18.725 SOLUTIONS TO PROBLEM SET 5

Read through all the problems. Write solutions to the "Required Problems", 1, 2, 3, and 4, together with 2 others of your choice to a total of 6 problems. One or two more optional problems may be added to the problem set soon.

Required Problem 1: Let $F : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a dominant, separated morphism of irreducible algebraic varieties, i.e., $F(X) \subset Y$ is dense. The morphism F is *generically finite* if the induced map of fields of fractions, $F^\# : k(Y) \to k(X)$, is a finite, algebraic field extension. The next two problems prove the following proposition. This proposition reduces to the case that $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are affine varieties.

Proposition 0.1. *For every generically finite morphism $F : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$, there exists a dense open subset $U \subset Y$ such that $F : F^{-1}(U) \to U$ is a finite morphism.*

(a) Prove it suffices to consider the case when $(Y, \mathcal{O}_Y)$ is an affine variety.

Solution: Assume the proposition is true when Y is affine. Let Y be an arbitary irreducible variety. Let $U \subset Y$ be a nonempty open affine subset. Because Y is irreducible, U is dense. Because F is dominant, $F^{-1}(U)$ is dense. The restriction $F : F^{-1}(U) \to U$ is a dominant, separated morphism of irreducible algebraic varieties. And the induced morphisms $k(Y) \to k(U)$, $k(X) \to k(F^{-1}(U))$ are isomorphisms, i.e., the extension $k(F^{-1}(U))/k(U)$ is isomorphic to $k(X)/k(Y)$. Therefore $F : F^{-1}(U) \to U$ is generically finite. By hypothesis, there exists a dense open subset $V \subset U$ such that $F : F^{-1}(V) \to V$ is finite. Because V is relatively open in an open subset of Y, V is an open subset of Y. Because V is dense in U and U is dense in Y, V is dense in Y. So the proposition holds for Y.

(b) Prove the following lemma.

Lemma 0.2. *Let $F : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a separated morphism. If $Z \subset X$ is a locally closed subset such that $F|_Z : (Z, \mathcal{O}_Z) \to (Y, \mathcal{O}_Y)$ is proper, then $Z \subset X$ is closed.*

Sketch: Prove $\Delta_{X/Y} : X \to X \times_Y X$ is closed and $\pi_1 : X \times_Y Z \to X$ are closed. Deduce that $\Delta_{X/Y}|_Z : Z \to X \times_Y Z$ is closed, thus $Z = \pi_1(\Delta_{X/Y}|_Z(Z)) \subset X$ is closed.

Solution: Because F is separated, $\Delta_{X/Y} : X \to X \times_Y X$ is a closed morphism. So the restriction, $\Delta_{X/Y} : \Delta_{X/Y}^{-1}(X \times_Y Z) \to X \times_Y Z$ is a closed morphism. But this is precisely $\Delta_{X/Y}|_Z : Z \to X \times_Y Z$, in particular $\Delta_{X/Y}(Z) \subset X \times_Y Z$ is relatively closed. Because $F|_Z : Z \to Y$ is proper, the base-change by $F : X \to Y$ is a closed morphism. The base-change is precisely, $\pi_1 : X \times_Y Z \to X$. Therefore the image under π_1 of the closed subset $\Delta_{X/Y}(Z)$ is closed, i.e., $Z \subset X$ is closed.

(c) Back to the proposition, let $V \subset X$ be a dense open such that $F|_V : V \to Y$ is finite. By Required Problem 4(c) from Problem Set 4, $F|_V$ is proper. Use

(b) to prove $V \subset X$ is open and closed, thus all of X. Deduce it suffices to prove the proposition after replacing X by a dense open affine $W \subset X$ (with $V = W \cap F^{-1}(U)$).

Solution: By (b), $V \subset X$ is open and closed, thus all of X. Let $W \subset X$ be a dense open affine. By a similar argument to (a), $F|_W : W \to Y$ is separated, dominant and generically finite. Assume the proposition holds for $F|_W$, i.e., there exists a dense open subset $U \subset Y$ such that $F|_W : (F|_W)^{-1}(U) \to U$ is finite. Define $V = (F|_W)^{-1}(U) = W \cap F^{-1}(U)$. By the argument above, $V \subset F^{-1}(U)$ is open and closed, thus all of $F^{-1}(U)$. Therefore $F : F^{-1}(U) \to U$ is finite.

Required Problem 2: This is the follow-up to Required Problem 1. You may assume all parts of that problem. Thus $F : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a generically finite morphism of affine varieties. Let $a_1, \dots, a_n \in k[X]$ be generators for $k[X]$ as a k-algebra. Because $k(X)/k(Y)$ is an algebraic extension, each a_i satisfies a polynomial equation with coefficients in $k[Y]$. Clearing denominators, each a_i satisfies a polynomial equation $f_i(t)$ with coefficient in $k[Y]$,

$$f_i(t) = c_{i,d_i} t^{d_i} + \cdots + c_{i,1} t + c_{i,0},$$

where each $c_{i,d_i} \neq 0$. Define $c = c_{1,d_1} \cdots c_{n,d_n}$. Prove that for $U = D(c) \subset Y$ and $F^{-1}(U) = D(F^{\#}(c)) \subset X$, $F : F^{-1}(U) \to U$ is a finite morphism.

Solution: The coordinate ring of $F^{-1}(D(c)) = D(F^{\#}c)$ is generated as a k-algebra by $a_1/1, \dots, a_n/1$ and $1/F^{\#}c$. Of course $1/F^{\#}c$ satisfies the monic polynomial over $k[D(c)]$, $t - 1/c$. For every $i = 1, \dots, n$, define $c_i' = c/c_{i,d_i} = \prod_{j \neq i} c_{j,d_j}$. Then $a_i/1$ satisfies the monic polynomial equation over $k[D(c)]$,

$$g_i(t) = t^{d_i} + (c_i'/c) c_{i,d_i-1} t^{d_i-1} + \cdots + (c_i'/c) c_{i,1} t + (c_i'/c) c_{i,0}.$$

Therefore each of the generators $a_1/1, \dots, a_n/1$ and $1/F^{\#}c$ are integral over $k[D(c)]$. By standard commutative algebra, this implies that $k[F^{-1}(D(c))]$ is an integral extension of $k[D(c)]$. By Prop. 14.18, $F : F^{-1}(D(c)) \to D(c)$ is a finite morphism.

Required Problem 3: Let $F : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a quasi-compact, separated morphism of algebraic varieties. For every $p \in X$, denote $X_p = F^{-1}(F(p))$. For every integer $e \geq 0$, define $U_e(X, F) = \{p \in X | \dim(X_p, p) \leq e\}$. In this problem, you will prove the following proposition of Chevalley. This problem has many parts. Do the parts you can; it will be graded generously.

Proposition 0.3. *For every integer $e \geq 0$, $U_e(X, F) \subset X$ is an open subset.*

(a) Prove it suffices to consider the case where $(Y, \mathcal{O}_Y)$ is affine and $(X, \mathcal{O}_X)$ is affine.

