely through an R-module epimorphism

$$c_{\Sigma,p,q} : \bigwedge_R^p (M') \otimes_R \bigwedge_R^q (M'') \to F_{p+q}^p / F_{p+q}^{p+1}.$$

In case there exists a splitting of Σ, prove that every epimorphism $c_{\Sigma,p,q}$ is an isomorphism. On the other hand, find an example where Σ is not split and some morphism $c_{\Sigma,p,q}$ is not a monomorphism (there exist such examples for $R = \mathbb{C}[x,y]$).

(h) Continuing the previous problem, assume that M'' is isomorphic to R as an R-module (or, more generally, projective of constant rank 1), so that Σ is split. For every nonnegative integer p, conclude that there exists a short exact sequence,

$$\Sigma_{p,1} :\ 0 \longrightarrow \bigwedge_R^{p+1}(M') \xrightarrow{\bigwedge_R^{p+1}(u)} \bigwedge_R^{p+1}(M) \xrightarrow{c_{\Sigma,p,1}^{-1}} \bigwedge_R^p (M') \otimes_R M'' \longrightarrow 0,$$

that is split. Check that this is compatible with the product structure and, thus, defines a short exact sequence of graded (left) $\bigwedge_R^\bullet(M)$-modules,

$$\bigwedge_R^\bullet(\Sigma): \ 0 \longrightarrow \bigwedge_R^\bullet(M') \xrightarrow{\bigwedge_R^\bullet(u)} \bigwedge_R^\bullet(M) \xrightarrow{c_\Sigma^{-1}} \bigwedge_R^\bullet(M')\otimes_R M''[+1] \longrightarrow 0.$$

(i) Now, let $\phi : M \to R$ be an R-module homomorphism. Denote by $\phi' : M' \to R$ the restriction $\phi\circ u$. These morphisms define structures of differential graded R-algebra, $K_\bullet(M,\phi)$ on $\bigwedge_R^\bullet(M)$, and $K_\bullet(M',\phi')$ on $\bigwedge_R^\bullet(M')$. Moreover, the morphism $\bigwedge_R^\bullet(u)$ above is a morphism of differential graded R-modules,

$$K(u) : K_\bullet(M',\phi') \to K_\bullet(M,\phi).$$

Prove that the induced morphism

$$c_\Sigma^{-1} : K_\bullet(M,\phi) \to K_\bullet(M',\phi')\otimes_R M''[+1]$$

is a morphism of cochain complexes. Moreover, for a choice of splitting $s : M'' \to M$, for the induced morphism $\phi'' : M'' \to R$, $\phi'' = \phi\circ s$, for the induced morphism of cochain complexes,

$$\mathrm{Id}_{K_\bullet(M',\phi')}\otimes\phi'' : K_\bullet(M',\phi')\otimes_R M'' \to K_\bullet(M',\phi'),$$

prove that there is a unique commutative diagram of short exact sequences,

$$\begin{array}{ccccccccc}
T_{\mathrm{Id}\otimes\phi''}: & 0 & \longrightarrow & K_\bullet(M',\phi') & \xrightarrow{q_{\mathrm{Id}\otimes\phi''}} & \mathrm{Cone}(\mathrm{Id}\otimes\phi'') & \xrightarrow{p_{\mathrm{Id}\otimes\phi''}} & K_\bullet(M',\phi')\otimes_R M''[+1] & \longrightarrow & 0 \\
\downarrow{\scriptstyle\widetilde{s}} & & & \downarrow{\scriptstyle\mathrm{Id}} & & \downarrow{\scriptstyle\widetilde{s}} & & \downarrow{\scriptstyle\mathrm{Id}} & & \\
K(\Sigma) & 0 & \longrightarrow & K_\bullet(M',\phi') & \xrightarrow{K_\bullet(u)} & K_\bullet(M,\phi) & \xrightarrow{c_\Sigma^{-1}} & K_\bullet(M',\phi')\otimes_R M'' & \longrightarrow & 0.
\end{array}$$

(j) With the same hypotheses as above, conclude that there is an exact sequence of homology (remember the shift $[+1]$ above is cohomological),

$$H_0(K_\bullet(M',\phi'))\otimes_R M'' \xrightarrow{\mathrm{Id}\otimes\phi''} H_0(K_\bullet(M',\phi')) \xrightarrow{K_0(u)} H_0(K_\bullet(M,\phi)) \to 0,$$

i.e., $H_0(K_\bullet(M,\phi)) \cong H_0(K_\bullet(M,\phi))/\phi(M'')\cdot H_0(K_\bullet(M,\phi))$ as a quotient algebra of R. Also, for every $n>0$, conclude the existence of a short exact sequence of Koszul homologies,

$$0 \to K_n(M',\phi')\otimes_R R/\mathrm{Im}(\phi'') \xrightarrow{\psi''} K_n(M,\phi) \to K_{n-1}(M',\phi';M'')_{\mathrm{Im}(\phi'')} \to 0,$$

where for every R-module N, $N_{\mathrm{Im}(\phi'')}$ denotes the submodule of elements that are annihilated by the ideal $\mathrm{Im}(\phi'')\subset R$. As graded modules over the associative, unital, graded commutative R-algebra $K_*(M',\phi') = H_*(K_\bullet(M',\phi'))$, this is a short exact sequence,

$$0 \to K_*(M',\phi')\otimes_R R/\mathrm{Im}(\phi'') \xrightarrow{\psi''} K_*(M,\phi) \to K_{*-1}(M',\phi';M'')_{\mathrm{Im}(\phi'')} \to 0,$$

As a special case, if $K_\bullet(M',\phi')$ is acyclic, and if the morphism

$$H_0(K_\bullet(M',\phi'))\otimes_R M'' \xrightarrow{\mathrm{Id}\otimes\phi''} H_0(K_\bullet(M',\phi'))$$

is injective, conclude that also $K_\bullet(M,\phi)$ is acyclic.

(k) Repeat this exercise for the cohomological Koszul complexes $K^\bullet(M,\phi)$.

17 Adjoint Pairs of Simplicial and Cosimplicial Objects

Constant Cosimplicial Objects and the Right Adjoint. Please read the basic definitions of cosimplicial objects in a category $\mathcal{C}$. In particular, for the small category Δ of totally ordered finite sets with nondecreasing morphisms, read the equivalent characterization of a (covariant) functor

$$C : \Delta \to \mathcal{C},$$

via the specification for every integer $r \geq 0$ of an object C^r of $\mathcal{C}$, the specification for every integer $r \geq 0$ and every integer $i = 0, \dots, r+1$, of a morphism,

$$\partial_r^i : C^r \to C^{r+1},$$

and the specification for every integer $r \geq 0$ and every integer $i = 0, \dots, r$, of a morphism,

$$\sigma_{r+1}^i : C^{r+1} \to C^r,$$

satisfying the *cosimplicial identities*: for every $r \geq 0$, for every $0 \leq i < j \leq r+2$,

$$\partial_{r+1}^j \circ \partial_r^i = \partial_{r+1}^i \circ \partial_r^{j-1},$$

for every $0 \leq i \leq j \leq r$,

$$\sigma_{r+1}^j \circ \sigma_{r+2}^i = \sigma_{r+1}^i \circ \sigma_{r+2}^{j+1},$$

and for every $0 \leq i \leq r+1$ and $0 \leq j \leq r$,

$$\sigma_{r+1}^j \circ \partial_r^i = \begin{cases} \partial_{r-1}^i \circ \sigma_r^{j-1}, & i < j, \\ \mathrm{Id}_{C^r}, & i = j, i = j+1, \\ \partial_{r-1}^{i-1} \circ \sigma_r^j, & i > j+1 \end{cases}$$

Moreover, for cosimplicial objects $C^\bullet = (C^r, \partial_r^i, \sigma_{r+1}^i)$ and $\widetilde{C}^\bullet = (\widetilde{C}^r, \widetilde{\partial}_r^i, \widetilde{\sigma}_{r+1}^i)$, read about the equivalent specification of a natural transformation $\alpha^\bullet : C^\bullet \to \widetilde{C}^\bullet$ as the specification for every integer $r \geq 0$ of a $\mathcal{C}$-morphism $\alpha^r : C^r \to \widetilde{C}^r$ such that for every r and i,

$$\widetilde{\partial}_r^i \circ \alpha^r = \alpha^{r+1} \circ \partial_r^i, \quad \widetilde{\sigma}_{r+1}^i \circ \alpha^{r+1} = \alpha^r \circ \sigma_{r+1}^i.$$

Finally, for every pair of morphisms of cosimplicial objects, $\alpha^\bullet, \beta^\bullet : C^\bullet \to \widetilde{C}^\bullet$, a *cosimplicial homotopy* is a specification for every integer $r \geq 0$ and for every integer $i = 0, \dots, r$ of a $\mathcal{C}$-morphism,

$$g_{r+1}^i : C^{r+1} \to \widetilde{C}^r,$$

satisfying the following *cosimplicial homotopy identities*: for every $r \geq 0$,

$$g_{r+1}^0 \circ \partial_r^0 = \alpha^r, \quad g_{r+1}^r \circ \partial_r^{r+1} = \beta^r,$$

$$g^j_{r+1} \circ \partial^i_r = \begin{cases} \widetilde{\partial}^i_{r-1} \circ g^{j-1}_r, & 0 \le i < j \le r, \\ g^{i-1}_{r+1} \circ \partial^i_r, & 0 < i = j \le r, \\ \widetilde{\partial}^{i-1}_{r-1} \circ g^j_r, & 1 \le j+1 < i \le r+1. \end{cases}$$

$$g^j_r \circ \sigma^i_{r+1} = \begin{cases} \widetilde{\sigma}^i_r \circ g^{j+1}_{r+1}, & 0 \le i \le j \le r-1, \\ \widetilde{\sigma}^{i-1}_r \circ g^j_{r+1}, & 0 \le j < i \le r. \end{cases}$$

(a)(Constant Cosimplicial Objects) For every object C of $\mathcal{C}$, define $\mathrm{const}(C)$ to be the rule that associates to every integer $r \ge 0$ the object C of $\mathcal{C}$, and that associates to (r, i) the morphisms $\partial^i_r = \mathrm{Id}_C$, $\sigma^i_{r+1} = \mathrm{Id}_C$. **Prove** that $\mathrm{const}(C)$ is a cosimplicial object of $\mathcal{C}$. For every morphism of objects $\alpha : C \to \widetilde{C}$, **prove** that the specification for every integer $r \ge 0$ of the morphism $\alpha : C \to \widetilde{C}$ defines a morphism of cosimplicial objects,

$$\mathrm{const}(\alpha) : \mathrm{const}(C) \to \mathrm{const}(\widetilde{C}).$$

Prove that $\mathrm{const}(\mathrm{Id}_C)$ is the identity morphism of $\mathrm{const}(C)$. For a pair of morphisms, $\alpha : C \to \widetilde{C}$ and $\beta : \widetilde{C} \to \widehat{C}$, **prove** that $\mathrm{const}(\beta \circ \alpha)$ equals $\mathrm{const}(\beta) \circ \mathrm{const}(\alpha)$. Conclude that these rules define a functor

$$\mathrm{const} : \mathcal{C} \to \mathbf{Fun}(\Delta, \mathcal{C}).$$

Prove that this is functorial in $\mathcal{C}$, i.e., given a functor $F : \mathcal{C} \to \mathcal{D}$, for the associated functor,

$$\mathbf{Fun}(\Delta, F) : \mathbf{Fun}(\Delta, \mathcal{C}) \to \mathbf{Fun}(\Delta, \mathcal{D}),\ (C^r, \partial^i_r, \sigma^i_{r+1}) \mapsto (F(C^r), F(\partial^i_r), F(\sigma^i_{r+1})),$$

$\mathbf{Fun}(\Delta, F) \circ \mathrm{const}_{\mathcal{C}}$ strictly equals $\mathrm{const}_{\mathcal{D}} \circ F$.

(b)(Morphisms from a Constant Cosimplicial Object) For every integer $r \ge 1$ and for every pair of distinct morphisms $[0] \to [r]$, **prove** that there exists a unique Δ-morphism $F : [1] \to [r]$ such that the two morphisms are $F \circ \partial^0_0$ and $F \circ \partial^1_0$. Let $C^\bullet = (C^r, \partial^i_r, \sigma^i_{r+1})$ be a cosimplicial object in $\mathcal{C}$. For every object A of $\mathcal{C}$ and for every morphism, $\alpha^\bullet : \mathrm{const}(A) \to C^\bullet$, of cosimplicial objects, **prove** that $\alpha^0 : A \to C^0$ is a morphism such that $\partial^0_0 \circ \alpha^0$ equals $\partial^1_0 \circ \alpha^0$. **Prove** that the morphism $\alpha^\bullet$ is uniquely determined by α^0, i.e., for every $r \ge 0$, and for every morphism $f : [0] \to [r]$, $\alpha^r : A \to C^r$ equals $C(f) \circ \alpha^0$. Conversely, for every morphism $\alpha^0 : A \to C^0$ such that $\partial^0_0 \circ \alpha^0$ equals $\partial^1_0 \circ \alpha^0$, **prove** that the morphisms $\alpha^r := C(f) \circ \alpha^0$ are well-defined and define a morphism $\alpha^\bullet : \mathrm{const}(A) \to C^\bullet$ of cosimplicial objects. Conclude that the set map,

$$\mathrm{Hom}_{\mathbf{Fun}(\Delta,\mathcal{C})}(\mathrm{const}(A), C^\bullet) \to \{\alpha^0 \in \mathrm{Hom}_{\mathcal{C}}(A, C^0) | \partial^0_0 \circ \alpha^0 = \partial^1_0 \circ \alpha^0\},\ \alpha^\bullet \mapsto \alpha^0,$$

is a bijection. **Prove** that this bijection is natural in both A and in $C^\bullet$. In particular, conclude that the functor,

$$\mathrm{const} : \mathcal{C} \to \mathbf{Fun}(\Delta, \mathcal{C}),$$

is fully faithful. Finally, for every pair of morphisms, $\alpha^0, \beta^0 : A \to C^0$ equalizing ∂^0_0 and ∂^1_0, **prove** that there exists a cosimplicial homotopy $g^i_{r+1} : A \to C^r$ from $\alpha^\bullet$ to $\beta^\bullet$ if and only if β^0 equals α^0, and in this case there is a unique cosimplicial homotopy given by $g^i_{r+1} = \alpha^r = \beta^r$.

(c)(Equalizers in Cartesian Categories) Let $\Delta_{\le 1}$ be the category of totally ordered sets of cardinality ≤ 1. Prove that a functor $C^\bullet : \Delta_{\le 1} \to \mathcal{C}$ is equivalent to the data of a pair of objects C^0, C^1, a pair of morphisms $\partial_0^0, \partial_0^1 : C^0 \to C^1$, and a morphism $\sigma_1^0 : C^1 \to C^0$ such that $\sigma_1^0 \circ \partial_0^0 = \sigma_1^0 \circ \partial_0^1 = \text{Id}_{C^0}$. Let,

$$Z^0 : \mathbf{Fun}(\Delta_{\le 1}, \mathcal{C}) \to \mathcal{C},$$

be a functor and let,

$$\eta : \text{const} \circ Z^0 \Rightarrow \text{Id}_{\mathbf{Fun}(\Delta_{\le 1}, \mathcal{C})},$$

be a natural transformation such that $(\text{const}, Z^0, \eta)$ extends to an adjoint pair of functors $(\text{const}, Z^0, \theta, \eta)$. **Prove** that the natural transformation θ is a natural isomorphism. **Prove** that for every object $C^\bullet$ of $\mathbf{Fun}(\Delta_{\le 1}, \mathcal{C})$, the morphism $\eta_{C^\bullet} : Z^0(C^\bullet) \to C^0$ satisfies $\partial_0^0 \circ \eta_{C^\bullet} = \partial_0^1 \circ \eta_{C^\bullet}$ and is final among all such morphisms. **Prove** that if $\alpha^\bullet, \beta^\bullet : C^\bullet \to \widetilde{C}^\bullet$ are two morphisms of cosimplicial objects, and if $(g_{r+1}^i : C^{r+1} \to \widetilde{C}^r)$ is a cosimplicial homotopy from $\alpha^\bullet$ to $\beta^\bullet$, then $Z^0(\alpha^\bullet)$ equals $Z^0(\beta^\bullet)$.