Solution: For every open subset $V \subset Y$ and every open subset $W \subset F^{-1}(V)$, denoting $F|_W : W \to V$, $W_p = X_p \cap W$ is an open subset, thus $\dim(W_p, p) = \dim(X_p, p)$. Therefore $U_e(W, F|_W) = U_e(X, F) \cap W$. Let V_α be a covering of Y by open affine subsets. For every α, let $(W_{\alpha,\beta})$ be a covering of $F^{-1}(V_\alpha)$ by open affine subsets. If each $U_e(W_{\alpha,\beta}, F|_W) \subset W_{\alpha,\beta}$ is open, then each $U_e(W_{\alpha,\beta}, F|_W) \subset X$ is open, and therefore,

$$U_e(X, F) = \cup_{(\alpha,\beta)} (U_e(X, F) \cap W_{\alpha,\beta}) = \cup_{(\alpha,\beta)} U_e(W_{\alpha,\beta}, F|_W),$$

is an open subset of X.

(a) Prove the following simple lemma from topology.

Lemma 0.4. *Let $U \subset X$ be a topological space such that for every closed subset $C \subset X$ containing $X - U$ and intersecting U, there is a nonempty relatively open subset $O \subset C$ contained in $U \cap C$. Then $U \subset X$ is open.*

Hint: For the closure C of $X - U$, prove $C \cap U = \emptyset$, hence $U = X - C$ is open.

Solution: Denote by C the closure of $X - U$ in X. This certainly contains $X - U$. If $C \cap U \neq \emptyset$, then by hypothesis there exists a nonempty relatively open subset $O \subset C$ contained in $U \cap C$. Let $D = C - O$. This is relatively closed in C which is closed in X. Thus D is a closed subset of X. And D contains $X - U$. Thus $C \subset D$. But, by construction, D is propertly contained in C. This contradiction proves $C \cap U = \emptyset$. So $C = X - U$, i.e., $U = X - C$ is an open subset of X.

(b) Let $Z \subset X$ be a closed set containing $X - U_e(X, F)$. Prove $U_e(Z, F|_Z) = U_e(X, F) \cap Z$. Combined with (a), reduce Proposition 0.3 to the proposition.

Proposition 0.5. *For every integer $e \geq 0$, if $U_e(X, F) \subset X$ is nonempty, it contains a nonempty open subset of X.*

Sketch: Denote $V = X - Z$. For every $p \in X$ and every irreducible component $T \subset X_p$ containing p, if $T \cap V$ is nonempty, prove $\dim(T) \leq e$. Conclude if $p \in U_e(Z, F|_Z)$, every irreducible component of X_p has dimension $\leq e$.

Solution: For every $p \in X$ and every irreducible component $T \subset X_p$, if there exists $q \in T \cap V$, then $T \subset X_q$ is an irreducible component passing through q. Because $q \in V \subset U_e(X, F)$, $\dim(T, q) \leq e$. By Corollary 18.6, $\dim(T) \leq e$, in particular $\dim(T, p) \leq e$.

Let $p \in U_e(Z, F|_Z)$ and let $T \subset X_p$ be an irreducible component. If T intersects V, by the last paragraph $\dim(T, p) \leq e$. If $T \cap V = \emptyset$, then $T \subset Z$ and thus $T \subset Z_p$. By hypothesis, $\dim(Z_p, p) \leq e$, thus $\dim(T, p) \leq e$. Since this holds for every irreducible component T of X_p containing p, $\dim(X_p, p) \leq e$, i.e., $p \in U_e(X, F)$.

Assume Proposition 0.5. Then, for every closed subset $Z \subset X$ containing $X - U_e(X, F)$, if $Z \cap U_e(X, F) = U_e(Z, F|_Z)$ is nonempty, it contains a nonempty open subset of Z. By Lemma 0.4, $U_e(X, F)$ is an open subset, i.e., Proposition 0.3 holds. Therefore it suffices to prove Proposition 0.5.

(c) By considering the restriction of F to each irreducible component of X, reduce Proposition 0.5 to the case that X and Y are irreducible and F is dominant.

Solution: This is straightforward.

(d), $e = 0$ If $U_0(X, F)$ is nonempty, use Corollary 17.4 to prove $\dim(X) = \dim(Y)$, thus $k(X)/k(Y)$ is algebraic. Assuming Required Problem 2, prove $U_0(X, F)$ contains a nonempty open subset of X. Deduce Proposition 0.3 for $e = 0$.

Solution: Because F is a dominant morphism of irreducible varieties, there is an induced map $F^\# : k(Y) \to k(X)$. Let $p \in U_0(X, F)$. By Corollary 17.4, $\dim(X, p) \leq \dim(Y, F(p))$. Thus $\text{tr.deg}_k(k(X)) \leq \text{tr.deg}_k(k(Y))$ by Corollary 18.5, i.e., $F^\#$ is an algebraic field extension. Since it is also a finitely generated field extension, it is a finite algebraic field extension, hence F is a generically finite morphism. By Required Problem 2, there exists a dense open subset $V \subset Y$ such that $F : F^{-1}(V) \to V$ is finite. Therefore $F^{-1}(V) \subset X$ is a nonempty open subset contained in $U_0(X,F)$. Together with (a), (b) and (c) this proves Proposition 0.3 for $e=0$.

(e), $e>0$ Let $p\in U_e(X,F)$. Let $f_1,\dots,f_e\in\mathfrak{m}_p\mathcal{O}_{X_p,p}$ be a system of parameters. There exists an open subset $p\in V\subset X$ and elements $g_1,\dots,g_e\in\mathcal{O}_X(V)$ whose images in $\mathcal{O}_{X_p,p}$ are $f_1,\dots,f_e$. Define $G=(g_1,\dots,g_e):V\to\mathbb{A}^e_k$, and consider $F\times G:V\to Y\times\mathbb{A}^e_k$. Prove $p\in U_0(X,F\times G)\subset U_e(X,F)$. Assuming (d) which proves Proposition 0.3 for $e=0$, prove $U_e(X,G)$ contains an open subset of X, thus proving Proposition 0.3.

Solution: By Krull's Hauptidealsatz, Theorem 17.1, there exists a system of parameters $f_1,\dots,f_e\in\mathfrak{m}_p\mathcal{O}_{X_p,p}$. There exists an open affine subset $p\in V\subset X$ and elements $g_1,\dots,g_e\in\mathcal{O}_X(V)$ germs at p are $f_1,\dots,f_e$. Define $G=(g_1,\dots,g_e):V\to\mathbb{A}^e_k$ and consider $F\times G:V\to Y\times\mathbb{A}^e_k$. By construction, $F\times G(p)=(F(p),0)$. Of course $(F\times G)^{-1}(F(p),0)=F^{-1}(F(p))\cap G^{-1}(0)$, i.e., $F^{-1}(F(p))\cap\mathbb{V}(g_1,\dots,g_e)$. Because $f_1,\dots,f_e\in\mathfrak{m}_p\mathcal{O}_{X_p,p}$ is a system of parameters, $\mathfrak{m}_p\subset\mathcal{O}_{X_p}(V\cap X_p)$ is a minimal prime over $\langle g_1,\dots,g_p\rangle\mathcal{O}_{X_p}(V\cap X_p)$. By the ideal-variety correspondence, $\{p\}$ is an irreducible component of $(F\times G)^{-1}(F(p),0)$, i.e., $p\in U_0(V,F\times G)$. By (d), $U_0(V,F\times G)\subset V$ is open, thus it is nonempty and open. Let $q\in U_0(V,F\times G)$. By Corollary 17.4 applied to $G:(F^{-1}(F(q)),q)\to(\mathbb{A}^e_k,G(q))$, $\dim(X_q,q)\le e$, i.e., $q\in U_e(X,F)$. Thus $U_0(V,F\times G)\subset U_e(X,F)$ is a subset that is a nonempty open subset of X. Together with (a), (b) and (c), this proves Proposition 0.3.

Required Problem 4: Before solving this problem, read through Problem 5 (although you don't have to solve it). Let X be an algebraic variety. Let $n\ge 0$ be an integer.

Definition 0.6. An Abelian cone ζ is a *vector bundle of rank n on X* if for every point $p\in X$ there exists an open subset $p\in U\subset X$ such that $\pi^{-1}(U)$ is isomorphic to $U\times\mathbb{A}^n$ as an Abelian cone over U.

(a) Let $\zeta=(\pi:E\to X,+,\cdot,0)$ be a vector bundle of rank n on X and let $F:Y\to X$ be a regular morphism. Denote $E_Y:=Y\times_X E$, and $\pi_Y:E_Y\to Y$ is the projection. Prove there are "natural" choices of $+$, $\cdot$, and 0 such that $(\pi_Y:E_Y\to Y,+,\cdot,0)$ is a vector bundle of rank n on Y. This is the *pullback vector bundle*, denoted $F^*\zeta$. Indicate why F^* is a functor from the category of vector bundles on X to the category of vector bundles on Y (but you don't have to prove this). Of course the same is true for Abelian cones as well – prove this if you prefer.

Solution: Comparing universal properties, the following morphisms are inverse isomorphisms,

$$(\mathrm{pr}_Y\circ\mathrm{pr}_1)\times((\mathrm{pr}_E\circ\mathrm{pr}_1)\times(\mathrm{pr}_E\circ\mathrm{pr}_2)):(Y\times_X E)\times_Y(Y\times_X E)\\ \to Y\times_X(E\times_X E),$$

$$(\mathrm{pr}_Y\times(\mathrm{pr}_1\circ\mathrm{pr}_{E\times_X E}))\times(\mathrm{pr}_Y\times(\mathrm{pr}_2\circ\mathrm{pr}_{E\times_X E})):Y\times_X(E\times_X E)\\ \to(Y\times_X E)\times_Y(Y\times_X E).$$

Denote the first morphism by α. Define $+_Y:E_Y\times_Y E_Y\to E_Y$ to be the composition of α with,

$$\mathrm{pr}_Y\times(+_X\circ\mathrm{pr}_{E\times_X E}):Y\times_X(E\times_X E)\to Y\times_X E.$$

Similarly, define $\cdot_Y:\mathbb{A}^1_k\times E_Y\to E_Y$ to be,

$$\mathrm{pr}_Y\times(\cdot_X\circ(\mathrm{pr}_{\mathbb{A}^1_k}\times\mathrm{pr}_E)):\mathbb{A}^1_k\times Y\times_X E\to Y\times_X E.$$

Finally, define $0_Y : Y \to E_Y$ to be,

$$\mathrm{Id}_Y \times (0_X \circ F) : Y \to Y \times_X E.$$

By definition, each of $+_Y$, $\cdot_Y$ and 0_Y is compatible with projection to Y. Moreover, for every $y \in Y$ the fiber over y of $(\pi_Y : E_Y \to Y, +_Y, \cdot_Y, 0_Y)$ is canonically isomorphic to the fiber over $F(y)$ of $(\pi : E \to X, +_X, \cdot_X, 0_X)$, which by hypothesis is the canonical structure on $\mathbb{A}^n_k$ for some $n \geq 0$. Thus $(\pi_Y : E_Y \to Y, +_Y, \cdot_Y, 0_Y)$ is an Abelian cone.

Let $\phi : E_1 \to E_2$ be a homomorphism of Abelian cones on X. Define $F^*\phi : Y \times_X E_1 \to Y \times_X E_2$ to be $\mathrm{pr}_Y \times (\phi \circ \mathrm{pr}_{E_1})$. By definition, this is compatible with projection to Y. Moreover, for every $y \in Y$ the restriction of $F^*\phi$ to the fiber over y agrees with the restriction of ϕ to the fiber over $F(y)$, from which it immediately follows that $F^*\phi$ is compatible with $+_Y$, $\cdot_Y$ and 0_Y for $Y \times_X E_1$ and $Y \times_X E_2$. (N.B.: This argument would not be sufficient if Y and X were *schemes*. Then more diagram-chasing would be involved.)

Together $\zeta \mapsto F^*\zeta$ and $\phi \mapsto F^*\phi$ define a functor from the category of Abelian cones over X to the category of Abelian cones over Y. Moreover, for every $n \geq 0$, the variety $F^*(X \times \mathbb{A}^n_k) = Y \times_X (X \times \mathbb{A}^n_k)$ is canonically isomorphic to $Y \times \mathbb{A}^n_k$ and this isomorphism is an isomorphism of Abelian cones over Y.

Now suppose that E is a vector bundle of rank n on X. For every $p \in Y$, there exists an open subset $F(p) \in V \subset X$ such that $E|_V$ is isomorphic to $V \times \mathbb{A}^n_k$. Therefore $F^*E|_{F^{-1}(V)} = (F|_V)^*(E|_V)$ is isomorphic to $(F|_V)^*(V \times \mathbb{A}^n_k)$, which is canonically isomorphic to $F^{-1}(V) \times \mathbb{A}^n_k$. Therefore F^*E is a vector bundle of rank n on Y.

(b) Given a second regular morphism $G : Z \to Y$, prove there is a natural isomorphism of functors from the category of vector bundles on X to the category of vector bundles on Z, $\theta_{G,F}(\zeta) : G^*F^*\zeta \to (F \circ G)^*\zeta$. Given a third regular morphism $H : W \to Z$, prove that,

$$\theta_{H,G\circ F}(\zeta) \circ H^*(\theta_{G,F}(\zeta)) = \theta_{H\circ G,F}(\zeta) \circ \theta_{H,G}(F^*\zeta).$$

Solution: There are canonical inverse isomorphisms,

$$\begin{gathered}\mathrm{pr}_Z \times (\mathrm{pr}_E \circ \mathrm{pr}_{Y\times_X E}) : Z \times_Y (Y \times_X E) \to Z \times_X E,\\ \mathrm{pr}_Z \times ((G \circ \mathrm{pr}_Z) \times \mathrm{pr}_E) : Z \times_X E \to Z \times_Y (Y \times_X E).\end{gathered}$$

Denote the first morphism by $\theta_{G,F}(E)$. In fact this is defined for every morphism $\pi : E \to X$, not just for the projection morphism of an Abelian cone. For every pair $\pi_1 : E_1 \to X$, $\pi_2 : E_2 \to X$ and every morphism $\phi : E_1 \to E_2$ such that $\pi_2 \circ \phi = \pi_1$, define $F^*\phi = \mathrm{pr}_Y \times (\phi \circ \mathrm{pr}_{E_1}) : Y \times_X E_1 \to Y \times_X E_2$ as above. By definition, $\theta_{G,F}(E_2) \circ G^*(F^*\phi)$ equals,

$$(\mathrm{pr}_Z \times (\mathrm{pr}_{E_2} \circ \mathrm{pr}_{Y\times_X E_2})) \circ (\mathrm{pr}_Z \times ((\mathrm{pr}_Y \times (\phi \circ \mathrm{pr}_{E_1})) \circ \mathrm{pr}_{Y\times_X E_1})),$$

which equals $\mathrm{pr}_Z \times (\phi \circ \mathrm{pr}_{E_1} \circ \mathrm{pr}_{Y\times_X E_1})$. On the other hand, this equals,

$$(\mathrm{pr}_Z \times (\phi \circ \mathrm{pr}_{E_1})) \circ (\mathrm{pr}_Z \times (\mathrm{pr}_{E_1} \circ \mathrm{pr}_{Y\times_X E_1})).$$

Therefore $\theta_{G,F}(\phi)$ is a natural transformation of functors.