Assume that $\mathcal{C}$ has finite products. For every pair of objects N^0 and N^1 of $\mathcal{C}$ and for every pair of morphisms $d_0^0, d_0^1 : N^0 \to N^1$, define $C^0 = N^0$, define $C^1 = N^0 \times N^1$, define $\partial_0^0 = (\text{Id}_{C^0}, d_0^0)$, define $\partial_0^1 = (\text{Id}_{C^0}, d_0^1)$, and define $\sigma_1^0 = \text{pr}_{N^0}$. **Prove** that $C^\bullet$ is an object of $\mathbf{Fun}(\Delta_{\le 1}, \mathcal{C})$, and **prove** that $\eta_{C^\bullet} : Z^0(C^\bullet) \to C^0$ is an equalizer of $d_0^0, d_0^1 : N^0 \to N^1$. In particular, if $\mathcal{C}$ has both finite products and Z^0, **prove** that $\mathcal{C}$ has all equalizers of a pair of morphisms. For every pair of morphisms $f_0^0 : M_0^0 \to N^1$ and $f_0^1 : M_1^0 \to N^1$ in $\mathcal{C}$, for $N^0 = M_0^0 \times M_1^0$, and for $d_0^0 = f_0^0 \circ \text{pr}_{M_0^0}$ and $d_0^1 = f_0^1 \circ \text{pr}_{M_1^0}$, **prove** that the equalizer of $d_0^0, d_0^1 : N^0 \to N^1$ is a fiber product of f_0^0 and f_0^1. Conclude that $\mathcal{C}$ has all finite fiber products, i.e., $\mathcal{C}$ is a *Cartesian category*. Conversely, assuming that $\mathcal{C}$ is a Cartesian category, then, up to some form of the Axiom of Choice, prove that there exists a functor Z^0 and a natural transformation η such that $(\text{const}, Z^0, \eta)$ extends to an adjoint pair of functors.

(d)(The Right Adjoint to the Constant Cosimplicial Object) Assume now that there exists a functor

$$Z^0 : \mathbf{Fun}(\Delta_{\le 1}, \mathcal{C}) \to \mathcal{C},$$

and a natural transformation,

$$\eta : \text{const} \circ Z^0 \Rightarrow \text{Id}_{\mathbf{Fun}(\Delta_{\le 1}, \mathcal{C})},$$

such that $(\text{const}, Z^0, \eta)$ extends to an adjoint pair of functors. For every cosimplicial object $C^\bullet : \Delta \to \mathcal{C}$, for the equalizer $\eta : Z^0(C^\bullet) \to C^0$ of ∂_0^0 and ∂_0^1, use (b) above to prove that there exists a unique extension $\eta^\bullet : \text{const}(Z^0) \to C^\bullet$ of η to a morphism of cosimplicial objects of $\mathcal{C}$. **Prove** that this defines a functor,

$$Z^0 : \mathbf{Fun}(\Delta, \mathcal{C}) \to \mathcal{C},$$

and a natural transformation,

$$\eta^\bullet : \text{const} \circ Z^0 \Rightarrow \text{Id}_{\mathbf{Fun}(\Delta, \mathcal{C})},$$

such that $(\text{const}, Z^0, \eta^\bullet)$ extends uniquely to an adjoint pair of functors, $(\text{const}, Z^0, \eta^\bullet, \theta)$. **Prove** that θ is a natural isomorphism. **Prove** that if $\alpha^\bullet, \beta^\bullet : C^\bullet \to \widetilde{C}^\bullet$ are two morphisms of cosimplicial objects, and if $(g_{r+1}^i : C^{r+1} \to \widetilde{C}^r)$ is a cosimplicial homotopy from $\alpha^\bullet$ to $\beta^\bullet$, then $Z^0(\alpha^\bullet)$ equals $Z^0(\beta^\bullet)$.

18 Topology Adjoint Pairs

Categories of Topologies on a Fixed Set Exercise. Recall from Problem 1(iv) on Problem Set 3, for every partially ordered set there is an associated category. For a set P, form the partially ordered set $\mathcal{P}(P)$ of subsets S of P. Then for objects S, S' of the category $\mathcal{P}(P)$, i.e., for subsets of P, the Hom set $\mathrm{Hom}_{\mathcal{P}(P)}(S, S')$ is nonempty if and only if $S' \subset S$, in which case the Hom set is a singleton set. In particular, this category has arbitrary (inverse) limits, namely unions, and it has arbitrary colimits (direct limits), namely intersections. Moreover, it has a final object, $\varnothing$, and it has an initial object, P.

Now let X be a set, and let P be $\mathcal{P}(X)$, so that P is a lattice. Denote by Power_X the category from the previous paragraph. Thus, objects are subsets $S \subset \mathcal{P}(X)$, and there exists a morphism from S to S' if and only if $S' \subset S$, and then the morphism is unique. We say that S *refines* S'. There is a covariant functor

$$\cup : \mathcal{P}(P) \to P, \cup S = \{x \in X | \exists p \in S, x \in p\},$$

and a contravariant functor

$$\cap : \mathcal{P}(P)^{\mathrm{opp}} \to P, \cap S = \{x \in X | \forall p \in S, x \in p\}.$$

By convention, $\cup\varnothing = \varnothing$ and $\cap\varnothing = X$.

A *topology* on X is a subset $\tau \subset \mathcal{P}(X)$ such that (i) $\varnothing \in \tau$ and $X \in \tau$, (ii) for every finite subset $S \subset \tau$, also $\cap S$ is in τ, and (iii) for every $S \subset \tau$ (possibly infinite), the set $\cup S$ is in τ. Denote by Top_X the full subcategory of Power_X whose objects are topologies on X. A *topological basis* on X is a subset $B \subset \mathcal{P}(X)$ such that for every finite subset S of B, the set $V = \cap S$ equals $\cup B_V$, where $B_V = \{U \in B : U \subset V\}$. Denote by Basis_X the full subcategory of Power_X whose objects are topological bases on X.

(a) Prove that Top_X is stable under colimits, i.e., for every collection of topologies, there is a topology that is refined by every topology in the collection and that refines every topology that is refined by every topology in the collection. **Prove** that Top_X is a full subcategory of Basis_X. For every topological basis B on X, define $\mathcal{T}(B)$ to consist of all elements $\cup S$ for $S \subset B$. **Prove** that $\mathcal{T}(B)$ is a topology on X. **Prove** that this uniquely extends to a functor

$$\mathcal{T} : \mathrm{Basis}_X \to \mathrm{Top}_X,$$

and **prove** that $\mathcal{T}$ is a right adjoint of the full embedding. Moreover, for every subset $S \subset \mathcal{P}(X)$, define $\mathcal{B}(S)$ to consist of all elements $\cap R$ for $R \subset S$ a *finite* subset. In particular, $\cap\varnothing = X$ is an element of $\mathcal{B}(S)$. **Prove** that $\mathcal{B}(S)$ is topological basis on X. **Prove** that this uniquely extends to a functor

$$\mathcal{B} : \mathrm{Power}_X \to \mathrm{Basis}_X,$$

and **prove** that $\mathcal{T} \circ \mathcal{B}$ is a right adjoint to the full embedding of Basis_X in Power_X.

(b) Prove that for every adjoint pair of functors, the left adjoint functor preserves colimits (direct limits), and the right adjoint functor preserves limits (inverse limits). Conclude that Top_X is stable under limits, i.e., for every collection of topologies, there is a topology that refines every topology in the collection and that is refined by every topology that refines every topology in the collection.

(c) Let $f : Y \to X$ be a set map. Denote by

$$\mathcal{P}^f : \mathcal{P}(X) \to \mathcal{P}(Y)$$

the functor that associates to every subset S of X the preimage subset $f^{-1}(S)$ of Y, and denote by

$$\mathcal{P}_f : \mathcal{P}(Y) \to \mathcal{P}(X)$$

the functor that associates to every subset T of Y the image subset $f(T)$ of X. **Prove** that $(\mathcal{P}^f, \mathcal{P}_f)$ extends uniquely to an adjoint pair of functors. In particular, define

$$\text{Power}_f : \text{Power}_X \to \text{Power}_Y$$

to be $\mathcal{P}_{\mathcal{P}^f}$, i.e., for every subset $S \subset \mathcal{P}(X)$, $\text{Power}_f(S) \subset \mathcal{P}(Y)$ is the set of all subsets $f^{-1}(U) \subset Y$ for subsets $U \subset X$ that are in S. Similarly, define

$$\text{Power}^f : \text{Power}_Y \to \text{Power}_X,$$

to be $\mathcal{P}^{\mathcal{P}^f}$, i.e., for every subset $T \subset \mathcal{P}(Y)$, $\text{Power}^f(Y) \subset \mathcal{P}(X)$ is the set of all subsets $U \subset X$ such that the subset $f^{-1}(U) \subset Y$ is in T. **Prove** that $(\text{Power}^f, \text{Power}_f)$ extends uniquely to an adjoint pair of functors. **Prove** that Power_f and Power^f restrict to functors $\text{Top}_X \to \text{Top}_Y$. For a given topology σ on Y and τ on X, f is *continuous* with respect to σ and τ if σ refines $\text{Power}_f(\tau)$, i.e., for every τ-open subset U of X, also $f^{-1}(U)$ is σ-open in Y. For a given topology τ on X, for every topology σ on Y, σ refines $\text{Power}_f(\tau)$ if and only if f is continuous with respect to σ and τ. Similarly, for a given topology σ on Y, for every topology τ on X, $\text{Power}^f(\sigma)$ refines τ if and only if f is continuous with respect to σ and τ.

Adjoint Pair for the Category of Topological Spaces Exercise. A topological space is a pair (X, τ) of a set X and a topology τ on X. For topological spaces (X, τ) and (Y, σ), a *continuous map* is a function $f : X \to Y$ such that for every subset V of Y that is in σ, the inverse image subset $f^{-1}(V)$ of X is in τ, i.e., σ refines $\text{Power}_f(\tau)$ and τ is refined by $\text{Power}^f(\sigma)$.

(a) Prove that for every topological space (X, τ), the identity function $\text{Id}_X : X \to X$ is a continuous map from (X, τ) to (X, τ). For every pair of continuous maps $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \rho)$, **prove** that the composition $g \circ f : (X, \tau) \to (Z, \rho)$ is a continuous map. With this notion of composition of continuous map, check that the topological spaces and continuous maps form a category, **Top**.

(b) For every topological space (X, τ), define $\Phi(X)$ to be the set X. For every continuous map of topological spaces, $f : (X, \tau) \to (Y, \sigma)$, define $\Phi(f) : \Phi(X) \to \Phi(Y)$ to be $f : X \to Y$. **Prove** that this defines a covariant functor,

$$\Phi : \textbf{Top} \to \textbf{Sets}.$$

(c) For every set X, define $L(X) = (X, \mathcal{P}(X))$, i.e., every subset of X is open. **Prove** that $\mathcal{P}(X)$ satisfies the axioms for a topology on X. This is called the *discrete topology* on X. For every set map, $f : X \to Y$, **prove** that $f : (X, \mathcal{P}(X)) \to (Y, \mathcal{P}(Y))$ is a continuous map, denoted $L(f)$. **Prove** that this defines a functor,

$$L : \mathbf{Sets} \to \mathbf{Top}.$$

For every set X, define $\theta_X : X \to \Phi(L(X))$ to be the identity map on X. **Prove** that θ is a natural equivalence $\mathrm{Id}_{\mathbf{Sets}} \Rightarrow \Phi \circ L$. For every topological space (X, τ), **prove** that Id_X is a continuous map $(X, \mathcal{P}(X)) \to (X, \tau)$, denoted $\eta_{(X,\tau)}$. **Prove** that η is a natural transformation $L \circ \Phi \Rightarrow \mathrm{Id}_{\mathbf{Top}}$. **Prove** that (L, Φ, θ, η) is an adjoint pair of functors. In particular, Φ preserves monomorphisms and limits (inverse limits).

(d) For every set X, define $R(X) = (X, \{\varnothing, X\})$. **Prove** that $\{\varnothing, X\}$ satisfies the axioms for a topology on X. This is called the *indiscrete topology* on X. For every set map $f : X \to Y$, **prove** that $f : R(X) \to R(Y)$ is a continuous map, denoted $R(f)$. **Prove** that this defines a functor,

$$R : \mathbf{Sets} \to \mathbf{Top}.$$

For every set topological space (X, τ), **prove** that Id_X is a continous map $(X, \tau) \to R(\Phi(X, \tau))$, denoted $\alpha_{(X,\tau)}$. **Prove** that α is a natural transformation $\mathrm{Id}_{\mathbf{Top}} \Rightarrow R \circ \Phi$. For every set S, denote by $\beta_X : \Phi(R(X)) \to X$ the identity morphism. **Prove** that β is a natural equivalence $\Phi \circ R \Rightarrow \mathrm{Id}_{\mathbf{Sets}}$. **Prove** that (Φ, R, α, β) is an adjoint pair of functors. In particular, Φ preserves epimorphisms and colimits (direct limits).

(e) Use the method of Problem 0 to prove that **Top** has (small) limits and colimits. Finally, **prove** that the projective objects in **Top** are precisely the discrete topological spaces, and the injective objects in **Top** are precisely the nonempty indiscrete topological spaces.

Adjoint Pair of Direct Image and Inverse Image Presheaves. Let (X, τ_X) be a topological space. As above, consider τ_X as a category whose objects are open sets U of the topology, and where for open sets U and V, there is a unique morphism from U to V if $U \supseteq V$, and otherwise there is no morphism. Let $\mathcal{C}$ be a category. A **presheaf** on (X, τ_X) of objects of $\mathcal{C}$ is a functor,

$$A : \tau_X \to \mathcal{C},$$

i.e., a τ_X-family as in Problem 0. By Problem 0, the τ-families form a category $\mathbf{Fun}(\tau_X, \mathcal{C})$, called the category of presheaves of objects of $\mathcal{C}$. For every continuous map $f : (Y, \tau_Y) \to (X, \tau_X)$, define

$$f^{-1} : \tau_X \to \tau_Y,$$

as in Problem 1(c), i.e., $U \mapsto f^{-1}(U)$. The corresponding functor

$$*_{f^{-1}} : \mathbf{Fun}(\tau_Y, \mathcal{C}) \to \mathbf{Fun}(\tau_X, \mathcal{C})$$

is called the *direct image functor* and is denoted f_*, i.e., for every presheaf $\mathcal{F}$ on (Y, τ_Y), $f_*\mathcal{F}$ is a presheaf on (X, τ_X) given by $(f_*\mathcal{F})(U) = \mathcal{F}(f^{-1}(U))$.

(a) Denote by σ_f the category whose objects are pairs (U,V) of an object U of τ_X and an object V of τ_V such that V is contained in $f^{-1}(U)$. For objects (U,V) and (U',V'), there is a morphism from (U,V) to (U',V') if and only if there is a morphism $U \supseteq U'$ in τ_X and a morphism $V \supseteq V'$ in τ_Y, and in this case the morphism for (U,V) to (U',V') is unique. **Prove** that this is a category. **Prove** that the map on objects,

$$x : \sigma_f \to \tau_X,\ (U,V) \mapsto U,$$

extends uniquely to a functor that is essentially surjective (in fact strictly surjective on objects). **Prove** that the following maps on objects,

$$\ell x : \tau_X \to \sigma_f,\ U \mapsto (U, f^{-1}(U)),$$

$$rx : \tau_X \to \sigma_f,\ U \mapsto (U, \varnothing)$$

extend uniquely to functors, and **prove** that $(\ell x, x)$ and (x, rx) extend uniquely to adjoint functors, i.e., $(U, f^{-1}(U))$, resp. $(U,\varnothing)$, is the initial object, resp. final object, in the fiber category $(\sigma_f)_{x,U}$. **Prove** that the map on objects

$$y : \sigma_f \to \tau_Y,\ (U,V) \mapsto V$$

extends uniquely to a functor that is essentially surjective (in fact strictly surjective on objects). **Prove** that the following map on objects,

$$\ell y : \tau_Y \to \sigma_f,\ V \mapsto (X,V),$$

extends uniquely to a functor, and **prove** that $(\ell y, y)$ extends uniquely to an adjoint functor, i.e., (X,V) is the initial object in the fiber category $(\sigma_f)_{y,V}$. Prove that $y \circ \ell_x$ is the functor $f^{-1} : \tau_X \to \tau_Y$ from above. **Find** an example where y does not admit a right adjoint.

Assume now that $\mathcal{C}$ has colimits. Apply Problem 0(g) to conclude that there are adjoint pairs of functors $(\star_x, \star_{\ell x})$, $(\star_{rx}, \star_x)$, $(\star_y, \star_{\ell y})$, and $(L_y, \star_y)$. Compose these adjoint pairs to obtain an adjoint pair $(L_y \circ \star_x, \star_{\ell x} \circ \star_y)$. Also, by functoriality of $\star_z$ in z, $\star_{\ell x} \circ \star_y$ equals $\star_{y \circ \ell x}$, and this equals $\star_{f^{-1}}$. Thus, this is an adjoint pair $(L_y \circ \star_x, f_\star)$. Unwind the defintions from Problem 0(g) to **check** that for every presheaf A on X and for every V an object of τ_Y, $L_y \circ \star_x(A)$ on V is the colimit over the fiber category $(\sigma_f)_{y,V}$ of all U an object of τ_X with $V \subseteq f^{-1}(U)$ of $A(U)$. The functor $L_y \circ \star_x$ is the *inverse image functor* for presheaves,

$$f^{-1} : \mathbf{Fun}(\tau_X, \mathcal{C}) \to \mathbf{Fun}(\tau_Y, \mathcal{C}).$$

Čech Cosimplicial Object of a Covering Exercise. Let (X,τ_X) be a topological space. For every object U of τ_X, **prove** that the topology τ_U on U associated to $i : U \to X$ via Problem 1(c) is a full, upper subcategory of τ_X that has an initial object $\odot = U$. For every U, an *open covering* of U is a set $\mathfrak{U}$ and a set map $\iota_{\mathfrak{U}} : \mathfrak{U} \to \tau_U$ such that $\cup\text{Image}(\iota_{\mathfrak{U}})$ equals U. Define σ to be the category whose objects are pairs $(U,\mathfrak{U})$ of an open U in τ_X and an open covering $\iota_{\mathfrak{U}} : \mathfrak{U} \to \tau_U$. For objects $(U,\mathfrak{U})$ and $(V,\mathfrak{V})$, a σ-morphism from $(U,\mathfrak{U})$ to $(V,\mathfrak{V})$ is a pair $U \supseteq V$ of a morphism in τ_X and a *refinement* $\phi : \mathfrak{U} \geq \mathfrak{V}$, i.e., a set function $\phi : \mathfrak{V} \to \mathfrak{U}$ such that for every V_0 in $\mathfrak{V}$, $\iota_{\mathfrak{U}}(\phi(V_0))$

contains $\iota_{\mathfrak{V}}(V_0)$. In particular, for every object $(U, \iota_{\mathfrak{U}} : \mathfrak{U} \to \tau_U)$ of σ, define $\mathfrak{V} = \text{Image}(\iota_{\mathfrak{U}})$ with its natural inclusion $\iota_{\mathfrak{V}} : \mathfrak{V} \hookrightarrow \tau_U$. Up to the Axiom of Choice, **prove** that there exists a refinement $\phi : (U, \mathfrak{U}) \geq (U, \mathfrak{V})$. Thus, the open coverings with ι a monomorphism are cofinal in the category σ.

(a)(Category of Open Coverings) For every pair of refinements, $\phi : (U, \mathfrak{U}) \geq (V, \mathfrak{V})$ and $\psi : (V, \mathfrak{V}) \geq (W, \mathfrak{W})$, **prove** that the composition $\phi \circ \psi : \mathfrak{W} \to \mathfrak{U}$ is a refinement, $\phi \circ \psi : (U, \mathfrak{U}) \to (W, \mathfrak{W})$. Also **prove** that $\text{Id}_{\mathfrak{U}} : \mathfrak{U} \to \mathfrak{U}$ is a refinement $(U, \mathfrak{U}) \to (U, \mathfrak{U})$. Conclude that these rules define a category σ whose objects are open coverings $(U, \mathfrak{U})$ of opens U in τ_X and whose morphisms are refinements. Define $x : \sigma \to \tau_X$ to be the rule that associates to every $(U, \mathfrak{U})$ the open U and that associates to every refinement $\phi : (U, \mathfrak{U}) \geq (V, \mathfrak{V})$ the inclusion $U \supseteq V$. **Prove** that this is a strictly surjective functor. **Prove** that the map on objects,

$$\ell x : \tau_X \to \sigma, \ U \mapsto (U, \{U\}),$$

extends uniquely to a functor, and **prove** that $(\ell x, x)$ extends uniquely to an adjoint pair of functors, i.e., $(U, \{U\})$ is the initial object in the fiber category $\sigma_{x,U}$. Typically x does not admit a right adjoint.

For every open covering $\iota_{\mathfrak{U}} : \mathfrak{U} \to \tau_U$, for every integer $r \geq 0$, define the following set map,

$$\iota_{\mathfrak{U}^{r+1}} : \mathfrak{U}^{r+1} \to \tau_U, \ (U_0, U_1, \dots, U_r) \mapsto \iota_{\mathfrak{U}}(U_0) \cap \iota_{\mathfrak{U}}(U_1) \cap \cdots \cap \iota_{\mathfrak{U}}(U_r).$$

Let $\mathcal{C}$ be a category, and let A be an $\mathcal{C}$-presheaf on (X, τ_X). Let $(U, \mathfrak{U})$ be an object of σ. Recall that for every object T of $\mathcal{C}$, there is a Yoneda functor,

$$h_T : \mathcal{C}^{\text{opp}} \to \mathbf{Sets}, \ S \mapsto \mathbf{Hom}_{\mathcal{C}}(S, T),$$

and this is covariant in T. For every integer $r \geq 0$, define

$$h_{A,\mathfrak{U},r} : \mathcal{C}^{\text{opp}} \to \mathbf{Sets}, \ S \mapsto \prod_{(U_0,\dots,U_r) \in \mathfrak{U}^{r+1}} h_{A(\iota(U_0,\dots,U_r))}(S),$$

together with the projections,

$$\pi_{(U_0,\dots,U_r)} : h_{A,\mathfrak{U},r} \to h_{A(\iota(U_0,\dots,U_r))}.$$

For every integer $r \geq 0$, and for every integer $i = 0, \dots, r+1$, define

$$\partial_r^i : h_{A,\mathfrak{U},r} \to h_{A,\mathfrak{U},r+1},$$

to be the unique natural transformation such that for every $(U_0, \dots, U_{r+1}) \in \mathfrak{U}^{r+2}$, $\pi_{(U_0,\dots,U_{r+1})} \circ \partial_r^i$ equals the composition of the projection,

$$\pi_{(U_0,\dots,U_{i-1},U_{i+1},\dots,U_{r+1})} : h_{A,\mathfrak{U},r} \to h_{A(\iota(U_0,\dots,U_{i-1},U_{i+1}\cap,\cap U_{r+1}))},$$

with the natural transformation of Yoneda functors arising from the restriction morphism

$$A(\iota(U_0) \cap \cdots \cap \iota(U_{i-1}) \cap \iota(U_{i+1}) \cap \cdots \cap \iota(U_{r+1})) \to A(\iota(U_0) \cap \cdots \cap \iota(U_{r+1})).$$

Similarly, for every $i = 0, \dots, r$, define

$$\sigma^i_{r+1} : h_{A,\mathfrak{U},r+1} \to h_{A,\mathfrak{U},r}$$

to be the unique natural transformation such that for every $(U_0, \dots, U_r) \in \mathfrak{U}^{r+1}$, $\pi_{(U_0,\dots,U_{r+1})} \circ \sigma^i_{r+1}$ equals the projection $\pi_{(U_0,\dots,U_{i-1},U_i,U_i,U_{i+1},\dots,U_r)}$.

(b)(Cosimplicial Identities) **Prove** that these natural transformations satisfy the *cosimplicial identities*: for every $r \geq 0$, for every $0 \leq i < j \leq r+2$,

$$\partial^j_{r+1} \circ \partial^i_r = \partial^i_{r+1} \circ \partial^{j-1}_r,$$

for every $0 \leq i \leq j \leq r$,

$$\sigma^j_{r+1} \circ \sigma^i_{r+2} = \sigma^i_{r+1} \circ \sigma^{j+1}_{r+2},$$

and for every $0 \leq i \leq r+1$ and $0 \leq j \leq r$,

$$\sigma^j_{r+1} \circ \partial^i_r = \begin{cases} \partial^i_{r-1} \circ \sigma^{j-1}_r, & i < j, \\ \mathrm{Id}, & i = j, i = j+1, \\ \partial^{i-1}_{r-1} \circ \sigma^j_r, & i > j+1 \end{cases}$$

In the case that $\mathcal{C}$ is an additive category, define

$$d^r : h_{A,\mathfrak{U},r} \to h_{A,\mathfrak{U},r+1}, \quad d^r = \sum_{i=0}^{r+1} \partial^i_r.$$

Prove that $d^{r+1} \circ d^r$ equals 0.

(c)(Refinements and Cosimplicial Homotopies) For every refinement, $\phi : (U, \mathfrak{U}) \geq (V, \mathfrak{V})$, for every integer $r \geq 0$, define

$$h_{A,\phi,r} : h_{A,\mathfrak{U},r} \to h_{A,\mathfrak{V},r}$$

to be the unique natural transformation such that for every $(V_0, \dots, V_r) \in \mathfrak{V}^{r+1}$, the composition $\pi_{(V_0,\dots,V_r)} \circ h_{A,q,r}$ equals the composition of projection

$$\pi_{(\phi(V_0),\dots,\phi(V_r))} : h_{A,\mathfrak{U},r} \to h_{A(\phi(V_0)\cap\cdots\cap\phi(V_r))}$$

with the natural transformation of Yoneda functors arising from the restriction morphism

$$A(\iota\phi(V_0) \cap \cdots \cap \iota\phi(V_r)) \to A(\iota(V_0) \cap \cdots \cap \iota(V_r)).$$

Prove that the natural transformations $(h_{A,\phi,r})_{r\geq 0}$ are compatible with the natural transformations ∂^i_r and σ^i_{r+1}. For every pair of refinements, $\phi : (U, \mathfrak{U}) \geq (V, \mathfrak{V})$ and $\psi : (V, \mathfrak{V}) \geq (W, \mathfrak{W})$, for the composition refinement $\phi \circ \psi : (U, \mathfrak{U}) \geq (W, \mathfrak{W})$, **prove** that $h_{A,\phi\circ\psi,r}$ equals $h_{A,\psi,r} \circ h_{A,\phi,r}$, and also **prove** that $h_{A,\mathrm{Id}_{\mathfrak{U}},r}$ equals $\mathrm{Id}_{h_{A,\mathfrak{U},r}}$. Thus $h_{A,\phi,r}$ is functorial in ϕ.

Let $\phi : (U, \mathfrak{U}) \geq (V, \mathfrak{V})$ and $\psi : (U, \mathfrak{U}) \geq (V, \mathfrak{V})$ be refinements. For every integer $r \geq 0$, for every integer $i = 0, \dots, r$, define

$$g^i_{A,\phi,\psi,r+1} : h_{A,\mathfrak{U},r+1} \to h_{A,\mathfrak{V},r}$$

to be the unique natural transformation such that for every $(V_0,\dots,V_r) \in \mathfrak{V}^{r+1}$, $\pi_{(V_0,\dots,V_r)} \circ g^i_{A,\phi,\psi,r+1}$ equals the composition of the projection,

$$\pi_{\psi(V_0),\dots,\psi(V_i),\phi(V_i),\dots,\phi(V_r)} : h_{A,\mathfrak{U},r+1} \to h_{A(\iota(\psi(V_0),\dots,\psi(V_i),\phi(V_i),\dots,\phi(V_r)))},$$

with the natural transformation of Yoneda functors arising from the restriction morphism

$$A(\iota\psi(V_0) \cap \dots \cap \iota\psi(V_i) \cap \iota\phi(V_i) \cap \dots \cap \iota\phi(V_r)) \to A(\iota(V_0) \cap \dots \cap \iota(V_i) \cap \dots \cap \iota(V_r)).$$

Prove the following identities (cosimplicial homotopy identities),

$$g^0_{A,\phi,\psi,r+1} \circ \partial^0_{A,\mathfrak{U},r} = h_{A,\phi,r}, \quad g^r_{A,\phi,\psi,r+1} \circ \partial^{r+1}_{A,\mathfrak{U},r} = h_{A,\psi,r},$$

$$g^j_{A,\phi,\psi,r+1} \circ \partial^i_{A,\mathfrak{U},r} = \begin{cases} \partial^i_{A,\mathfrak{V},r-1} \circ g^{j-1}_{A,\phi,\psi,r}, & 0 \le i < j \le r, \\ g^{i-1}_{A,\phi,\psi,r+1} \circ \partial^i_{A,\mathfrak{U},r}, & 0 < i = j \le r, \\ \partial^{i-1}_{A,\mathfrak{V},r-1} \circ g^j_{A,\phi,\psi,r}, & 1 \le j+1 < i \le r+1. \end{cases}$$

$$g^j_{A,\phi,\psi,r} \circ \sigma^i_{A,\mathfrak{U},r+1} = \begin{cases} \sigma^i_{A,\mathfrak{V},r} \circ g^{j+1}_{A,\phi,\psi,r+1}, & 0 \le i \le j \le r-1, \\ \sigma^{i-1}_{A,\mathfrak{V},r} \circ g^j_{A,\phi,\psi,r+1}, & 0 \le j < i \le r. \end{cases}$$

For the identity refinement $\mathrm{Id}_{\mathfrak{U}} : \mathfrak{U} \ge \mathfrak{U}$, **prove** that $g^j_{A,\mathrm{Id},\mathrm{Id},r+1}$ equals $\sigma^j_{A,\mathfrak{U},r+1}$. Also **prove** that for refinements $\chi : \mathfrak{V} \to \mathfrak{W}$ and $\xi : \mathfrak{T} \to \mathfrak{U}$, $g^j_{A,\phi\circ\chi,\psi\circ\chi,r+1}$ equals $h_{A,\chi,r} \circ g^j_{A,\phi,\psi,r+1}$ and $g^j_{A,\xi\circ\phi,\xi\circ\psi,r+1}$ equals $g^j_{A,\phi,\psi,r+1} \circ h_{A,\xi,r+1}$.

(d)(Functoriality in A) For every morphism of $\mathcal{C}$-presheaves, $\alpha : A \to A'$, define

$$h_{\alpha,\mathfrak{U},r} : h_{A,\mathfrak{U},r} \to h_{B,\mathfrak{U},r},$$

to be the unique natural transformation whose postcomposition with each projection $\pi_{B,(U_0,\dots,U_r)}$ equals the composition of $\pi_{A,(U_0,\dots,U_r)}$ with the natural transformation induced by the morphism

$$\alpha_{\iota(U_0,\dots,U_r)} : A(\iota(U_0,\dots,U_r)) \to A'(\iota(U_0,\dots,U_r)).$$

Prove that these maps are compatible with the cosimplicial operations ∂^i_r and σ^i_{r+1}, as well as the operations $h_{A,\phi,r}$ associated to a refinement $\phi : \mathfrak{U} \ge \mathfrak{V}$, and the cosimplicial homotopies $g^i_{A,\phi,\psi,r+1}$ associated to a pair of refinements, $\phi, \psi : \mathfrak{U} \ge \mathfrak{V}$. **Prove** that this is functorial in α. Conclude that (up to serious set-theoretic issues), for every open cover $\mathfrak{U}$, morally these rules define a functor from the category of $\mathcal{C}$-presheaves to the "category" of cosimplicial objects in the category of contravariant functors from $\mathcal{C}$ to **Sets**. Stated differently, to every open cover $\mathfrak{U}$ there is an associated cosimplicial object in the category $\mathbf{Fun}(\mathcal{C} - \mathrm{Presh}, \mathbf{Fun}(\mathcal{C}, \mathbf{Sets}))$ of covariant functors from the category of $\mathcal{C}$-presheaves to the category of contravariant functors $\mathcal{C} \to$ **Sets**. This rule is covariant for refinement of open covers. Moreover, up to simplicial homotopy, it is independent of the choice of refinement.

(e)(Coadjunction of Sections) As a particular case, for the left adjoint ℓx of x, observe that there is a canonical refinement

$$\eta_{U,\mathfrak{U}} : \ell x \circ x(U,\mathfrak{U}) \geq (U,\mathfrak{U}), \text{ i.e., } (U,\{U\}) \geq (U,\mathfrak{U}).$$

Prove that $h_{A,\{U\},r}$ is the constant / diagonal cosimplicial object that for every r associates $h_{A(U)}$ and with ∂^i and σ^i equal to the identity morphism. Conclude that for every cover $(U,\mathfrak{U})$ in σ, there is a natural coaugmentation,

$$g^r_{A,\mathfrak{U}} : h_{A(U)} \to h_{A,\mathfrak{U},r},$$

that is functorial in A, functorial in $(U,\mathfrak{U})$ with respect to refinements, and that equalizes the simplicial homotopies associated to a pair of refinements in the sense that

$$g^j_{A,\phi,\psi,r+1} \circ g^{r+1}_{A,\mathfrak{U}} = g^r_{A,\mathfrak{V}} \circ h_{A^U_V}.$$

Define the functor

$$\text{const} : \mathbf{Fun}(\sigma,\mathcal{C}) \to \mathbf{Fun}(\Delta \times \sigma,\mathcal{C})$$

that associates to a functor $B : \sigma \to \mathcal{C}$ the functor $\text{const}_B : \sigma \to \mathbf{Fun}(\Delta,\mathcal{C})$ whose value on every $(U,\mathfrak{U})$ is the constant / diagonal cosimplicial object $r \mapsto B(U,\mathfrak{U})$ for every r with every ∂^i and σ^i defined to be the identity morphism. Conclude that the rule $U \mapsto (r \mapsto h_{A(U)})$ above is the Yoneda functor associated to $\text{const} \circ \star_x(A)$.