To prove that,

$$\theta_{H,G\circ F}(\zeta) \circ H^*(\theta_{G,F}(\zeta)) = \theta_{H\circ G,F}(\zeta) \circ \theta_{H,G}(F^*\zeta),$$

observe that both equal $\mathrm{pr}_W \times (\mathrm{pr}_E \circ \mathrm{pr}_{Y\times_X E} \circ \mathrm{pr}_{Z\times_Y(Y\times_X E)})$.

(c) For every integer $n \geq 0$, let $X = \mathbb{P}^n_k$, let ζ be the trivial vector bundle of rank $n+1$, i.e., $\mathbb{P}^n_k \times \mathbb{A}^{n+1}_k$, and denote,

$$E = \{([a_0,\dots,a_n],(b_0,\dots,b_n)) \in \mathbb{P}^n_k \times \mathbb{A}^{n+1}_k | \text{for every } 0 \leq i < j \leq n,\ a_j b_i - a_i b_j = 0\}.$$

Prove that E is a sub-Abelian cone of ζ and is, in fact, a vector bundle of rank 1 on X. This is the *tautological rank 1 subbundle* on $\mathbb{P}^n_k$.

Solution: Given a variety X and an Abelian cone $\pi_1 : E_1 \to X$, to prove a subvariety $E_2 \subset E_1$ is a sub-Abelian cone, it suffices to prove the following,

(i) $0(X) \subset E_2$,
(ii) $+(E_2 \times_X E_2) \subset E_2$, and
(iii) $\cdot(\mathbb{A}^1_k \times E_2) \subset E_2$.

Let $E_1 = \mathbb{P}^n_k \times \mathbb{A}^{n+1}_k$ and let $E_2 = E$. In this case, $\mathbb{P}^n_k \times \{0\}$ clearly satisfies the defining equations of E, so $0(\mathbb{P}^n_k) \subset E$. For every element $p = [a_0,\dots,a_n] \in \mathbb{P}^n_k$, the fiber of E over p is just

$$\{(\lambda a_0,\dots,\lambda a_n) \in \mathbb{A}^{n+1}_k | \lambda \in \mathbb{A}^1_k\}.$$

This is clearly stable under addition of pairs of elements and stable under scalar multiplication. Thus $E \subset \mathbb{P}^n_k \times \mathbb{A}^{n+1}_k$ satisfies (i), (ii) and (iii), i.e., it is a sub-Abelian cone.

For every integer $i = 0,\dots,n$, there is a morphism $\phi_n : D_+(x_i) \times \mathbb{A}^1_k \to E|_{D_+(x_i)}$ defined by,

$$([a_0,\dots,a_n],\lambda) \mapsto ([a_0,\dots,a_n],(\lambda a_0/a_i,\dots,\lambda a_n/a_i)).$$

It is straightforward to check this is an isomorphism of Abelian cones. Therefore, since the sets $D_+(x_i)$ cover $\mathbb{P}^n_k$, E is a vector bundle of rank 1 on $\mathbb{P}^n_k$.

Problem 5, Abelian cones: Let X be an algebraic variety.

Definition 0.7. An *Abelian cone over* X is a datum $\zeta = (\pi, +, \cdot, 0)$ of a regular morphism of algebraic varieties $\pi : E \to X$, a regular morphism $+ : E \times_X E \to E$, denoted $(e_1, e_2) \mapsto e_1 + e_2$, a regular morphism $\cdot : \mathbb{A}^1_k \times E \to E$, denoted $(\lambda, e) \mapsto \lambda \cdot e$, and a regular morphism $0 : X \to E$, denoted $x \mapsto 0_x$, satisfying the following axioms.

(i) For every $(e_1, e_2) \in E \times_X E$, $\pi(e_1 + e_2) = \pi(e_1) = \pi(e_2)$; for every $e \in E$ and $\lambda \in \mathbb{A}^1_k$, $\pi(\lambda \cdot e) = \pi(e)$; and for every $x \in X$, $\pi(0_x) = x$.
(ii) For every $x \in X$, denoting $E_x = \pi^{-1}(x)$, there exists an integer $n \geq 0$ (depending on x) such that the datum $(E_x, +, \cdot, 0_x)$ is isomorphic to the standard datum $(\mathbb{A}^n, +, \cdot, 0)$.

The variety E is called the *total space*, the morphism π is the *projection*, and the morphism 0 is the *zero section*. If ζ, η are Abelian cones over X, a *homomorphism of cones from* ζ *to* η is a regular morphism $F : E_\zeta \to E_\eta$ such that $\pi_\eta \circ F = \pi_z eta$, such that $F(e_1 + e_2) = F(e_1) + F(e_2)$, and such that $F(\lambda \cdot e) = \lambda \cdot F(e)$, for every e_1, e_2, e and λ.

(a) For every integer $n \geq 0$, for $E = X \times \mathbb{A}^n_k$ and $\pi = \pi_1 : X \times \mathbb{A}^n_k \to X$, prove there is a "natural" choice of $+$, $\cdot$ and 0 so that $(\pi, +, \cdot, 0)$ is an Abelian cone. This is called the *trivial vector bundle of rank* n.

Solution: Let $F : X \to \mathbb{A}_k^0$ be the unique morphism. The datum $(\pi : \mathbb{A}_k^n \to \mathbb{A}_k^0, +, \cdot, 0)$ is an Abelian cone over $\mathbb{A}_k^0$, where $+$, $\cdot$ and 0 are the usual morphisms. By Problem 4(a), there is a natural structure of vector bundle on $F^*\mathbb{A}_k^n = \mathbb{A}_k^n \times X$.

Let ζ be an Abelian cone over X. The *sheaf of sections of* ζ is the sheaf of sets $\mathcal{E}_{\text{sec}}$ on X whose sections over each open $U \subset X$ are the regular morphisms $s : U \to \pi^{-1}(U)$ such that $\pi \circ s = \text{Id}_U$. The *sheaf of functionals of* ζ is the sheaf of sets $\mathcal{E}_{\text{func}}$ on X whose sections over each open $U \subset X$ are the Abelian cone morphisms $F : \pi^{-1}(U) \to U \times \mathbb{A}_k^1$.