(f)(Čech cosimplicial object) Assume now that $\mathcal{C}$ has all finite products. Thus, for every open covering $(U,\mathfrak{U})$ and for every integer $r \geq 0$, there exists an object

$$\check{C}^r(\mathfrak{U}, A) = \prod_{(U_0,\dots,U_r)\in\mathfrak{U}} A(U_0 \cap \dots \cap U_r),$$

such that $h_{A,\mathfrak{U},r}$ equals $h_{\check{C}^r(\mathfrak{U},A)}$. Use the Yoneda Lemma to **prove** that there are associated morphisms in $\mathcal{C}$,

$$\partial^i_{A,\mathfrak{U},r} : \check{C}^r(\mathfrak{U}, A) \to \check{C}^{r+1}(\mathfrak{U}, A),$$
$$\sigma^i_{A,\mathfrak{U},r+1} : \check{C}^{r+1}_{\mathfrak{U}}(A) \to \check{C}^r_{\mathfrak{U}}(A),$$
$$\check{C}^r(\phi, A) : \check{C}^r(\mathfrak{U}, A) \to \check{C}^r(\mathfrak{V}, A),$$
$$\check{C}^{r+1,i}(\phi,\psi, A) : \check{C}^{r+1}(\mathfrak{U}, A) \to \check{C}^r(\mathfrak{V}, A),$$
$$\check{C}^r(\mathfrak{U}, \alpha) : \check{C}^r(\mathfrak{U}, A) \to \check{C}^r(\mathfrak{U}, A'),$$

whose associated morphisms of Yoneda functors equal the morphisms defined above. Thus, in this case, $\check{C}^*(\mathfrak{U}, A)$ is a cosimplicial object in $\mathcal{C}$. **Prove** that this defines a covariant functor

$$\check{C}(\mathfrak{U}, -) : \mathbf{Fun}(\tau_X,\mathcal{C}) \to \mathbf{Fun}(\Delta,\mathcal{C}).$$

Incorporating the role of $\mathfrak{U}$, **prove** that this defines a functor

$$\check{C} : \mathbf{Fun}(\tau_X,\mathcal{C}) \to \mathbf{Fun}(\Delta \times \sigma,\mathcal{C}).$$

Prove that this is, typically, *not* equivalent to the composite functor,

$$\text{const} \circ *_x : \mathbf{Fun}(\tau_X, \mathcal{C}) \to \mathbf{Fun}(\sigma, \mathcal{C}) \to \mathbf{Fun}(\Delta \times \sigma, \mathcal{C}).$$

However, **prove** that the coadjunction in the last part does give rise to a natural transformation,

$$g : \text{const} \circ *_x \Rightarrow \check{C}.$$

(g) Assume now that there exists a functor,

$$Z^0 : \mathbf{Fun}(\Delta_{\le 1}, \mathcal{C}) \to \mathcal{C},$$

and a natural transformation,

$$\eta : \text{const} \circ Z^0 \Rightarrow \text{Id}_{\mathbf{Fun}(\Delta_{\le 1}, \mathcal{C})},$$

such that $(\text{const}, Z^0, \eta)$ extends to an adjoint pair of functors, i.e., assume that $\mathcal{C}$ is a Cartesian category. Use Problem 4(d) to conclude that there exists a functor,

$$Z^0 : \mathbf{Fun}(\Delta \times \sigma, \mathcal{C}) \to \mathbf{Fun}(\sigma, \mathcal{C}),$$

and a natural transformation,

$$\eta : \text{const} \circ Z^0 \Rightarrow \text{Id}_{\mathbf{Fun}(\Delta \times \sigma, \mathcal{C})},$$

such that $(\text{const}, Z^0, \eta)$ extends to an adjoint pair of functors, $(\text{const}, Z^0, \eta, \theta)$ such that θ is a natural isomorphism. Moreover, for every $A^\bullet : \Delta \times \sigma \to \mathcal{C}$, for every object $(U, \mathfrak{U})$ of σ, **prove** that $\eta : Z^0(A^\bullet(\mathfrak{U}) \to A^0(\mathfrak{U}))$ is an equalizer of $\partial_0^0, \partial_0^1 : A^0(\mathfrak{U}) \to A^1(\mathfrak{U})$. Finally, the composition of natural transformations, $(Z^0 \circ g) \circ (\theta \circ *_x)$, is a natural transformation

$$Z^0(g) : *_x \Rightarrow Z^0 \circ \text{const} \circ *_x \Rightarrow Z^0 \circ \check{C}.$$

In particular, conclude that for a refinement $\phi : (U, \mathfrak{U}) \ge (V, \mathfrak{V})$, the induced morphism $Z^0(\check{C}^\bullet(\mathfrak{U}, A)) \to Z^0(\check{C}^\bullet(\mathfrak{V}, A))$ is independent of the choice of refinement.

(h) Let $(U, \iota : \mathfrak{U} \to \tau_U)$ be an object of σ. Let $\phi : (U, \mathfrak{U}) \ge (U, \{U\})$ be a refinement, i.e., $* = \phi(U)$ is an element of $\mathfrak{U}$ such that $\iota(*)$ equals U. Thus, $(U, \mathfrak{U})$ admits both the identity refinement of $(U, \mathfrak{U})$ and also the composite of ϕ with the canonical refinment from (e), $\eta_{U,\mathfrak{U}} circ \phi$. Using (c), **prove** that the identity on $\check{C}^\bullet(\mathfrak{U}, -)$ is homotopy equivalent to $\check{C}(\eta_{U,\mathfrak{U}}, -) \circ \check{C}(\phi, -)$. On the other hand, the refinement $\phi \circ \eta_{U,\mathfrak{U}}$ of $(U, \{U\})$ is the identity refinement. Thus the composite $\check{C}(\phi, -) \circ \check{C}(\eta_{U,\mathfrak{U}}, -)$ equals the identity on $\check{C}^\bullet(\{U\}, -)$. **Prove** that $\check{C}^\bullet(\mathfrak{U}, A)$ is homotopy equivalent to the constant simplicial object $\text{const}_{A(U)}$, and these homotopy equivalences are natural in A and open coverings $(U, \mathfrak{U})$ that refine to $(U, \{U\})$.

Sheaves Exercise. Let (X, τ_X) be a topological space. Let $\mathcal{C}$ be a category. A $\mathcal{C}$-*sheaf* on (X, τ_X) is a $\mathcal{C}$-presheaf A such that for every open subset U in τ_X, for every open covering $\iota : \mathfrak{U} \to \tau_U$ of U, the associated sequence of Yoneda functors,

$$h_{A(U)} \xrightarrow{g^0_{A,\mathfrak{U}}} h_{A,\mathfrak{U},0} \rightrightarrows h_{A,\mathfrak{U},1},$$

is exact, where the two arrows are $\partial^0_{A,\mathfrak{U},0}$ and $\partial^1_{A,\mathfrak{U},0}$. Stated more concretely, for every object S of $\mathcal{C}$, for every collection $(s_{U_0} : S \to A(\iota(U_0)))_{U_0 \in \mathfrak{U}}$ of $\mathcal{C}$-morphisms such that for every $(U_0, U_1) \in \mathfrak{U}^2$, the following two compositions are equal,

$$S \xrightarrow{s_{U_0}} A(\iota(U_0)) \xrightarrow{A^{\iota(U_0)}_{\iota(U_0)\cap\iota(U_1)}} A(\iota(U_0) \cap \iota(U_1)), \quad S \xrightarrow{s_{U_1}} A(\iota(U_1)) \xrightarrow{A^{\iota(U_1)}_{\iota(U_0)\cap\iota(U_1)}} A(\iota(U_0) \cap \iota(U_1)),$$

there exists a unique morphism $s_U : S \to A(U)$ such that for every $U_0 \in \mathfrak{U}$, s_{U_0} equals $A^U_{\iota(U_0)} \circ s_U$.

(a)(Sheaf Axiom via Čech Objects) For simplicity, assume that $\mathcal{C}$ is a Cartesian category that has all small products. In particular, assume that the functors $\check{C}$ and Z^0 of the previous exercise are defined. **Prove** that a $\mathcal{C}$-presheaf on (X, τ_X) is a sheaf if and only if the morphism

$$Z^0(g) : \ast_x(A) \to Z^0(\check{C}(A))$$

of objects in $\mathbf{Fun}(\sigma, \mathcal{C})$ is an isomorphism.

(b)(Associated Sheaf / Sheafification Functor) Now assume that $\mathcal{C}$ has all small colimits. In particular, assume that there exists a functor

$$L_x : \mathbf{Fun}(\sigma, \mathcal{C}) \to \mathbf{Fun}(\tau_X, \mathcal{C}),$$

such that $(L_x, \ast_x)$ extends to an adjoint pair of functors. Using Exercise 0(g), **prove** that for every open U in τ_X and for every functor,

$$B : \sigma \to \mathcal{C},$$

$L_x(B)(U)$ is the colimit of the restriction of B to the fiber category $\sigma_{x,U}$. In particular, since open coverings $(U, \iota : \mathfrak{U} \to U)$ such that ι is a monomorphism are cofinal in the category $\sigma_{x,U}$, it suffices to compute the colimit over such open coverings. For every functor,

$$A : \tau_X \to \mathcal{C},$$

prove that $L_x \circ \ast_x(A) \to A$ is a natural isomorphism. Denote by $\mathrm{Sh} : \mathbf{Fun}(\tau_X, \mathcal{C}) \to \mathbf{Fun}(\tau_X, \mathcal{C})$ the composite functor,

$$L_x \circ Z^0 \circ \check{C} : \mathbf{Fun}(\tau_X, \mathcal{C}) \to \mathbf{Fun}(\tau_x, \mathcal{C}).$$

Prove that there exists a unique natural transformation,

$$\mathrm{sh} : \mathrm{Id}_{\mathbf{Fun}(\tau_x, \mathcal{C})} \Rightarrow \mathrm{Sh},$$

whose composition with the natural isomorphism above equals $L_x(Z^0(g))$. For every sheaf A, **prove** that

$$\mathrm{sh} : A \to \mathrm{Sh}(A)$$

is an isomorphism.

(c)(The Associated Sheaf is a Sheaf) Let $(U, \iota : \mathfrak{U} \to \tau_U)$ an object of σ, and let,

$$(\iota(U_0), \kappa_{U_0} : \mathfrak{V}_{U_0} \to \tau_{\iota(U_0)}),$$

be a collection of open coverings of each $\iota(U_0)$. For every pair $(U_0, U_1) \in \mathfrak{U}^2$, let

$$(\iota(U_0, U_1), \kappa_{U_0,U_1} : \mathfrak{V}_{U_0,U_1} \to \tau_{\iota(U_0,U_1)}),$$

be an open covering together with refinements

$$\phi_0^0 : (\iota(U_0), \mathfrak{V}_{U_0}) \geq (\iota(U_0, U_1), \mathfrak{V}_{U_0,U_1}), \quad \phi_0^1 : (\iota(U_1), \mathfrak{V}_{U_1}) \geq (\iota(U_0, U_1), \mathfrak{V}_{U_0,U_1}).$$

Define

$$\mathfrak{V} := \left(\sqcup_{U_0\in\mathfrak{U}}\mathfrak{V}_{U_0}\right) \sqcup \left(\sqcup_{(U_0,U_1)\in\mathfrak{U}^2}\mathfrak{V}_{U_0,U_1}\right),$$

define

$$\kappa : \mathfrak{V} \to \tau_U,$$

to be the unique set map whose restriction to every $\mathfrak{V}_{U_0}$ equals κ_{U_0} and whose restriction to every $\mathfrak{V}_{U_0,U_1}$ equals κ_{U_0,U_1}. For every $U_0 \in \mathfrak{U}$, define

$$\phi_{U_0} : (U, \kappa : \mathfrak{V} \to \tau_U) \geq (\iota(U_0), \kappa_{U_0} : \mathfrak{V}_{U_0} \to \tau_{\iota(U_0)}),$$

to be the obvious refinement. For every $U_0 \in \mathfrak{U}$, define $Z(U_0, A) = Z^0(\check{C}^\bullet(\mathfrak{V}_{U_0}, A))$. For every $(U_0, U_1) \in \mathfrak{U}^2$, define $Z^0(U_0, U_1, A) = Z^0(\check{C}^\bullet(\mathfrak{V}_{U_0,U_1}, A))$. Define

$$Z^0(\mathfrak{U}, A) := \prod_{U_0\in\mathfrak{U}} Z^0(U_0, A),$$

$$Z^1(\mathfrak{U}, A) := \prod_{(U_0,U_1)\in\mathfrak{U}^2} Z^0(U_0, U_1, A),$$

$$\partial_0^0 : Z^0(\mathfrak{U}, A) \to Z^1(\mathfrak{U}, A), \ \partial_0^i(z_{U_0}) = (A^{U_i}_{U_0\cap U_1}(z_{U_i}))_{U_0,U_1}.$$

Prove that the restriction morphism,

$$Z^0(\phi^\bullet) : Z^0(\mathfrak{V}, A) \to Z^0(Z^\bullet(\mathfrak{U}, A)),$$

is a $\mathfrak{C}$-isomorphism. Conclude that $\text{Sh}(A)$ is a sheaf. Denote by,

$$\Phi : \mathcal{C} - \text{Sh}_{(X,\tau_X)} \to \mathcal{C} - \text{Presh}_{(X,\tau_X)},$$

the full embedding of the category of sheaves in the category of presheaves. Thus, Sh is a functor,

$$\text{Sh} : \mathcal{C} - \text{Presh}_{(X,\tau_X)} \to \mathcal{C} - \text{Sh}_{(X,\tau_X)},$$

and sh is a natural transformation $\text{Id}_{\mathcal{C}-\text{Presh}_X} \Rightarrow \Phi \circ \text{Sh}$. Conclude that $(\text{Sh}, \Phi, \text{sh})$ extends to an adjoint pair of functors.

(d)(Pushforward and Inverse Image) For a continuous map $f : (X, \tau_X) \to (Y, \tau_Y)$, **prove** that the composite functor,

$$\mathcal{C} - \text{Sh}_{(X,\tau_X)} \xrightarrow{\Phi} \mathcal{C} - \text{Presh}_{(X,\tau_X)} \xrightarrow{f_*} \mathcal{C} - \text{Presh}_{(Y,\tau_Y)},$$

factors uniquely through $\Phi : \mathcal{C} - \text{Sh}_{(Y,\tau_Y)} \to \mathcal{C} - \text{Presh}_{(Y,\tau_Y)}$, i.e., there is a functor

$$f_* : \mathcal{C} - \text{Sh}_{(X,\tau_X)} \to \mathcal{C} - \text{Sh}_{(Y,\tau_Y)},$$

such that $f_* \circ \Phi$ equals $\Phi \circ f_*$. On the other hand, **prove** by example that the composite

$$\mathcal{C} - \text{Sh}_{(Y,\tau_Y)} \xrightarrow{\Phi} \mathcal{C} - \text{Presh}_{(Y,\tau_Y)} \xrightarrow{f^{-1}} \mathcal{C} - \text{Presh}_{(X,\tau_X)}$$

need not factor through Φ. Define

$$f^{-1} : \mathcal{C} - \text{Sh}_{(Y,\tau_Y)} \to \mathcal{C} - \text{Sh}_{(X,\tau_X)},$$

to be the composite of the previous functor with $\text{Sh} : \mathcal{C} - \text{Presh}_{(X,\tau_X)} \to \mathcal{C} - \text{Sh}_{(X,\tau_X)}$. **Prove** that the functors (f^{-1}, f_*) extend to an adjoint pair of functors between $\mathcal{C} - \text{Sh}_{(X,\tau_X)}$ and $\mathcal{C} - \text{Sh}_{(Y,\tau_Y)}$.

Espace Étalé Exercise. Let (X,τ_X) be a topological space. A *space over* X is a continuous map of topological spaces, $f : (Y,\tau_Y) \to (X,\tau_X)$. For spaces over X, $f : (Y,\tau_Y) \to (X,\tau_X)$ and $g : (Z,\tau_Z) \to (X,\tau_X)$, a *morphism of spaces over* X from f to g is a continuous map $u : (Y,\tau_Y) \to (Z,\tau_X)$ such that $g \circ u$ equals f.

(a)(The Category of Spaces over X) For every space over X, $f : (Y,\tau_Y) \to (X,\tau_X)$, prove that $\text{Id}_Y : (Y,\tau_Y) \to (Y,\tau_Y)$ is a morphism from f to f. For spaces over X, $f : (Y,\tau_Y) \to (X,\tau_X)$, $g : (Z,\tau_Z) \to (X,\tau_X)$ and $h : (W,\tau_W) \to (X,\tau_X)$, for every morphism from f to g, $u : (Y,\tau_Y) \to (Z,\tau_Z)$, and for every morphism from g to h, $v : (Z,\tau_Z) \to (W,\tau_W)$, prove that the composition $v \circ u : (Y,\tau_Y) \to (W,\tau_W)$ is a morphism from f to h. Conclude that these notions form a category, denoted $\mathbf{Top}_{(X,\tau_X)}$.

(b)(The Sheaf of Sections) For every space over X, $f : (Y,\tau_Y) \to (X,\tau_X)$, for every open U of τ_X, define $\text{Sec}_f(U)$ to be the set of continuous functions $s : (U,\tau_U) \to (Y,\tau_Y)$ such that $f \circ s$ is the inclusion morphism $(U,\tau_Y) \to (X,\tau_X)$. For every inclusion of τ_X-open subsets, $U \supseteq V$, for every s in $\text{Sec}_f(U)$, define $s|_V$ to be the restriction of s to the open subset V. **Prove** that $s|_V$ is an element of $\text{Sec}_f(V)$. **Prove** that these rules define a functor

$$\text{Sec}_f : \tau_X \to \mathbf{Sets}.$$

Prove that this functor is a sheaf of sets on (X,τ_X).

(c)(The Sections Functor) For spaces over X, $f(Y,\tau_Y) \to (X,\tau_X)$ and $g : (Z,\tau_Z) \to (X,\tau_X)$, for every morphism from f to g, $u : (Y,\tau_Y) \to (Z,\tau_Z)$, for every τ_X-open set U, for every s in $\text{Sec}_f(U)$, **prove** that $u \circ s$ is an element of $\text{Sec}_g(U)$. For every inclusion of τ_X-open sets, $U \supseteq V$, **prove** that $u \circ (s|_V)$ equals $(u \circ s)|_V$. Conclude that these rules define a morphism of sheaves of sets,

$$\text{Sec}_u : \text{Sec}_f \to \text{Sec}_g.$$

Prove that Sec_{Id_Y} is the identity morphism of Sec_f. For spaces over X, $f : (Y,\tau_Y) \to (X,\tau_X)$, $g : (Z,\tau_Z) \to (X,\tau_X)$ and $h : (W,\tau_W) \to (X,\tau_X)$, for every morphism from f to g, $u : (Y,\tau_Y) \to (Z,\tau_Z)$, and for every morphism from g to h, $v : (Z, \tau_Z) \to (W, \tau_W)$, **prove** that $\text{Sec}_{v \circ u}$ equals $\text{Sec}_v \circ \text{Sec}_u$. Conclude that these rules define a functor,

$$\text{Sec} : \mathbf{Top}_{(X,\tau_X)} \to \mathbf{Sets} - \text{Sh}_{(X,\tau_X)}.$$

(d)(The Éspace Étalé) For every presheaf of sets over X, $\mathcal{F}$, define $\text{Esp}_{\mathcal{F}}$ to be the set of pairs (x, ϕ_x) of an element x of X and an element ϕ_x of the stalk $\mathcal{F}_x = \text{colim}_{x \in U} \mathcal{F}(U)$; such an element is called a *germ* of $\mathcal{F}$ at x. Denote by

$$\pi_{\mathcal{F}} : \text{Esp}_{\mathcal{F}} \to X,$$

the set map sending (x, ϕ_x) to x. For every open subset U of X and for every element ϕ of $\mathcal{F}(U)$, define $B(U, \phi) \subset \text{Esp}_{\mathcal{F}}$ to be the image of the morphism,

$$\widetilde{\phi} : U \to \text{Esp}_{\mathcal{F}}, \ x \mapsto \phi_x.$$

Let (U, ψ) and (V, χ) be two such pairs. Let (x, ϕ_x) be an element of both $B(U, \psi)$ and $B(V, \chi)$. **Prove** that there exists an open subset W of $U \cap V$ containing x such that $\psi|_W$ equals $\chi|_W$. Denote this common restriction by $\phi \in \mathcal{F}(W)$. Conclude that (x, ϕ_x) is contained in $B(W, \phi)$, and this is contained in $B(U, \psi) \cap B(V, \chi)$. Conclude that the collection of all subset $B(U, \phi)$ of $\text{Esp}_{\mathcal{F}}$ is a topological basis. Denote by $\tau_{\mathcal{F}}$ the associated topology on $\text{Esp}_{\mathcal{F}}$. **Prove** that $\tau_{\mathcal{F}}$ is the finest topology on $\text{Esp}_{\mathcal{F}}$ such that for every τ_X-open set U and for every $\phi \in \mathcal{F}(U)$, the set map $\widetilde{\phi}$ is a continuous map $(U, \tau_U) \to (\text{Esp}_{\mathcal{F}}, \tau_{\mathcal{F}})$. In particular, since every composition $\pi_{\mathcal{F}} \circ \widetilde{\phi}$ is the continuous inclusion of (U, τ_U) in (X, τ_X), conclude that every $\widetilde{\phi}$ is continuous for the topology $\pi_{\mathcal{F}}^{-1}(\tau_X)$ on $\text{Esp}_{\mathcal{F}}$. Since $\tau_{\mathcal{F}}$ refines this topology, **prove** that

$$\pi_{\mathcal{F}} : (\text{Esp}_{\mathcal{F}}, \tau_{\mathcal{F}}) \to (X, \tau_X)$$

is a continuous map, i.e., $\pi_{\mathcal{F}}$ is a space over X.

(e)(The Éspace Functor) For every morphism of presheaves of sets over X, $\alpha : \mathcal{F} \to \mathcal{G}$, for every (x, ϕ_x) in $\text{Esp}_{\mathcal{F}}$, define $\text{Esp}_\alpha(x, \phi_x)$ to be $(x, \alpha_x(\phi_x))$, where $\alpha_x : \mathcal{F}_x \to \mathcal{G}_x$ is the induced morphism of stalks. For every τ_X-open set U and every $\phi \in \mathcal{F}(U)$, **prove** tht the composition $\text{Esp}_\alpha \circ \widetilde{\phi}$ equals $\widetilde{\alpha_U(\phi)}$ as set maps $U \to \text{Esp}_{\mathcal{G}}$. By construction, $\widetilde{\alpha_U(\phi)}$ is continuous for the topology $\tau_{\mathcal{G}}$. Conclude that $\widetilde{\phi}$ is continuous for the topology $(\text{Esp}_\alpha)^{-1}(\tau_{\mathcal{G}})$ on $\text{Esp}_{\mathcal{F}}$. Conclude that $\tau_{\mathcal{F}}$ refines this topology, and thus Esp_α is a continuous function,

$$\text{Esp}_\alpha : (\text{Esp}_{\mathcal{F}}, \tau_{\mathcal{F}}) \to (\text{Esp}_{\mathcal{G}}, \tau_{\mathcal{G}}).$$

Prove that $\text{Esp}_{\text{Id}_{\mathcal{F}}}$ equals the identity map on $\text{Esp}_{\mathcal{F}}$. For morphisms of presheaves of sets over X, $\alpha : \mathcal{F} \to \mathcal{G}$ and $\beta : \mathcal{G} \to \mathcal{H}$, **prove** that $\text{Esp}_{\beta \circ \alpha}$ equals $\text{Esp}_\beta \circ \text{Esp}_\alpha$. Conclude that these rules define a functor,

$$\text{Esp} : \mathbf{Sets} - \text{Presh}_{(X,\tau_X)} \to \mathbf{Top}_{(X,\tau_X)}.$$

(f)(The Adjointness Natural Transformations) For every presheaf of sets over X, $\mathcal{F}$, for every τ_X-open set U, for every $\phi \in \mathcal{F}(U)$, **prove** that $\widetilde{\phi}$ is an element of $\text{Sec}_{\pi_{\mathcal{F}}}(U)$. For every τ_X-open subset $U \supseteq V$, **prove** that $\widetilde{\phi}|_V$ equals $\widetilde{\phi|_V}$. Conclude that $\phi \mapsto \widetilde{\phi}$ is a morphism of presheaves of sets over X,

$$\theta_{\mathcal{F}} : \mathcal{F} \to \text{Sec} \circ \text{Esp}(\mathcal{F}).$$

For every morphism of presheaves of sets over X, $\alpha : \mathcal{F} \to \mathcal{G}$, for every τ_X-open set U, for every $\phi \in \mathcal{F}(U)$, **prove** that $\text{Esp}_\alpha \circ \theta_{\mathcal{F},U}(\phi)$ equals $\widetilde{\alpha_U(\phi)}$, and this in turn equals $\theta_{\mathcal{G},U} \circ \alpha_U(\phi)$. Conclude that $\text{Sec} \circ \text{Esp}(\alpha) \circ \theta_{\mathcal{F}}$ equals $\theta_{\mathcal{G}} \circ \alpha$. Therefore θ is a natural transformation of functors,

$$\theta : \text{Id}_{\mathbf{Sets}-\text{Presh}_{(X,\tau_X)}} \Rightarrow \text{Sec} \circ \text{Esp}.$$

(g)(Alternative Description of Sheafification) Since $\text{Sec} \circ \text{Esp}(\mathcal{F})$ is a sheaf, **prove** that there exists a unique morphism

$$\widetilde{\theta}_{\mathcal{F}} : \text{Sh}(\mathcal{F}) \to \text{Sec} \circ \text{Esp}(\mathcal{F})$$

factoring $\theta_{\mathcal{F}}$. For every element $t \in \text{Sec} \circ \text{Esp}(\mathcal{F})(U)$, a *$t$-pair* is a pair (U_0, s_0) of a τ_X-open subset $U \supseteq U_0$ and an element $s_0 \in \mathcal{F}(U_0)$ such that $t^{-1}(B(U_0, s_0))$ equals U_0. Define $\mathfrak{U}$ to be the set of t-pairs, and define $\iota : \mathfrak{U} \to \tau_U$ to be the set map $(U_0, s_0) \mapsto U_0$. **Prove** that $(U, \iota : \mathfrak{U} \to \tau_U)$ is an open covering. For every pair of t-pairs, (U_0, s_0) and (U_1, s_1), for every $x \in U_0 \cap U_1$, prove that there exists a τ_X-open subset $U_{0,1} \subset U_0 \cap U_1$ containing x such that $s_0|_{U_{0,1}}$ equals $s_1|_{U_{0,1}}$. **Prove** that this data gives rise to a section $s \in \text{Sh}(\mathcal{F})(U)$ such that $\widetilde{\theta}_{\mathcal{F}}(s)$ equals t. Conclude that $\widetilde{\theta}$ is an epimorphism. On the other hand, for every $r, s \in \mathcal{F}(U)$, if $\theta_{\mathcal{F},x}(r_x)$ equals $\theta_{\mathcal{F},x}(s_x)$, **prove** that $\widetilde{r}(x)$ equals $\widetilde{s}(x)$, i.e., r_x equals s_x. Conclude that every morphism $\widetilde{\theta}_x$ is a monomorphism, and hence $\widetilde{\theta}$ is a monomorphism of sheaves. Thus, finally **prove** that $\widetilde{\theta}_{\mathcal{F}}$ is an isomorphism of sheaves. Conclude that $\widetilde{\theta}$ is a natural isomorphism of functors,

$$\widetilde{\theta} : \text{Sh} \Rightarrow \text{Sec} \circ \text{Esp}.$$

(h) For every space over X, $f : (Y, \tau_Y) \to (X, \tau_X)$, for every τ_X-open U, for every $s \in \text{Sec}_f(U)$, and for every $x \in U$, define a set map,

$$\eta_{f,U,x} : \text{Sec}_f(U) \to Y, \ s \mapsto s(x).$$

Prove that for every τ_X-open subset $U \supseteq V$ that contains x, $\eta_{f,V,x}(s|_V)$ equals $\eta_{f,U,x}(s)$. Conclude that the morphisms $\eta_{f,U,x}$ factor through set maps,

$$\eta_{f,x} : (\text{Sec}_f)_x \to Y, \ s_x \mapsto s(x).$$

Define a set map,

$$\eta_f : \text{Esp}_{\text{Sec}_f} \to Y, \ (x, s_x) \mapsto \eta_{f,x}(s_x).$$

Prove that $\eta_f \circ \widetilde{s}$ equals s as set maps $U \to Y$. Since s is continuous for τ_Y, conclude that $\widetilde{s}$ is continuous for the inverse image topology $(\eta_f)^{-1}(\tau_Y)$ on $\text{Esp}_{\text{Sec}_f}$. Conclude that τ_{Sec_f} refines this topology, and thus η_f is a continuous map,

$$\eta_f : (\text{Esp}_{\text{Sec}_f}, \tau_{\text{Sec}_f}) \to (Y, \tau_Y).$$

Also **prove** that $f \circ \eta_f$ equals π_{Sec_f}. Conclude that η_f is a morphism of spaces over X. Finally, for spaces over X, $f : (Y, \tau_Y) \to (X, \tau_X)$ and $g : (Z, \tau_Z) \to (X, \tau_X)$, and for every morphism from f to g, $u : (Y, \tau_Y) \to (Z, \tau_Z)$, **prove** that $u \circ \eta_f$ equals $\eta_g \circ \mathrm{Esp} \circ \mathrm{Sec}(u)$. Conclude that $f \mapsto \eta_f$ defines a natural transformation of functors,

$$\eta : \mathrm{Esp} \circ \mathrm{Sec} \Rightarrow \mathrm{Id}_{\mathbf{Top}_{(X,\tau_X)}}.$$

(i)(The Adjoint Pair) **Prove** that $(\mathrm{Esp}, \mathrm{Sec}, \theta, \eta)$ is an adjoint pair of functors.

Alternative Description of Inverse Image Exercise. Let $f : (Y, \tau_Y) \to (X, \tau_X)$ be a continuous function of topological spaces. Since the category of topological spaces is a Cartesian category (by Problem 2(e) on Problem Set 8), for every space over X, $g : (Z, \tau_Z) \to (X, \tau_X)$, there is a fiber product diagram in **Top**,

$$\begin{array}{ccc} (Z, \tau_Z) \times_{(X,\tau_X)} (Y, \tau_Y) & \xrightarrow{g^*f} & (Z, \tau_Z) \\ {\scriptstyle f^*g}\downarrow & & \downarrow{\scriptstyle g} \\ (Y, \tau_Y) & \xrightarrow[f]{} & (X, \tau_X) \end{array}.$$

Denote the fiber product by $f^*(Z, \tau_Z)$.

(a) For spaces over X, $g : (Z, \tau_Z) \to (X, \tau_X)$ and $h : (W, \tau_W) \to (X, \tau_X)$, for every morphism of spaces over X, $u : (Z, \tau_Z) \to (W, \tau_W)$, **prove** that there is a unique morphism of topological spaces,

$$f^*u : f^*(Z, \tau_Z) \to f^*(W, \tau_W),$$

such that $f^*h \circ f^*u$ equals f^*g and $h^*f \circ f^*u$ equals $u \circ g^*f$. **Prove** that $f^*\mathrm{Id}_Z$ is the identity morphism of $f^*(Z, \tau_Z)$. For spaces over X, $g : (Z, \tau_Z) \to (X, \tau_X)$, $h : (W, \tau_W) \to (X, \tau_X)$ and $i : (M, \tau_M) \to (X, \tau_X)$, for every morphism from g to h, $u : (Z, \tau_Z) \to (W, \tau_W)$, and for every morphism from h to i, $v : (W, \tau_W) \to (M, \tau_M)$, **prove** that $f^*(v \circ u)$ equals $f^*v \circ f^*u$. Conclude that these rules define a functor,

$$f^*_{\mathrm{Sp}} : \mathbf{Top}_{(X,\tau_X)} \to \mathbf{Top}_{(Y,\tau_Y)}.$$

Prove that this functor is contravariant in f. In particular, there is a composite functor,

$$f^*_{\mathrm{Sp}} \circ \mathrm{Esp}_{(X,\tau_X)} : \mathbf{Sets} - \mathrm{Sh}_{(X,\tau_X)} \to \mathbf{Top}_{(Y,\tau_Y)}.$$

(b) Consider the composite functor,

$$f_* \circ \mathrm{Sec}_{(Y,\tau_Y)} : \mathbf{Top}_{(Y,\tau_Y)} \to \mathbf{Sets} - \mathrm{Sh}_{(Y,\tau_Y)} \to \mathbf{Sets} - \mathrm{Sh}_{(X,\tau_X)}.$$

Prove directly (without using the inverse image functor on sheaves) that $(f^*_{\mathrm{Sp}} \circ \mathrm{Esp}_{(X,\tau_X)}, f_* \circ \mathrm{Sec}_{(Y,\tau_Y)})$ extends to an adjoint pair of functors. Use this to conclude that the composite $\mathrm{Sec}_{(Y,\tau_Y)} \circ f^*_{\mathrm{Sp}} \circ \mathrm{Esp}_{(X,\tau_X)}$ is naturally isomorphic to the inverse image functor on sheaves of sets.

19 The Adjoint Pair of Discontinuous Sections (Godement Resolution)

Flasque Sheaves Exercise. Let (X, τ_X) be a topological space, and let $\mathcal{C}$ be a category. A $\mathcal{C}$-presheaf F on (X, τ_X) is *flasque* (or *flabby*) if for every inclusion of τ_X-open sets, $U \supseteq V$, the restriction morphism $A_V^U : A(U) \to A(V)$ is an epimorphism.

(a)(Pushforward Preserves Flasque Sheaves) For every continuous function $f : (X, \tau_X) \to (Y, \tau_Y)$, for every flasque $\mathcal{C}$-presheaf F on (X, τ_X), **prove** that f_*F is a flasque $\mathcal{C}$-presheaf on (Y, τ_Y).

(b)(Restriction to Opens Preserves Flasque Sheaves) For every τ_X-open subset U, for the continuous inclusion $i : (U, \tau_U) \to (X, \tau_X)$, for every flasque $\mathcal{C}$-presheaf F on (X, τ_X), **prove** that $i^{-1}F$ is a flasque $\mathcal{C}$-presheaf. Also, for every $\mathcal{C}$-sheaf F on (X, τ_X), **prove** that the presheaf inverse image $i^{-1}F$ is already a sheaf, so that the sheaf inverse image agrees with the presheaf inverse image.