(b) Prove that for every open subset $U \subset X$, the morphisms $+$ and $\cdot$ naturally determine a structure of $\mathcal{O}_X(U)$-module on $\mathcal{E}_{\text{sec}}(U)$ and $\mathcal{E}_{\text{func}}(U)$, and for every inclusion $V \subset U$, the restriction maps $\mathcal{E}_{\text{sec}}(U) \to \mathcal{E}_{\text{sec}}(V)$ and $\mathcal{E}_{\text{func}}(U) \to \mathcal{E}_{\text{func}}(V)$ are homomorphisms of $\mathcal{O}_X(U)$-modules. Such a sheaf is called a *sheaf of* $\mathcal{O}_X$*-modules.*

Solution: First consider $\mathcal{E}_{\text{sec}}(U)$. For every pair of sections $s_1, s_2 \in \mathcal{E}_{\text{sec}}(U)$, the morphism $s_1 + s_2 := + \circ (s_1 \times s_2)$ is a section in $\mathcal{E}_{\text{sec}}(U)$. For every element $f \in \mathcal{O}_X(U)$ and every section $s \in \mathcal{E}_{\text{sec}}(U)$, the element $f \cdot s := \cdot \circ (f \times s)$ is a section in $\mathcal{E}_{\text{sec}}(U)$. In particular, define $s_1 - s_2 := s_1 + (-1 \cdot s_2)$. The restriction of 0 to U is a section in $\mathcal{E}_{\text{sec}}(U)$. It remains to prove the axioms for an $\mathcal{O}_X(U)$-module: associativity, commutativity and cancellation for addition, distributivity of scalar multiplication and addition in $\mathcal{O}_X(U)$ and in $\mathcal{E}_{\text{sec}}(U)$, and distributivity of scalar multiplication and multiplication in $\mathcal{O}_X(U)$. Each of these is implied by a corresponding axiom for the morphisms $+$, $\cdot$ and 0, e.g., distributivity of scalar multiplication and addition in $\mathcal{E}_{\text{sec}}(U)$ is commutativity of the following diagram,

$$\begin{tikzcd}
\mathbb{A}_k^1 \times (E \times_X E) \arrow[rr, "(\cdot\circ \text{pr}_1 \circ \text{pr}_{E\times E}) \times (\cdot \circ \text{pr}_2 \circ \text{pr}_{E\times E})"] \arrow[d, "\text{Id}_{\mathbb{A}_k^1} \times +"'] & & E \times_X E \arrow[d, "+"] \\
\mathbb{A}_k^1 \times E \arrow[rr, "\cdot"] & & E
\end{tikzcd}$$

Every such compatibility is an equality of morphisms compatible with projection to the base X. To prove the morphisms are equal, it suffices to restrict to the fiber over every point in X and check equality of the restrictions. But then the corresponding axiom for $(\mathbb{A}_k^n, +, \cdot, 0)$ implies equality of the restrictions. Thus $\mathcal{E}_{\text{sec}}(U)$ is an $\mathcal{O}_X(U)$-module.

For $\mathcal{E}_{\text{func}}(U)$, given elements $F_1, F_2 \in \mathcal{E}_{\text{func}}(U)$, define $F_1 + F_2$ to be the composition of $\Delta_{E/X} : E \to E \times_X E$, of $\pi \times ((F_1 \circ \text{pr}_1) \times (F_2 \circ \text{pr}_2)) : E \times_X E \to X \times (\mathbb{A}_k^1 \times \mathbb{A}_k^1)$, and of $\text{pr}_1 \times (+ \circ \text{pr}_1) : X \times (\mathbb{A}_k^1 \times \mathbb{A}_k^1) \to X \times \mathbb{A}_k^1$. Given $f \in \mathcal{O}_X(U)$ and $F \in \mathcal{E}_{\text{func}}(U)$, define $f \cdot F$ to be the composition of $F : E \to X \times \mathbb{A}_k^1$, of $\text{pr}_1 \times ((f \circ \text{pr}_1) \times \text{pr}_2) : X \times \mathbb{A}_k^1 \to X \times (\mathbb{A}_k^1 \times \mathbb{A}_k^1)$, and of $\text{pr}_1 \times (\cdot \circ \text{pr}_2) : X \times (\mathbb{A}_k^1 \times \mathbb{A}_k^1) \to X \times \mathbb{A}_k^1$. Finally, define $0 : E \to X \times \mathbb{A}_k^1$ to be the composition of $\pi : E \to X$ and $\text{Id}_X \times 0 : X \to X \times \mathbb{A}_k^1$. As above, the axioms for an $\mathcal{O}_X(U)$-module reduce to equality of certain morphisms compatible with projection to X. Equality can be checked after restriction to fibers over points of X. And then equality follows from the corresponding axiom for $\mathbb{A}_k^n$. Thus $\mathcal{E}_{\text{func}}(U)$ is an $\mathcal{O}_X(U)$-module.

For every inclusion $V \subset U$, it is clear that the addition and scaling maps and the zero element are compatible with restriction.

(c) Let $F : E_\zeta \to E_\eta$ be a homomorphism of Abelian cones over X. Prove there are induced homomorphisms of sheaves of $\mathcal{O}_X$-modules, $F_* : \mathcal{E}_{\zeta,\text{sec}} \to \mathcal{E}_{\eta,\text{sec}}$ and $F^* : \mathcal{E}_{\eta,\text{func}} \to \mathcal{E}_{\zeta,\text{func}}$.

Solution: For every open subset $U \subset X$, define $F_*(U) : \mathcal{E}_{\zeta,\text{sec}}(U) \to \mathcal{E}_{\eta,\text{sec}}(U)$ by mapping an elements $s : U \to E_\zeta$ to $F \circ s : U \to E_\eta$. Similarly, define $F^* : \mathcal{E}_{\eta,\text{func}}(U) \to \mathcal{E}_{\zeta,\text{func}}(U)$ by mapping an element $\phi : U \times_X E_\eta \to U \times \mathbb{A}^1_k$ to $\phi \circ (\text{pr}_1 \times (F \circ \text{pr}_2)) : U \times_X E_\zeta \to U \times \mathbb{A}^1_k$. The axioms for F_* and F^* to be $\mathcal{O}_X(U)$-module homomorphisms are dealt with in a similar manner to (b). And these maps are clearly compatible with restriction for every inclusion $V \subset U$. Thus they are homomorphisms of sheaves of $\mathcal{O}_X$-modules.

(d) Let $X = \mathbb{A}^1_k$, let ξ be the trivial vector bundle of rank 1, $X \times \mathbb{A}^1_k$, whose total space is just $\mathbb{A}^2_k$. Denote by $E \subset \mathbb{A}^2_k$ the closed subvariety $\mathbb{V}(xy)$. Prove that E is a sub-Abelian cone of ζ. Denote this by η.

Solution: Quite generally, let $\zeta = (\pi_\zeta : E_\zeta \to X, +_\zeta, \cdot_\zeta, 0_\zeta)$ be an Abelian cone over X. Let $F \subset E_\zeta$ be a closed subvariety. If $+_\zeta(F \times_X F) \subset F$ and if $\cdot_\zeta(\mathbb{A}^1_k \times F) \subset F$, then $\eta := ((\pi_\zeta)|_F : F \to X, (+_\zeta)|_{F \times_X F}, (\cdot_\zeta)|_{\mathbb{A}^1_k \times F})$ is an Abelian cone over X: the *induced Abelian subcone*. In the case above, denote $F = \mathbb{V}(xy)$. For every $((a, b_1), (a, b_2)) \in F \times_X F$, $+((a, b_1), (a, b_2)) = (a, b_1 + b_2)$ and $a(b_1 + b_2) = ab_1 + ab_2 = 0 + 0 = 0$. Therefore $(a, b_1 + b_2) \in F$. Similarly, for every $c \in \mathbb{A}^1_k$, $\cdot(c, (a, b)) = (a, cb)$ and $a(cb) = c(ab) = c0 = 0$. Therefore $(a, cb) \in F$. So η is an Abelian subcone.