(c)(H^1-Acyclicity of Flasque Sheaves) Let $\mathcal{A}$ be an Abelian category realized as a full subcategory of the category of left R-modules (via the embedding theorem). Let

$$0 \longrightarrow A' \xrightarrow{q} A \xrightarrow{p} A'' \longrightarrow 0$$

be a short exact sequence of $\mathcal{A}$-sheaves on (X, τ_X). Let U be a τ_X-open set. Let $t : A''(U) \to T$ be a morphism in $\mathcal{A}$ such that $t \circ p(U)$ is the zero morphism. Assume that A' is flasque. **Prove** that t is the zero morphism as follows. Let $a'' \in A''(U)$ be any element. Let $\mathcal{S}$ be the set of pairs (V, a) of a τ_X-open subset $V \subseteq U$ and an element $a \in A(V)$ such that $p(V)(a)$ equals $a''|_V$. For elements (V, a) and $(\widetilde{V}, \widetilde{a})$ of $\mathcal{S}$, define $(V, a) \le (\widetilde{V}, \widetilde{a})$ if $V \subseteq V'$ and $\widetilde{a}|_V$ equals a. **Prove** that this defines a partial order on $\mathcal{S}$. Use the sheaf axiom for A to **prove** that every totally ordered subset of $\mathcal{S}$ has a least upper bound in $\mathcal{S}$. Use Zorn's Lemma to conclude that there exists a maximal element (V, a) in $\mathcal{S}$. For every x in U, since p is an epimorphism of sheaves, **prove** that there exists (W, b) in $\mathcal{S}$ such that $x \in W$. Conclude that on $V \cap W$, $a|_{V\cap W} - b|_{V\cap W}$ is in the kernel of $p(V \cap W)$. Since the sequence above is exact, **prove** that there exists unique $a' \in A'(V \cap W)$ such that $q(V \cap W)(a')$ equals $a|_{V\cap W} - b|_{V\cap W}$. Since A' is flasque, **prove** that there exists $a'_W \in A'(W)$ such that $a'_W|_{V\cap W}$ equals a'. Define $a_W = b + q(W)(a'_W)$. **Prove** that (W, a_W) is in $\mathcal{S}$ and $a|_{V\cap W}$ equals $a_W|_{V\cap W}$. Use the sheaf axiom for A once more to **prove** that there exists unique $(V \cap W, a_{V\cap W})$ in $\mathcal{S}$ with $a_{V\cap W}|_V$ equals a and $a_{V\cap W}|_W$ equals a_W. Since (V, a) is maximal, conclude that $W \subset V$, and thus x is in V. Conclude that V equals U. Thus, a'' equals $p(U)(a)$. Conclude that $t(a'')$ equals 0, and thus t is the zero morphism. (For a real challenge, modify this argument to avoid any use of the embedding theorem.)

(d)(H^r-Acyclicity of Flasque Sheaves) Let $C^\bullet = (C^q, d_C^q)_{q\ge0}$ be a complex of $\mathcal{A}$-sheaves on (X, τ_X). Assume that every C^q is flasque. Let $r \ge 0$ be an integer, and assume that the cohomology sheaves $h^q(C^\bullet)$ are zero for $q = 0, \dots, r$. Use (c) and induction on r to prove that for the associated complex in $\mathcal{C}$,

$$C^\bullet(U) = (C^q(U), d_C^q(U))_{q\ge0}$$

also $h^q(C^\bullet(U))$ is zero for $q = 0, \dots, r$.

Enough Injective $\Lambda - \Pi$-modules Exercise. Let (X, τ_X) be a topological space. Let Λ and Π be presheaves of associative, unital rings on (X, τ_X). The most common case is to take both Λ and Π to be the constant presheaf with values $\mathbb{Z}$. Assume, for simplicity, that $\Lambda(\varnothing)$ and $\Pi(\varnothing)$ are the zero ring. A *presheaf of* $\Lambda - \Pi$-*bimodules* on (X, τ_X) is a presheaf M of Abelian groups on (X, τ_X) together with a structure of $\Lambda(U) - \Pi(U)$-bimodule on every Abelian group $M(U)$ such that for every open subset $U \supseteq V$, relative to the restriction homomorphisms of associative, unital rings,

$$\Lambda_V^U : \Lambda(U) \to \Lambda(V), \quad \Pi_V^U : \Pi(U) \to \Pi(V),$$

every restriction homomorphism of Abelian groups,

$$M_V^U : M(U) \to M(V),$$

is a homomorphism of $\Lambda(U) - \Pi(U)$-bimodules. For presheaves of $\Lambda - \Pi$-bimodules on (X, τ_X), M and N, a *morphism of presheaves of* $\Lambda - Pi$-*bimodules* is a morphism of presheaves of Abelian groups $\alpha : M \to N$ such that for every open U, the Abelian group homomorphism,

$$\alpha(U) : M(U) \to N(U),$$

is a homomorphism of $\Lambda(U) - \Pi(U)$-bimodules.

(a)(The Category of Presheaves of $\Lambda - \Pi$-Bimodules) **Prove** that these notions form a category $\Lambda - \Pi - \text{Presh}_{(X,\tau_X)}$. Prove that this is an Abelian category that satisfies Grothendieck's axioms (AB1), (AB2), (AB3), (AB3*), (AB4) and (AB5).

(b)(Discontinuous $\Lambda - \Pi$-Bimodules) A *discontinuous* $\Lambda - \Pi$-bimodule is a specification K for every nonempty τ_X-open U of a $\Lambda(U) - \Pi(U)$-bimodule $K(U)$, but without any specification of restriction morphisms. For discontinuous $\Lambda - \Pi$-bimodules K and L, a *morphism* of discontinuous $\Lambda - \Pi$-bimodules $\alpha : K \to L$ is a specification for every nonempty τ_X-open U of a homomorphism $\alpha(U) : K(U) \to L(U)$ of $\Lambda(U) - \Pi(U)$-bimodules. **Prove** that with these notions, there is a category $\Lambda - \Pi - \text{Disc}_{(X,\tau_X)}$ of discontinuous $\Lambda - \Pi$-bimodules. Prove that this is an Abelian category that satisfies Grothendieck's axioms (AB1), (AB2), (AB3), (AB3*), (AB4), (AB4*) and (AB5).

(c)(The Presheaf Associated to a Discontinuous $\Lambda - \Pi$-Bimodule) For every discontinuous $\Lambda - \Pi$-bimodule K, for every nonempty τ_X-open subset U, define

$$\widetilde{K}(U) = \prod_{W \subseteq U} K(W)$$

as a $\Lambda(U) - \Pi(U)$-bimodule, where the product is over nonempty open subsets $W \subseteq U$ (in particular also $W = U$ is allowed), together with its natural projections $\pi_W^U : \widetilde{K}(U) \to K(W)$. Also define $\widetilde{K}(\varnothing)$ to be a zero object. For every inclusion of τ_X-open subsets $U \supseteq V$, define

$$\widetilde{K}_V^U : \prod_{W \subseteq U} K(W) \to \prod_{W \subseteq V} K(W),$$

to be the unique morphism of $\Lambda(U)-\Pi(U)$-bimodules such that for every $W \subset V$, $\pi^V_W \circ \widetilde{K}^U_V$ equals π^U_W. **Prove** that $\widetilde{K}$ is a presheaf of $\Lambda-\Pi$-bimodules. For discontinuous $\Lambda-\Pi$-bimodules K and L, for every morphism of discontinuous $\Lambda-\Pi$-bimodules, $\alpha : K \to L$, for every τ_X-open set U, define

$$\widetilde{\alpha}(U) : \prod_{W \subseteq U} K(W) \to \prod_{W \subseteq U} L(W)$$

to be the unique morphism of $\Lambda(U)-\Pi(U)$-bimodules such that for every $W \subseteq U$, $\pi^U_{L,W} \circ \widetilde{\alpha}(U)$ equals $\pi^U_{K,W}$. **Prove** that $\widetilde{\alpha}$ is a morphism of presheaves of $\Lambda-\Pi$-bimodules. **Prove** that these notions define a functor,

$$\widetilde{\ast} : \Lambda-\Pi-\text{Disc}_{(X,\tau_X)} \to \Lambda-\Pi-\text{Presh}_{(X,\tau_X)}.$$

Prove that this is an exact functor that preserves arbitrary limits and finite colimits.

(d)(The Čech Object of a Discontinuous $\Lambda-\Pi$-Bimodule is Acyclic) For every open covering $(U, \iota : \mathfrak{U} \to \tau_U)$, define

$$\tau_{\mathfrak{U}} = \bigcup_{U_0 \in \mathfrak{U}} \tau_{\iota(U_0)} = \{W \in \tau_U | \exists U_0 \in \mathfrak{U}, W \subset \iota(U_0)\}.$$

For every discontinuous $\Lambda-\Pi$-bimodule K, define

$$\widetilde{K}(\mathfrak{U}) := \prod_{W \in \tau_{\mathfrak{U}}} K(W)$$

together with its projections $\pi_W : \widetilde{K}(\mathfrak{U}) \to K(W)$. In particular, define

$$\pi^U_{\mathfrak{U}} : \widetilde{K}(\mathfrak{U}) \to \widetilde{K}(\mathfrak{U})$$

to be the unique $\Lambda(U)-\Pi(U)$-morphism such that for every $W \in \tau_{\mathfrak{U}}$, $\pi_W \circ \pi^U_{\mathfrak{U}}$ equals π_W.

For every nonempty $W \in \tau_{\mathfrak{U}}$, define

$$\mathfrak{U}^W := \{U_0 \in \mathfrak{U} | W \subset \iota(U_0)\}.$$

Prove that

$$\check{C}^r(\mathfrak{U}, \widetilde{K}) = \prod_{(U_0,\dots,U_r) \in \mathfrak{U}^{r+1}} \prod_{W \subseteq \iota(U_0,\dots,U_r)} K(W)$$

together with its projection $\pi_{(U_0,\dots,U_r;W)} : \check{C}^r(\mathfrak{U}, \widetilde{K}) \to K(W)$ for every nonempty $W \subset \iota(U_0,\dots,U_r)$; if $\iota(U_0,\dots,U_r)$ is empty, the corresponding factor is a zero object. For every integer $r \geq 0$, for every $i = 0,\dots,r+1$, **prove** that the morphism

$$\partial^i_r : \check{C}^r(\mathfrak{U}, \widetilde{K}) \to \check{C}^{r+1}(\mathfrak{U}, \widetilde{K}),$$

is the unique $\Lambda(U)$–$\Pi(U)$-morphism such that for every nonempty $W \in \tau_{\mathfrak{U}}$ and for every $(U_0,\dots,U_r,U_{r+1}) \in (\mathfrak{U}^W)^{r+2}$, $\pi_{U_0,\dots,U_r,U_{r+1};W} \circ \partial^i_r$ equals $\pi_{U_0,\dots,U_{i-1},U_{i+1},\dots,U_{r+1};W}$. For every integer $r \geq 0$ and for every $i = 0,\dots,r$, **prove** that the morphism

$$\sigma^i_{r+1} : \check{C}^{r+1}(\mathfrak{U}, \widetilde{K}) \to \check{C}^r(\mathfrak{U}, \widetilde{K}),$$

is the unique $\Lambda(U)$–$\Pi(U)$-morphism such that for every nonempty $W \in \tau_{\mathfrak{U}}$ and for every $(U_0, \dots, U_r) \in (\mathfrak{U}^W)^{r+1}$, $\pi_{U_0,\dots,U_r;W} \circ \sigma^i_{r+1}$ equals equals $\pi_{U_0,\dots,U_{i-1},U_i,U_i,U_{i+1},\dots,U_{r+1};W}$. For every integer $r \geq 0$, prove that the morphism

$$g^r_{\widetilde{K},\mathfrak{U}} : \widetilde{K}(\mathfrak{U}) \to \check{C}^r(\mathfrak{U}, \widetilde{K})$$

is the unique $\Lambda(U)$–$\Pi(U)$-morphism such that for every nonempty $W \in \tau_{\mathfrak{U}}$ and for every $(U_0, \dots, U_r) \in (\mathfrak{U}^W)^{r+1}$, $\pi_{U_0,\dots,U_r;W} \circ g^r$ equals π_W.

For every nonempty $W \in \tau_{\mathfrak{U}}$, for every $r \geq 0$, define

$$\check{C}^r(\mathfrak{U}, \widetilde{K})^W := \prod_{(U_0,\dots,U_r)\in(\mathfrak{U}^W)^{r+1}} K(W),$$

with its projections

$$\pi_{U_0,\dots,U_r|W} : \check{C}^r(\mathfrak{U}, \widetilde{K})^W \to K(W).$$

Define

$$\pi^r_{-;W} : \check{C}^r(\mathfrak{U}, \widetilde{K}) \to \check{C}^r(\mathfrak{U}, \widetilde{K})^W$$

to be the unique $\Lambda(U)$–$\Pi(U)$-morphism such that for every $(U_0, \dots, U_r) \in (\mathfrak{U}^W)^{r+1}$, $\pi_{U_0,\dots,U_r|W} \circ \pi^r_{-;W}$ equals $\pi_{U_0,\dots,U_r;W}$. For every integer $r \geq 0$ and for every $i = 0, \dots, r+1$, **prove** that there exists a unique $\Lambda(U) - \Pi(U)$-morphism

$$\partial^i_r : \check{C}^r(\mathfrak{U}, \widetilde{K})^W \to \check{C}^{r+1}(\mathfrak{U}, \widetilde{K})^W,$$

such that $\partial^i_r \circ \pi^r_{-;W}$ equals $\pi^{r+1}_{-;W} \circ \partial^i_r$, and **prove** that for every $(U_0, \dots, U_r, U_{r+1}) \in (\mathfrak{U}^W)^{r+2}$, $\pi_{U_0,\dots,U_r,U_{r+1}|W} \circ \partial^i_r$ equals $\pi_{U_0,\dots,U_{i-1},U_{i+1},\dots,U_{r+1}|W}$. For every integer $r \geq 0$ and for every $i = 0, \dots, r$, **prove** that there exists a unique $\Lambda(U) - \Pi(U)$-morphism

$$\sigma^i_{r+1} : \check{C}^{r+1}(\mathfrak{U}, \widetilde{K})^W \to \check{C}^r(\mathfrak{U}, \widetilde{K})^W,$$

such that $\sigma^i_{r+1} \circ \pi^{r+1}_{-;W}$ equals $\pi^r_{-;W} \circ \sigma^i_{r+1}$, and **prove** that for every $(U_0, \dots, U_r) \in (\mathfrak{U}^W)^{r+1}$, $\pi_{U_0,\dots,U_r|W} \circ \sigma^i_{r+1}$ equals equals $\pi_{U_0,\dots,U_{i-1},U_i,U_i,U_{i+1},\dots,U_{r+1}|W}$. For every integer $r \geq 0$, **prove** that there exists a unique $\Lambda(U) - \Pi(U)$-morphism

$$g^r : K(W) \to \check{C}^r(\mathfrak{U}, \widetilde{K})^W$$

such that $\pi^r_{-;W} \circ g^r$ equals $g^r \circ \pi_W$, and **prove** that for every $(U_0, \dots, U_r) \in (\mathfrak{U}^W)^{r+1}$, $\pi_{U_0,\dots,U_r|W} \circ g^r$ equals $\mathrm{Id}_{K(W)}$. Conclude that

$$\pi^\bullet_{-;W} : \check{C}^\bullet(\mathfrak{U}, \widetilde{K}) \to \check{C}^\bullet(\mathfrak{U}, \widetilde{K})^W$$

is a morphism of cosimplicial $\Lambda(U)$–$\Pi(U)$-bimodules that is compatible with the coaugmentations $g^\bullet$. **Prove** that these morphisms realize $\check{C}^\bullet(\mathfrak{U}, \widetilde{K})$ in the category $S^\bullet\Lambda(U) - \Pi(U) - \text{Bimod}$ as a product,

$$\check{C}^\bullet(\mathfrak{U}, \widetilde{K}) = \prod_{W\in\tau_{\mathfrak{U}}} \check{C}^\bullet(\mathfrak{U}, \widetilde{K})^W.$$

Using the Axiom of Choice, prove that there exists a set map

$$\phi : \tau_{\mathfrak{U}} \setminus \{\varnothing\} \to \mathfrak{U}$$

such that for every nonempty $W \in \tau_{\mathfrak{U}}$, $\phi(W)$ is an element in $\mathfrak{U}^W$. For every integer $r \geq 0$, define

$$\check{C}^r(\phi, \widetilde{K})^W : \check{C}^r(\mathfrak{U}, \widetilde{K})^W \to K(W)$$

to be $\pi_{\phi(W),\dots,\phi(W)|W}$. **Prove** that for every integer $r \geq 0$ and for every $i = 0, \dots, r+1$, $\check{C}^{r+1}(\phi, \widetilde{K})^W \circ \partial^i_r$ equals $\check{C}^r(\phi, \widetilde{K})^W$. **Prove** that for every integer $r \geq 0$ and for every $i = 0, \dots, r$, $\check{C}^r(\phi, \widetilde{K})^W \circ \sigma^i_{r+1}$ equals $\check{C}^{r+1}(\phi, \widetilde{K})^W$. Conclude that

$$\check{C}^\bullet(\phi, \widetilde{K})^W \to \mathrm{const}_{K(W)}$$

is a morphism of cosimplicial $\Lambda(U) - \Pi(U)$-bimodules. **Prove** that $\check{C}^\bullet(\phi, \widetilde{K})^W \circ g^\bullet$ equals the identity morphism of $\mathrm{const}_{K(W)}$. For every nonempty $W \in \tau_{\mathfrak{U}}$, for every integer $r \geq 0$, for every integer $i = 0, \dots, r$, define

$$g^i_{\phi,r+1} : \check{C}^{r+1}(\mathfrak{U}, \widetilde{K})^W \to \check{C}^r(\mathfrak{U}, \widetilde{K})^W$$

to be the unique $\Lambda(U)$–$\Pi(U)$-morphism such that for every $(U_0, \dots, U_r) \in (\mathfrak{U}^W)^{r+1}$, $\pi_{U_0,\dots,U_r|W} \circ g^i_{\phi,r+1}$ equals $\pi_{U_0,\dots,U_i,\phi(W),\dots,\phi(W)|W}$. **Prove** the following identities (cosimplicial homotopy identities),

$$g^0_{\phi,r+1} \circ \partial^0_r = g^r \circ \check{C}^r(\phi, \widetilde{K})^W, \quad g^r_{\phi,r+1} \circ \partial^{r+1}_r = \mathrm{Id}_{\check{C}^r(\mathfrak{U},\widetilde{K})^W},$$

$$g^j_{\phi,r+1} \circ \partial^i_r = \begin{cases} \partial^i_{r-1} \circ g^{j-1}_{\phi,r}, & 0 \leq i < j \leq r, \\ g^{i-1}_{\phi,r+1} \circ \partial^i_r, & 0 < i = j \leq r, \\ \partial^{i-1}_{r-1} \circ g^j_{\phi,r}, & 1 \leq j+1 < i \leq r+1. \end{cases}$$

$$g^j_{\phi,r} \circ \sigma^i_{r+1} = \begin{cases} \sigma^i_r \circ g^{j+1}_{\phi,r+1}, & 0 \leq i \leq j \leq r-1, \\ \sigma^{i-1}_r \circ g^j_{\phi,r+1}, & 0 \leq j < i \leq r. \end{cases}$$

Conclude that $g^\bullet$ and $\check{C}^\bullet(\phi, \widetilde{K})^W$ are homotopy equivalences between $\check{C}^\bullet(\mathfrak{U}, \widetilde{K})^W$ and $\mathrm{const}_{K(W)}$. Conclude that $\check{C}^\bullet(\mathfrak{U}, \widetilde{K})$ is homotopy equivalent to $\mathrm{const}_{\widetilde{K}(\mathfrak{U})}$. In particular, **prove** that the associated cochain complex of $\check{C}^\bullet(\mathfrak{U}, \widetilde{K})^W$ is acyclic with $\check{H}^0(\mathfrak{U}, \widetilde{K})^W$ equal to $K(W)$. Similarly, **prove** that the associated cochain complex of $\check{C}^\bullet(\mathfrak{U}, \widetilde{K})$ is acyclic with $\check{H}^0(\mathfrak{U}, \widetilde{K})$ equal to $K(\mathfrak{U})$.

(e)(The Forgetful Functor to Discontinuous $\Lambda - \Pi$-Bimodules; Preservation of Injectives) For every presheaf M of $\Lambda - \Pi$-bimodules on (X, τ_X), define $\Phi(M)$ to be the discontinuous $\Lambda - \Pi$-bimodule $U \mapsto M(U)$. For presheaves of $\Lambda - \Pi$-bimodules, M and N, for every morphism of presheaves of $\Lambda - \Pi$-bimodules, $\alpha : M \to N$, define $\Phi(\alpha) : \Phi(M) \to \Phi(N)$ to be the assignment $U \mapsto \alpha(U)$. **Prove** that these rules define a functor

$$\Phi : \Lambda - \Pi - \mathrm{Presh}_{(X,\tau_X)} \to \Lambda - \Pi - \mathrm{Disc}_{(X,\tau_X)}.$$

Prove that this is a faithful exact functor that preserves arbitrary limits and finite colimits. For every presheaf M of $\Lambda-\Pi$-bimodules, for every τ_X-open U, define

$$\theta_{M,U} : M(U) \to \prod_{W \subseteq U} M(W)$$

to be the unique homomorphism of $\Lambda(U)-\Pi(U)$-bimodules such that for every τ_X-open subset $W \subset U$, $\pi^U_W \circ \theta_{M,U}$ equals M^U_W. **Prove** that $U \mapsto \theta_{M,U}$ is a morphism of presheaves of $\Lambda-\Pi$-bimodules,

$$\theta_M : M \to \widetilde{\Phi(M)}.$$

For every morphism of presheaves of $\Lambda-\Pi$-bimodules, $\alpha : M \to N$, for every τ_X-open set U, **prove** that $\widetilde{\Phi(\alpha)} \circ \theta_M$ equals $\theta_N \circ \alpha$. Conclude that θ is a natural transformation of functors,

$$\theta : \mathrm{Id}_{\Lambda-\Pi-\mathrm{Presh}_{(X,\tau_X)}} \Rightarrow \widetilde{\ast} \circ \Phi.$$

For every discontinuous $\Lambda-\Pi$-bimodule K, for every τ_X-open U, define

$$\eta_{K,U} : \prod_{W \subseteq U} K(W) \to K(U)$$

to be π^U_W. **Prove** that $U \mapsto \eta_{K,U}$ is a morphism of discontinuous $\Lambda-\Pi$-bimodules. For every pair of discontinuous $\Lambda-\Pi$-bimodules, K and L, for every morphism of discontinuous $\Lambda-\Pi$-bimodules, $\beta : K \to L$, **prove** that $\eta_L \circ \Phi(\widetilde{\beta})$ equals $\beta \circ \eta_L$. Conclude that η is a natural transformation of functors,

$$\eta : \Phi \circ \widetilde{\ast} \Rightarrow \mathrm{Id}_{\Lambda-\Pi-\mathrm{Disc}_{(X,\tau_X)}}.$$

Prove that $(\Phi, \widetilde{\ast}, \theta, \eta)$ is an adjoint pair of functors. Since Φ preserves monomorphisms, use Problem 3(d), Problem Set 5 to **prove** that $\widetilde{\ast}$ sends injective objects to injective objects. Since the forgetful morphism from sheaves to presheaves preserves monomorphisms, **prove** that the sheafification functor Sh sends injective objects to injective objects. Conclude that $\mathrm{Sh} \circ \widetilde{\ast}$ sends injective objects to injective objects.

(f)(Enough Injectives) Recall from Problems 3 and 4 of Problem Set 5 that for every τ_X-open set U, there are enough injective $\Lambda(U)-\Pi(U)$-bimodules. Using the Axiom of Choice, conclude that $\Lambda-\Pi-\mathrm{Disc}_{(X,\tau_X)}$ has enough injective objects. In particular, for every presheaf M of $\Lambda-\Pi$-bimodules, for every open set U, let there be given a monomorphism of $\Lambda(U)-\Pi(U)$-bimodules,

$$\epsilon_U : M(U) \to I(U),$$

with $I(U)$ an injective $\Lambda(U)-\Pi(U)$-bimodule. Conclude that $\widetilde{I}$ is an injective presheaf of $\Lambda-\Pi$-bimodules, and the composition

$$M \xrightarrow{\theta_M} \widetilde{\Phi(M)} \xrightarrow{\widetilde{\epsilon}} \widetilde{I}$$

is a monomorphism of presheaves of $\Lambda-\Pi$-bimodules. If M is a sheaf, conclude that $\mathrm{Sh}(\widetilde{I})$ is an injective sheaf of $\Lambda-\Pi$-bimodules. Also, use (d) to prove that the composition

$$M \xrightarrow{\theta_M} \widetilde{\Phi(M)} \xrightarrow{\widetilde{\epsilon}} \widetilde{I} \xrightarrow{sh} \mathrm{Sh}(\widetilde{I})$$

is a monomorphism of sheaves of $\Lambda - \Pi$-bimodules. (**Hint:** Since $\sigma_{x,U}$ is a filtering small category, use Problem 0 to reduce to the statement that for every open covering $(U, \mathfrak{U})$, the morphism $M(U) \to \widetilde{M}(\mathfrak{U})$ is a monomorphism. Realize this a part of the Sheaf Axiom for M.) Conclude that both the category $\Lambda - \Pi - \text{Presh}_{(X,\tau_X)}$ and $\Lambda - \Pi - \text{Sh}_{(X,\tau_X)}$ have enough injective objects. In particular, for an additive, left-exact functor F, resp. G, on the category of presheaves of $\Lambda - \Pi$-bimodules, resp. the category of sheaves of $\Lambda - \Pi$-bimodules, there are right derived functors $((R^n F)_n, (\delta^n)_n)$, resp. $((R^n G)_n, (\delta^n)_n)$. Finally, since $\widetilde{*}$ is exact and sends injective objects to injective objects, use the Grothendieck Spectral Sequence (or universality of the cohomological δ-functor) to **prove** that $(R^n F) \circ \widetilde{*}$ is $R^n(F \circ \widetilde{*})$.

(g)(Enough Flasque Sheaves; Injectives are Flasque) Let K be a discontinuous $\Lambda - \Pi$-bimodule on X. For every τ_X-open set U, **prove** that $\widetilde{K}(U) \to \text{Sh}(\widetilde{K})(U)$ is the colimit over all open coverings $\mathfrak{U} \subset \tau_U$ (ordered by refinement as usual) of the morphism

$$\pi^U_{\mathfrak{U}} : \widetilde{K}(U) \to \widetilde{K}(\mathfrak{U}).$$

In particular, since every morphism $\widetilde{K}(U) \to \widetilde{K}(\mathfrak{U})$ is surjective (by the Axiom of Choice), conclude that also

$$\text{sh}(U) : \widetilde{K}(U) \to \text{Sh}(\widetilde{K})(U)$$

is surjective. Use this to **prove** that $\text{Sh}(\widetilde{K})$ is a flasque sheaf.

For every injective $\Lambda - \Pi$-sheaf I, for the monomorphism $\theta_I : I \to \text{Sh}(\widetilde{\Phi(I)})$, there exists a retraction $\rho : \text{Sh}(\widetilde{\Phi(I)}) \to I$. Also $\text{Sh}(\widetilde{\Phi(I)})$ is flasque. Use this to **prove** that also I is flasque.

(h)(Sheaf Cohomology; Flasque Sheaves are Acyclic) For every τ_X-open set U, prove that the functor

$$\Gamma(U, -) : \Lambda - \Pi - \text{Presh}_{(X,\tau_X)} \to \Lambda(U) - \Pi(U) - \text{Bimod}, \quad M \mapsto M(U)$$

is an exact functor. Also prove that the functor

$$\Gamma(U, -) : \Lambda - \Pi - \text{Sh}_{(X,\tau_X)} \to \Lambda(U) - \Pi(U) - \text{Bimod}$$

is an additive, left-exact functor. Use (g) to conclude that every sheaf M of $\Lambda - \Pi$-modules admits a resolution, $\epsilon : M \to I^\bullet$ by injective sheaves of $\Lambda - \Pi$-modules that are also flasque. Conclude that $\Gamma(U, -)$ extends to a universal cohomological δ-functor formed by the right derived functors, $((H^n(U, -))_n, (\delta^n)_n)$. Finally, assume that M is flasque. Use Problem 4(d) to **prove** that $I^\bullet(U)$ is an acyclic complex of $\Lambda(U) - \Pi(U)$-bimodules. Conclude that for every flasque sheaf M of $\Lambda - \Pi$-bimodules, for every $n \geq 0$, $H^n(U, M)$ is zero, i.e., flasque sheaves of $\Lambda - \Pi$-bimodules are acyclic for the right derived functors of $\Gamma(U, -)$.

(i)(Computation of Sheaf Cohomology via Flasque Resolutions; Canonical Resolutions; Independence of $\Lambda - \Pi$) Use (h) and the hypercohomlogy spectral sequence to **prove** that for every sheaf M of $\Lambda - \Pi$-bimodules, for every acyclic resolution $\epsilon_M : M \to M^\bullet$ of M by sheaves of $\Lambda - \Pi$-bimodules that are flasque, for every integer $n \geq 0$, there is a canonical isomorphism of $H^n(U, M)$

with $h^n(M^\bullet(U))$. In particular, the functor $\top = \text{Sh} \circ \widetilde{\times} \circ \Phi$, the natural transformation $\theta : \text{Id} \Rightarrow \top$, and the natural transformation

$$\text{Sh} \circ \widetilde{\times} \circ \eta \circ \Phi : \top\top \Rightarrow \top,$$

form a *triple* on the category $\Lambda - \Pi - \text{Sh}_{(X,\tau_X)}$. There is an associated cosimplicial functor,

$$L_\top : \Lambda - \Pi - \text{Sh}_{(X,\tau_X)} \to S^\bullet \Lambda - \Pi - \text{Sh}_{(X,\tau_X)}$$

and a functorial coaugmentation,

$$\theta_M : \text{const}^\bullet_M \to L^\bullet_\top(M).$$

The associated (unnormalized) cochain complex of this cosimplicial object is an acyclic resolution of M by flasque sheaves of $\Lambda - \Pi$-bimodules, and it is *canonical*, depending on no choices of injective resolutions.

Finally, let $\widehat{\Lambda} \to \Lambda$ and $\widehat{\Pi} \to \Pi$ be morphisms of presheaves of associative, unital rings. This induces a functor,

$$\Lambda - \Pi - \text{Sh}_{(X,\tau_X)} \to \widehat{\Lambda} - \widehat{\Pi} - \text{Sh}_{(X,\tau_X)}.$$

For every sheaf M of $\Lambda - \Pi$-bimodules, and for every acyclic resolution $\epsilon : M \to M^\bullet$ of M by flasque sheaves of $\Lambda - \Pi$-bimodules, this is also an acyclic, flasque resolution of M with the associated structure of sheaves of $\widehat{\Lambda} - \widehat{\Pi}$-bimodules. For the natural map of cohomological δ-functors from the derived functors of $\Gamma(U, -)$ on $\Lambda - \Pi - \text{Sh}_{(X,\tau_X)}$ to the derived functors of $\Gamma(U, -)$ on $\widehat{\Lambda} - \widehat{\Pi} - \text{Sh}_{(X,\tau_X)}$, **prove** that this natural map is a natural isomorphism of cohomological δ-functors. This justifies the notation $H^n(U, -)$ that makes no reference to the underlying presheaves Λ and Π, and yet is naturally a functor to $\Lambda(U) - \Pi(U) - \text{Bimod}$ whenever M is a sheaf of $\Lambda - \Pi$-bimodules.