(e) Denote by ζ the trivial vector bundle on X of rank 0, i.e., $X \times \mathbb{A}^0_k$, and denote by $F : \zeta \to \eta$ the unique homomorphism of Abelian cones over X. Prove that F_* is an isomorphism, but F^* is not an isomorphism. Because the sheaves of functionals "detect" homomorphisms of Abelian cones that sheaves of sections do not detect, they are used more often in algebraic geometry (sheaves of sections are frequently used for vector bundles, especially in other branches of geometry, but rarely used for Abelian cones that are not vector bundles).

Solution: Let $U \subset X$ be any open subset. Let $s : U \to U \times_X F$ be any morphisms. The subset $V = U \cap D(x) \subset U$ is open, and $s|_V$ is a morphism to $\mathbb{V}(y) \subset D(x) \times \mathbb{A}^1_k$, i.e., $s|_V$ is a morphism to $D(x) \times \{0\}$. Therefore $s|_V = 0_V$. Moreover $V \subset U$ is dense because X is irreducible and, of course, F is separated. Therefore $s = 0_U$. In particular, $F_*(U) : \mathcal{E}_{\zeta,\text{sec}}(U) \to \mathcal{E}_{\eta,\text{sec}}(U)$ is an isomorphism for every open $U \subset X$, i.e., F_* is an isomorphism of sheaves of $\mathcal{O}_X(U)$-modules.

On the other hand, the defining inclusion $i : U \times_X F \to U \times \mathbb{A}^1_k$ is an element of $\mathcal{E}_{\eta,\text{func}}(U)$. If $0 \in U$, then the restriction of i to $\pi^{-1}(0)$ is an isomorphism. In particulat, $i \neq 0_U$. But $F^*(U)(i) = F^*(U)(0_U) = 0_U$. So $F^*(U)$ is not injective, in particular it is not an isomorphism. Thus F^* is not an isomorphism of sheaves of $\mathcal{O}_X(U)$-modules. In fact, the stalk of $\mathcal{E}_{\eta,\text{func}}(U)$ at $0 \in X$ as a module over $\mathcal{O}_{X,0} = k[x]_{(x)}$ is isomorphic to the residue field $k = k[x]_{(x)}/\mathfrak{m}_{X,0}k[x]_{(x)}$. The sheaf $\mathcal{E}_{\eta,\text{func}}(U)$ is an example of a *skyscraper sheaf*.

Problem 6: Let X be a variety and let $\zeta = (\pi : E \to X, +, \cdot, 0)$ be an Abelian cone on X. For every open set $U \subset X$ there is a pairing $\langle -, - \rangle_U : \mathcal{E}_{\text{func}}(U) \times \mathcal{E}_{\text{sec}}(U) \to \mathcal{O}_X(U)$ which maps a pair (F, s) of a functional $F : \pi^{-1}(U) \to U \times \mathbb{A}^1_k$ and a section $s : U \to \pi^{-1}(U)$ to the regular function $\text{pr}_2 \circ F \circ s$.

(a) Prove $\langle -, - \rangle_U$ is bilinear for the action of $\mathcal{O}_X(U)$ on each module.

(b) Prove that for every inclusion $V \subset U$, $\langle F|_V, s|_V \rangle_V = (\langle F, s \rangle_U)|_V$. Deduce that for every element $x \in X$, there is a pairing of stalks $\langle -, - \rangle_x : (\mathcal{E}_{\text{func}})_x \times (\mathcal{E}_{\text{sec}})_x \to \mathcal{O}_{X,x}$.

(c) If ζ is a vector bundle, prove that for every $x \in X$, the pairing $\langle -, - \rangle_x$ is a *perfect pairing*, i.e., the following induced homomorphisms of $\mathcal{O}_{X,x}$-modules are isomorphisms,

$$(\mathcal{E}_{\text{sec}})_x \to \operatorname{Hom}_{\mathcal{O}_{X,x}}((\mathcal{E}_{\text{func}})_x, \mathcal{O}_{X,x}),$$
$$(\mathcal{E}_{\text{func}})_x \to \operatorname{Hom}_{\mathcal{O}_{X,x}}((\mathcal{E}_{\text{sec}})_x, \mathcal{O}_{X,x}).$$

Problem 7, The universal property of projective space: Let $n \geq 0$ be an integer, and let $\eta = (\pi : E \to \mathbb{P}^n, +, \cdot, 0)$ be the tautological rank 1 subbundle on $\mathbb{P}^n_k$, and let $\phi : E \to \mathbb{P}^n_k \times \mathbb{A}^{n+1}_k$ be the inclusion. This is a homomorphism of vector bundles on $\mathbb{P}^n_k$ such that for every $x \in \mathbb{P}^n_k$, the induced map $\phi_x : E_x \to \mathbb{A}^{n+1}_k$ is injective.

(a) Let X be a variety, let $\zeta = (\pi : L \to X, +, \cdot, 0)$ be a vector bundle of rank 1 on X, and let $\psi : L \to X \times \mathbb{A}^{n+1}_k$ be a homomorphism of vector bundles on X such that for every $x \in X$, the induced map $\phi_x : E_x \to \mathbb{A}^{n+1}_k$ is injective. Let $U = L - 0(X)$, the complement of the zero section, and denote by $G : U \to \mathbb{P}^n_k$ the composition,

$$U \xrightarrow{\phi} X \times (\mathbb{A}^{n+1}_k - \{0\}) \xrightarrow{\pi_2} (\mathbb{A}^{n+1}_k - \{0\}) \xrightarrow{\pi_{\mathbb{P}^n_k}} \mathbb{P}^n_k.$$

Prove there exists a unique morphism $F : X \to \mathbb{P}^n_k$ such that $F \circ \pi = G$. **Hint:** Use the gluing lemma to reduce to the case that $L \cong X \times \mathbb{A}^1_k$ and compose G with any section disjoint from the zero section.

Solution: Denote by $\mathcal{C} = \{(U, F_U)\}$ the collection of all pairs (U, F_U) of an open subset $U \subset X$ and a morphism $F_U : U \to \mathbb{P}^n_k$ such that $F_U \circ \pi_{\pi^{-1}(U)}$ equals $G|_{\pi^{-1}(U)}$. Because $\pi : \pi^{-1}(U) \to U$ is surjective, if there exists a morphism F_U such that $F_U \circ \pi_{\pi^{-1}(U)} = G|_{\pi^{-1}(U)}$, then F_U is unique. In particular, given $(U_1, F_{U_1}), (U_2, F_{U_2})$ in $\mathcal{C}$, since both $F_{U_1}|_{U_1 \cap U_2}$ and $F_{U_2}|_{U_1 \cap U_2}$ satisfy the condition for $F_{U_1 \cap U_2}$, $F_{U_1}|_{U_1 \cap U_2} = F_{U_2}|_{U_1 \cap U_2}$, i.e., the gluing lemma applies. So there exists an open subset $V \subset X$ and a morphism $F_V : V \to \mathbb{P}^n_k$ such that $F_V \circ \pi|_{\pi^{-1}(V)} = G|_{\pi^{-1}(V)}$ and such that for every open subset $U \subset X$ and every F_U, U is contained in V and $F_U = F_V|_U$. The claim is that $U = X$.

Let $U \subset X$ be an open subset, and assume there exists a section $s : U \to U \times_X L$ in $\mathcal{E}_{\text{sec}}(U)$ such that $s(U)$ is disjoint from $0_U(U)$. Define $\psi(U)(s) : U \to \mathbb{A}^{n+1}_k$ to be the composition of $s : U \to U \times_X L$ and of of $\operatorname{pr}_{\mathbb{A}^{n+1}_k} \circ \psi \circ \operatorname{pr}_2 : U \times_X L \to \mathbb{A}^{n+1}_k$. Because ψ is injective on fibers, $(\psi(U)(s))(X) \subset \mathbb{A}^{n+1}_k - \{0\}$. Denote by $\pi_{\mathbb{P}^n_k} : (\mathbb{A}^{n+1}_k - \{0\}) \to \mathbb{P}^n_k$ the projection. Denote by $\overline{\psi(U)(s)} : X \to \mathbb{P}^n_k$ the composition $\pi_{\mathbb{P}^n_k} \circ \psi(s)$. Then $\overline{\psi(U)(s)} \circ \pi = G|_{\pi^{-1}(U)}$, i.e., $F_U = \overline{\psi(U)(s)}$ is a morphism as in the previous paragraph. Therefore $U \subset V$ if there exists a section $s \in \mathcal{E}_{\text{sec}}(U)$ such that $s(U)$ is disjoint from $0_U(U)$.

Because L is a vector bundle, for every point $p \in X$, there exists an open set $p \in U \subset X$ such that $U \times_X L$ is isomorphic to $U \times \mathbb{A}^1_k$ as vector bundles over U. In particular, since $\operatorname{Id}_U \times 1 : U \to U \times \mathbb{A}^1_k$ is a section whose image is disjoint from $0_U(U)$, there exists a sections $s : U \to U \times_X L$ such that $s(U)$ is disjoint from

$0_U(U)$. By the last paragraph, $U \subset V$. In particular, $p \in U$ for every $p \in X$, i.e., $V = X$. This proves the claim.

(b) Prove there exists a unique isomorphism $\theta : L \to F^*E$ such that $\psi = F^*\phi \circ \theta$. This is the *universal property of projective space*: morphisms from a variety X to $\mathbb{P}^n_k$ are in natural bijective correspondence to the set of pairs (L, ψ) up to precomposing ψ with an automorphism of L.

Solution: Define $\mathcal{C} = \{(U, \theta_U)\}$ to be the collection of all pairs (U, θ_U) of an open subset $U \subset X$ and an isomorphism $\theta_U : U \times_X L \to (F|_U)^*E$ such that $\psi|_U = (F|_U)^*\phi \circ \theta_U$. For any two isomorphisms $\theta_U, \theta'_U : U \times_X L \to (F|_U)^*E$, there is a unique morphism $\gamma : U \to \mathbb{G}_m$ such that $\theta'_U = \gamma \cdot \theta_U$. Since $\psi|_U = (F|_U)^*\phi \circ \theta'_U = \gamma \cdot ((F|_U)^*\phi \circ \theta'_U) = \gamma \cdot \psi|_U$, and since $\psi|_U$ is injective on all fibers, γ is the constant morphism with value 1, i.e., $\theta'_U = \theta_U$. So, for the same reason as in (a), there is a maximal open subset $V \subset X$ and a maximal isomorphism θ_V. The claim is that $V = X$.

In fact there is no question what θ is. By base-change, the morphism θ is equivalent to a morphism $\tilde{\theta} : L \to E$. Recall E is a closed subvariety of $\mathbb{P}^n_k \times \mathbb{A}^{n+1}_k$. Define $\tilde{\theta} : L \to \mathbb{P}^n_k \times \mathbb{A}^{n+1}_k$ to be $(F \times \pi) \times (\mathrm{pr}_{\mathbb{A}^{n+1}_k} \circ \psi)$. By construction, the restriction of this morphism to $L - 0_X(X)$ maps into E. Since this is a dense subset of L, all of L maps into E. Moreover, $\theta : L \to F^*E$ is the unique morphism of Abelian cones such that $\psi = F^*\phi \circ \theta$. It only remains to prove that θ is an isomorphism.

Because ψ is injective on all fibers, also θ is injective on all fibers. For every $p \in X$, there exists an open subset $p \in U \subset X$ and isomorphisms $\alpha : U \times_X L \to U \times \mathbb{A}^1_k$ and $\beta : U \times_X F^*E \to U \times \mathbb{A}^1_k$. Then $\beta^{-1} \circ \theta|_U \circ \alpha : U \times \mathbb{A}^1_k \to U \times \mathbb{A}^1_k$ is a morphism that is injective on all fibers. In particular, precomposing with $\mathrm{Id}_U \times 1 : U \to U \times \mathbb{A}^1_k$, and then composing with the projection $\mathrm{pr}_{\mathbb{A}^1_k} : U \times \mathbb{A}^1_k \to \mathbb{A}^1_k$ gives a morphism $\gamma : U \to \mathbb{A}^1_k$ which is everywhere nonzero, i.e., γ is a morphism $U \to \mathbb{G}_m$. Therefore there is a multiplicative inverse δ of γ, i.e., $\delta \cdot \gamma = \gamma \cdot \delta$ is the constant morphism 1. Define θ_U^{-1} to be $\delta \cdot (\alpha^{-1} \circ \beta) : U \times_X F^*E \to U \times_X L$. Then it is straightforward to check that $\theta_U^{-1} \circ (\theta|_U)$ and $(\theta|_U) \circ \theta_U^{-1}$ are identity morphisms. Therefore $\theta|_U$ is an isomorphism. In particular, $p \in V$ for every $p \in X$, i.e., $V = X$.

Remark: Given two pairs (L_1, ψ_1), (L_2, ψ_2) as above, define an *isomorphism of pairs* to be an isomorphism of vector bundles $\eta : L_1 \to L_2$ such that $\psi_2 \circ \eta = \psi_1$. What we have proved is, for every pair (L, ψ) there is a unique morphism $F : X \to \mathbb{P}^n_k$ and a unique isomorphism $\theta : (L, \psi) \to F^*(E, \phi)$. Therefore the set of isomorphism classes of pairs (L, ψ) is in natural bijection with the set of morphisms $F : X \to \mathbb{P}^n_k$. This is the *universal property of projective space*. It satisfies many compatibilities: in particular this bijection is compatible with pullback by arbitrary morphisms $G : Y \to X$. Stated more precisely, there is a contravariant functor P : **Algebraic varieties** $\to$ **Sets** assigning to every variety X the set of isomorphism classes of pairs (L, ψ) on X, and assigning to every morphism $G : Y \to X$ the induced map of sets, $[(L, \psi)] \mapsto P(G)[(L, \psi)] := [(G^*L, G^*\psi)]$. The isomorphism class $[(E, \phi)] \in P(\mathbb{P}^n_k)$ *represents* the functor P, i.e., there is an induced natural transformation $[(E, \phi)]$ of contravariant functors,

$$[(E, \phi)]_X : \mathrm{Hom}_{\textbf{Alg. var.}}(X, \mathbb{P}^n_k) \to P(X),$$
$$(F : X \to \mathbb{P}^n_k) \mapsto F^*[(E, \phi)],$$

and for every variety X this is a bijection of sets.