Problem 6.(Flasque Sheaves are Čech-Acyclic) Let (X, τ_X) be a topological space. Let M be a presheaf of $\Lambda - \Pi$-bimodules on (X, τ_X). Let U be a τ_X-open set. Let $(U, \iota : \mathfrak{U} \to \tau_U)$ be an open covering. For every τ_X-open subset V, define $(V, \iota_V : \mathfrak{U} \to \tau_V)$ to be the open covering $\iota_V(U_0) = V \cap \iota(U_0)$. For simplicity, denote this by $(V, \mathfrak{U}_V)$. For every integer $r \geq 0$, define $\underline{\check{C}}^r(\mathfrak{U}, M)(V)$ to be the $\Lambda(V) - \Pi(V)$-bimodule $\check{C}^r(\mathfrak{U}_V, M)$. Moreover, define

$$\partial^i_{V,r} : \underline{\check{C}}^r(\mathfrak{U}, M)(V) \to \underline{\check{C}}^{r+1}(\mathfrak{U}, M)(V), \quad \sigma^i_{V,r+1} : \underline{\check{C}}^{r+1}(\mathfrak{U}, M)(V) \to \underline{\check{C}}^r(\mathfrak{U}, M)(V),$$

to be the face and degeneracy maps on $\check{C}^\bullet(\mathfrak{U}_V, M)$. Finally, let $\eta^r_V : M(V) \to \underline{\check{C}}^r(\mathfrak{U}, M)(V)$ be the coadjunction of sections from Problem 5(e), Problem Set 8. For every inclusion of τ_X-open subsets $W \cap V \cap U$, the identity map $\text{Id}_{\mathfrak{U}}$ is a refinement of open coverings,

$$\phi^V_W : (V, \iota_V : \mathfrak{U} \to \tau_V) \to (W, \iota_W : \mathfrak{U} \to \tau_W).$$

By Problem 5(f) from Problem Set 8, $\check{C}^r(\phi^V_W, M)$ is an associated morphism of $\Lambda(V) - \Pi(V)$-bimodules, denoted

$$\underline{\check{C}}^r(\mathfrak{U}, M)^V_W : \underline{\check{C}}^r(\mathfrak{U}, M)(V) \to \underline{\check{C}}^r(\mathfrak{U}, M)(W).$$

(a)(The Presheaf of Čech Objects) **Prove** that the rules $V \mapsto \underline{\check{C}}^r(\mathfrak{U}, M)(V)$ and $\underline{\check{C}}^r(\mathfrak{U}, M)^V_W$ define a presheaf $\underline{\check{C}}^r(\mathfrak{U}, M)$ of $\Pi - \Lambda$-bimodules on U. Moreover, **prove** that the rules $V \mapsto \partial^i_{V,r}$, resp. $V \mapsto \sigma^i_{V,r+1}$, $V \mapsto \eta^r_V$, define morphisms of presheaves of $\Lambda - \Pi$-bimodules,

$$\partial^i_r : \underline{\check{C}}^r(\mathfrak{U}, M) \to \underline{\check{C}}^{r+1}(\mathfrak{U}, M), \quad \sigma^i_{r+1} : \underline{\check{C}}^{r+1}(\mathfrak{U}, M) \to \underline{\check{C}}^r(\mathfrak{U}, M), \quad \eta^r : M|_U \to \underline{\check{C}}^r(\mathfrak{U}, M).$$

Use Problem 5(f) from Problem Set 8 again to prove that these morphisms define a functor,

$$\underline{\check{C}}^\bullet : \sigma \times \Lambda - \Pi - \text{Presh}_{(X,\tau_X)} \to S^\bullet\Lambda - \Pi - \text{Presh}_{(U,\tau_U)},$$

compatible with cosimplicial homotopies for pairs of refinements and together with a natural transformation of cosimplicial objects,

$$\eta^\bullet : \text{const}^\bullet_{M|_U} \to \underline{\check{C}}^\bullet(\mathfrak{U}, M).$$

(b)(The Čech Resolution Preserves Sheaves and Flasques) For every $(U_0, \dots, U_r)$ in $\mathfrak{U}^{r+1}$, denote by $i_{U_0,\dots,U_r} : (\iota(U_0, \dots, U_r), \tau_{\iota(U_0,\dots,U_r)}) \to (U, \tau_U)$ the continuous inclusion map. **Prove** that $\underline{\check{C}}^r(\mathfrak{U}, M)$ is isomorphic as a presheaf of $\Lambda - \Pi$-bimodules to

$$\prod_{(U_0,\dots,U_r)} (\iota_{U_0,\dots,U_r})_* \iota^{-1}_{U_0,\dots,U_r} M.$$

Use Problem 4(a) and (b) to **prove** that $\underline{\check{C}}^r(\mathfrak{U}, M)$ is a sheaf whenever M is a sheaf, and it is flasque whenever M is flasque.

(c)(Localy Acyclicity of the Čech Resolution) Assume now that M is a sheaf. For every τ_X-open subset $V \subset U$ such that there exists $* \in \mathfrak{U}$ with $V \subset \iota(*)$, conclude that $(V, \mathfrak{U}_V)$ refines to $(V, \{V\})$. Using Problem 5(h), Problem Set 8, **prove** that

$$\eta^\bullet_V : \text{const}^\bullet_{M(V)} \to \underline{\check{C}}^\bullet(\mathfrak{U}, M)(V)$$

is a homotopy equivalence. Conclude that for the cochain differential associated to this cosimplicial object,

$$d^r = \sum_{i=0}^r (-1)^i \partial^i_r,$$

the coaugmentation

$$\eta_V : M(V) \to \underline{\check{C}}^\bullet(\mathfrak{U}, M)(V)$$

is an acyclic resolution. Conclude that the coaugmentation of complexes of sheaves of $\Pi - \Lambda$-bimodules,

$$\eta : M|_U \to \underline{\check{C}}^\bullet(\mathfrak{U}, M)$$

is an acyclic resolution.

Now assume that M is flasque. **Prove** that η is a flasque resolution of the flasque sheaf $M|_U$. Using Problem 5(i), **prove** that the cohomology of the complex of $\Lambda(U)-\Pi(U)$-bimodules,

$$\check{H}^n(\mathfrak{U}, M) := h^n(\check{C}^\bullet(\mathfrak{U}, M), d^\bullet)$$

equals $H^\bullet(U, M)$. Using Problem 5(h), **prove** that $H^0(U, M)$ equals $M(U)$ and $H^n(U, M)$ is zero for every integer $n > 0$. Conclude that for every flasque sheaf M of $\Lambda-\Pi$-bimodules, for every open covering $(U, \mathfrak{U})$, $M(U) \to \check{H}^0(\mathfrak{U}, M)$ is an isomorphism and $\check{H}^n(\mathfrak{U}, M)$ is zero for every integer $n > 0$.

Čech Cohomology as a Derived Functor Exercise. Let (X, τ_X) be a topological space. Let U be a τ_X-open set. Let $(U, \iota : \mathfrak{U} \to \tau_U)$ be an open covering. For every presheaf A of $\Lambda-\Pi$-bimodules, denote by $\check{C}^\bullet(\mathfrak{U}, A)$ the object in $\mathbf{Ch}^{\geq 0}(\Lambda - \Pi - \text{Bimod})$ associated to the cosimplicial object.

(a)(Exactness of the Functor of Čech Complexes; The δ-Functor of Čech Cohomologies) Use Problem 5 of Problem Set 8 to **prove** that this is an additive functor

$$\check{C}^\bullet(\mathfrak{U}, -) : \Lambda - \Pi - \text{Presh}_{(X,\tau_X)} \to \text{Ch}^{\geq 0}(\Lambda - \Pi - \text{Bimod}).$$

Prove that for every short exact sequence of presheaves of $\Lambda-\Pi$-bimodules,

$$0 \longrightarrow A' \xrightarrow{q} A \xrightarrow{p} A'' \longrightarrow 0,$$

the associated sequence of cochain complexes,

$$0 \longrightarrow \check{C}^\bullet(\mathfrak{U}, A') \xrightarrow{\check{C}^\bullet(\mathfrak{U},q)} \check{C}^\bullet(\mathfrak{U}, A) \xrightarrow{\check{C}^\bullet(\mathfrak{U},p)} \check{C}^\bullet(\mathfrak{U}, A'') \longrightarrow 0,$$

is a short exact sequence. Use this to prove that the Čech cohomology functor $\check{H}^0(\mathfrak{U}, A) = h^0(\check{C}^\bullet(\mathfrak{U}, A))$ is an additive, left-exact functor, and the sequence of Čech cohomologies,

$$\check{H}^r(\mathfrak{U}, A) = h^r(\check{C}^\bullet(\mathfrak{U}, A)),$$

extend to a cohomological δ-functor from $\Lambda - \Pi - \text{Presh}_{(X,\tau_X)}$ to $\Lambda(U) - \Pi(U) - \text{Bimod}$.

(b)(Effaceability of Čech Cohomology) For every presheaf A of $\Lambda-\Pi$-bimodules, use Problem 5(e) and 5(f) to **prove** that $\theta_A : A \to \widetilde{\Phi(A)}$ is a natural monomorphism of presheaves of $\Lambda-\Pi$-bimodules. Use Problem 5(d) to prove that for every $r \geq 0$, $\check{H}^r(\mathfrak{U}, \widetilde{\Phi(A)})$ is zero. Conclude that $\check{H}^r(\mathfrak{U}, -)$ is effaceable. **Prove** that the cohomological δ-functor $((\check{H}^r(\mathfrak{U}, A))_r, (\delta^r)_r)$ is universal. Conclude that the natural transformation of cohomological δ-functors from the right derived functor of $\check{H}^0(\mathfrak{U}, -)$ to the Čech cohomology δ-functor is a natural isomorphism of cohomological δ-functors.

(c)(Hypotheses of the Grothendieck Spectral Sequence) Denote by

$$\Psi : \Lambda - \Pi - \text{Sh}_{(X,\tau_X)} \to \Lambda - \Pi - \text{Presh}_{(X,\tau_X)},$$

the additive, fully faithful embedding (since we are already using Φ for the forgetful morphism to discontinuous $\Lambda-\Pi$-bimodules). Recall from Problem 6(c) on Problem Set 8 that this extends to an adjoint pair of functors (Sh, Φ). Recall the construction of Sh as a filtering colimit of Čech cohomologies $\check{H}^0(\mathfrak{U}, -)$. Since $\check{H}^0(\mathfrak{U}, -)$ is left-exact, and since $\Lambda - \Pi - \mathrm{Presh}_{(X,\tau_X)}$ satisfies Grothendieck's condition (AB5), **prove** that Sh is left-exact. Use Problem 3(d), Problem Set 5 to **prove** that Ψ sends injective objects to injective objects. Use Problem 5(g) to **prove** that every injective sheaf I of $\Lambda - \Pi$-bimodules is flasque. Use Problem 6(c) to **prove** that $\Psi(I)$ is acyclic for $\check{H}^\bullet(\mathfrak{U}, -)$. **Prove** that the pair of functors Ψ and $\check{H}^0(\mathfrak{U}, -)$ satisfy the hypotheses for the Grothendieck Spectral Sequence. Conclude that there is a convergent, first quadrant cohomological spectral sequence,

$$^I E_2^{p,q} = \check{H}^p(\mathfrak{U}, R^q\Psi(A)) \Rightarrow H^{p+q}(U, A).$$

(d)(The Derived Functors of Ψ are the Presheaves of Sheaf Cohomologies) For every sheaf A of $\Lambda - \Pi$-bimodules, for every integer $r \geq 0$, for every τ_X-open set U, denote $\mathcal{H}^r(A)(U)$ the additive functor $H^r(U, A)$. In particular, $\mathcal{H}^0(A)(U)$ is canonically isomorphic to $A(U)$. Thus, for all τ_X-open sets, $V \subset U$, there is a natural transformation

$$*|_V^U : \mathcal{H}^0(-)(U) \to \mathcal{H}^0(-)(V).$$

Use universality to **prove** that this uniquely extends to a morphism of cohomological δ-functors,

$$*|_V^U : ((\mathcal{H}^r(-)(U))_r, (\delta^r)_r) \to ((\mathcal{H}^r(-)(V))_r, (\delta^r)_r).$$

Prove that for all τ_X-open sets, $W \subset V \subset U$, both the composite morphism of cohomological δ-functors,

$$*|_W^V \circ *|_V^U : ((\mathcal{H}^r(-)(U))_r, (\delta^r)_r) \to ((\mathcal{H}^r(-)(V))_r, (\delta^r)_r) \to ((\mathcal{H}^r(-)(W))_r, (\delta^r)_r),$$

and the morphism of cohomological δ-functors,

$$*|_W^U : ((\mathcal{H}^r(-)(U))_r, (\delta^r)_r) \to ((\mathcal{H}^r(-)(W))_r, (\delta^r)_r),$$

extend the functor $*|_W^V \circ *|_V^U = *|_W^U$ from $\mathcal{H}^0(-)(U)$ to $\mathcal{H}^0(-)(W)$. Use the uniqueness in the universality to conclude that these two morphisms of cohomological δ-functors are equal. **Prove** that $((\mathcal{H}^r(-))_r, (\delta^r)_r)$ is a cohomological δ-functor from $\Lambda - \Pi - \mathrm{Sh}_{(X,\tau_X)}$ to $\Lambda - \Pi - \mathrm{Presh}_{(X,\tau_X)}$. Use Problem 5(h) to **prove** that every flasque sheaf is acyclic for this cohomological δ-functor. Combined with Problem 5(i), **prove** that the higher functors are effaceable, and thus this cohomological δ-functor is universal. Conclude that this the canonical morphism of cohomological δ-functors from the right derived functors of Ψ to this cohomological δ-functor is a natural isomorphism of cohomological δ-functors. In particular, combined with the last part, this gives a convergent, first quadrant spectral sequence,

$$^I E_2^{p,q} = \check{H}^p(\mathfrak{U}, \mathcal{H}^q(A)) \Rightarrow H^{p+q}(U, A).$$

This is the *Čech-to-Sheaf Cohomology Spectral Sequence*. In particular, conclude the existence of monomorphic abutment maps,

$$\check{H}^r(\mathfrak{U}, A) \to H^r(U, A).$$

as well as abutment maps,

$$H^r(U,A) \to H^0(\mathfrak{U},\mathcal{H}^r(A)).$$

(e)(The Colimit of Čech Cohomology with Respect to Refinement) Since Čech complexes are compatible with refinement, and the refinement maps are well-defined up to cosimplicial homotopy, the induced refinement maps on Čech cohomology are independent of the choice of refinement. Use this to define a directed system of Čech cohomologies. Denote the colimit of this direct system as follows,

$$\check{H}^\bullet(U,-) = \operatorname*{colim}_{\mathfrak{U}\in\sigma_{x,U}} \check{H}^\bullet(\mathfrak{U},-).$$

Prove that this extends uniquely to a cohomological δ-functor such that for every open covering $(U,\mathfrak{U})$, the induced sequence of natural transformations,

$$*|_U^{\mathfrak{U}} : ((\check{H}^r(\mathfrak{U},-))_r,(\delta^r)_r) \to ((\check{H}^r(U,-))_r,(\delta^r)_r),$$

is a natural transformation of cohomological δ-functors. Repeat the steps above to deduce the existence of a unique convergent, first quadrant spectral sequence,

$${}^I E_2^{p,q} = \check{H}^p(U,\mathcal{H}^q(A)) \Rightarrow H^{p+q}(U,A),$$

such that for every open covering $(U,\mathfrak{U})$, the natural maps

$$*|_U^{\mathfrak{U}} : \check{H}^p(\mathfrak{U},\mathcal{H}^q(A)) \to \check{H}^p(U,\mathcal{H}^q(A))$$

extend uniquely to a morphism of spectral sequences. In particular, conclude the existence of monomorphic abutment maps

$$\check{H}^r(U,A) \to H^r(U,A)$$

as well as abutment maps

$$H^r(U,A) \to \check{H}^0(U,\mathcal{H}^r(A)).$$

Use the first abutment maps to define subpresheaves $\check{\mathcal{H}}^r(A)$ of $\mathcal{H}^r(A)$ by $V \mapsto \check{H}^r(V,A)$.

(f)(Reduction of the Spectral Sequence; $\check{H}^1(U,A)$ equals $H^1(U,A)$) For every $r>0$, **prove** that the associated sheaf of $\mathcal{H}^r(A)$ is a zero sheaf. (**Hint**. Prove the stalks are zero by using commutation of sheaf cohomology with filtered colimits combined with exactness of the stalks functor.) Conclude that $\check{H}^0(U,\mathcal{H}^r(A))$ is zero. In particular, conclude that the natural abutment map,

$$\check{H}^1(U,A) \to H^1(U,A)$$

is an isomorphism. Thus, also $\check{\mathcal{H}}^1(A) \to \mathcal{H}^1(A)$ is an isomorphism. Use this to produce a "long exact sequence of low degree terms" of the spectral sequence,

$$0 \to \check{H}^2(U,A) \to H^2(U,A) \to \check{H}^1(U,\check{\mathcal{H}}^1(A)) \xrightarrow{\delta} \check{H}^3(U,A).$$

(g)(Sheaves that Are Čech-Acyclic for "Enough" Covers are Acyclic for Sheaf Cohomology) Let $\mathcal{B} \subset \tau_X$ be a basis that is stable for finite intersection. For every open U in $\mathcal{B}$, let Cov_U be a collection of open coverings of U by sets in $\mathcal{B}$ such that Cov_U is cofinal with respect to refinement in $\sigma_{x,U}$. Let A be such that for every U in $\mathcal{B}$, for every $(U, \mathfrak{U})$ in Cov_U, for every $r > 0$, $\check{H}^r(\mathfrak{U}, A)$ is zero. **Prove** that $\mathcal{H}^r(U, A)$ is zero. Use the spectral sequence to inductively **prove** that for every $r > 0$, $\mathcal{H}^r(A)(U)$ is zero, $H^r(U, A)$ is zero and $\mathcal{H}^r(A)(U)$ is zero. Conclude that for every open covering $(X, \iota : \mathfrak{V} \to \mathcal{B})$, the Čech-to-Sheaf Cohomology Spectral Sequence relative to $\mathfrak{V}$ degenerates to isomorphisms

$$\check{H}^r(\mathfrak{V}, A) \to H^r(X, A).$$

If you are an algebraic geometer, let $(X, \mathcal{O}_X)$ be a separated scheme, let $\Lambda = \Pi = \mathcal{O}_X$, let $\mathcal{B}$ be the basis of open affine subsets, let Cov_U be the collection of basic open affine coverings, and let A be a quasi-coherent sheaf. Read the proof that for every basic open affine covering $(U, \mathfrak{U})$ of an affine scheme, for every quasi-coherent sheaf A, $\check{H}^r(\mathfrak{U}, A)$ is zero for all $r \geq 0$ (this is essentially exactness of the Koszul cochain complex for a regular sequence, combined with commutation with colimits). Use this to conclude that quasi-coherent sheaves are acyclic for sheaf cohomology on any affine scheme. Conclude that, on a separated scheme, for every quasi-coherent sheaf, sheaf cohomology is computed as Čech cohomology of any open affine covering.