Problem 8: Let X be a variety, and let ζ, resp. ζ and η, be vector bundles on X. For each of the following functors, resp. bifunctors, on the category of k-vector spaces, define a natural analogue for ζ, resp. ζ and η, i.e., an analogous functor, resp. bifunctor, on the category of vector bundles on X. Observe that even if X, the total space of ζ and the total space of η are quasi-projective, it is not obvious that the total space of each of the new vector bundles is quasi-projective (although this turns out to be true). This is one justification of working in the larger category of algebraic varieties.

(a) Duals, $V \mapsto V^\vee := \mathrm{Hom}_k(V, k)$.

(b) Hom, $(V, W) \mapsto \mathrm{Hom}_k(V, W)$.

(c) Tensor product, $(V, W) \mapsto V \otimes_k W$.

Problem 9, Another definition of dimension I: Let $X \subset \mathbb{P}^n_k$ be a nonempty projective algebraic subset.

(a) Prove that if $\dim(X) > 0$, then for every $f \in k[x_0, \ldots, x_n]_1 - \{0\}$, the corresponding hyperplane $H = \mathbb{V}(f) \subset \mathbb{P}^n_k$ intersects X. **Hint:** If $X \cap H = \emptyset$ then $X \subset D(f) \cong \mathbb{A}^n_k$, which, combined with universal closedness, implies X is a finite set.

Solution: As proved some time ago, $D_+(f)$ is isomorphic to an affine space $\mathbb{A}^n_k$ as a variety. So if $X \cap H = \emptyset$, then the inclusion $i : X \to D_+(f)$ is a closed immersion. For every coordinate function $y : D_+(f) \to \mathbb{A}^1_k$, consider $\mathbb{A}^1_k$ as an open subvariety of $\mathbb{P}^1_k$ and form the composition $y \circ i : X \to \mathbb{P}^1_k$. The morphism $(y \circ i) \times i : X \to \mathbb{P}^1_k \times \mathbb{P}^n_k$ is a closed immersion, because composition with pr_2 is a closed immersion. Because $\mathrm{pr}_1 : \mathbb{P}^1_k \times \mathbb{P}^n_k \to \mathbb{P}^1_k$ is closed, it follows that $(y \circ i)(X) \subset \mathbb{P}^1_k$ is closed. On the other hand, $(y \circ i)(X) \subset \mathbb{A}^1_k$. Therefore $(y \circ i)(X)$ is a finite set, i.e., $y \circ i$ is constant on every connected component of X. Since this holds for every y, $i : X \to \mathbb{A}^n_k$ is constant on every connected component of X, i.e., $i(X) \subset \mathbb{A}^n_k$ is a finite subset. Since i is a closed immersion, X is a finite set. Hence $\dim(X) = 0$. Therefore, if $\dim(X) > 0$, then $H \cap X \neq \emptyset$.

(b) If $\dim(X) = d$, combine (a) with Krull's Hauptidealsatz to conclude that for every d hyperplanes $H_1, \ldots, H_d \subset \mathbb{P}^n_k$, the intersection $H_1 \cap \cdots \cap H_d \cap X$ is nonempty.

Solution: Let $d = \dim(X)$. The claim is that for every integer $1 \leq e \leq d$, and every e hyperplanes $H_1, \ldots, H_e \subset \mathbb{P}^n_k$, the intersection $H_1 \cap \cdots \cap H_e \cap X$ is nonempty. If $d = e = 1$, this follows from (a). Thus assume $d > 1$ and, by way of induction, assume the result for $d - 1$. If $e = 1$, then again the result follows by (a). Thus assume $e > 1$ and, by way of induction, assume the result for $e - 1$. Let $H_1, \ldots, H_e \subset \mathbb{P}^n_k$ be hyperplanes. By (a), $Y = H_e \cap X$ is nonempty. By Krull's Hauptidealsatz, either $\dim(Y) = d$ or $\dim(Y) = d - 1$. In the first case, then by induction on e, $H_1 \cap \cdots \cap H_{e-1} \cap Y$ is nonempty. In the second case, by induction on d, $H_1 \cap \cdots \cap H_{e-1} \cap Y$ is nonempty. In either case, $H_1 \cap \cdots \cap H_e \cap X$ is nonempty. So the claim is proved by induction on e and d. In particular, for $e = d$, $H_1 \cap \cdots \cap H_d \cap X$ is nonempty.

Problem 10, Another definition of dimension II: This continues Problem 9, which you may now assume. If $\dim(X) = d$, prove there exist hyperplanes

$H_1, \dots, H_{d+1} \subset \mathbb{P}^n_k$ such that $H_1 \cap \cdots \cap H_{d+1} \cap X$ is empty. Deduce the following,

$$\dim(X) = \min\{d \geq 0 | \exists \text{ hyperplanes } H_1, \dots, H_{d+1} \subset \mathbb{P}^n_k \text{ s.t. } H_1 \cap \cdots \cap H_{d+1} \cap X = \emptyset\}.$$

Hint: Work by induction on d. Let $S \subset X$ be a finite set of elements of X intersecting every irreducible component of X, and let $H \subset \mathbb{P}^n_k$ be a hyperplane not intersecting S. Prove that $H \cap X$ is either empty (if $d = 0$), or else has smaller dimension than X.

Solution: The claim is that for every integer $0 \leq e \leq d$, there exist e hyperplanes $H_1, \dots, H_e \subset \mathbb{P}^n_k$ such that $H_1 \cap \cdots \cap H_e \cap X$ has dimension $d - e$. The claim is proved by induction on d. If $d = 0$, the statement is vacuous. Thus assume $d > 0$ and, by way of induction, assume the claim for all smaller values of d. The claim for d is proved by induction on e. For $e = 0$ the statement is vacuous. Thus assume $e > 0$ and, by way of induction, assume the claim for all smaller values of e. The projective algebraic set X has finitely many irreducible components $X_1 \cup \cdots \cup X_r$. For every $i = 1, \dots, r$, let $p_i \in X_i$ be an element. The cones over these elements are finitely many 1-dimensional subspaces of $\mathbb{A}^{n+1}_k$. For any finite collection of 1-dimensional subspaces, there exists a linear functions $f \in k[x_0, \dots, x_n]_1$ that is nonzero on all of them. So $H_e = \mathbb{V}(f)$ is a hyperplane in $\mathbb{P}^n_k$ such that $\{p_1, \dots, p_r\} \cap H_e = \emptyset$. Denote $Y = H_e \cap X$. By Krull's Hauptidealsatz, either $\dim(Y) = d$ or $\dim(Y) = d - 1$. But for every $i = 1, \dots, r$, $Y \cap X_i \subset X_i$ does not contain p_i, thus $Y \cap X_i \neq X_i$. Therefore,

$$\dim(Y) = \max\{\dim(Y \cap X_i) | i = 1, \dots, r\} < \max\{\dim(X_i) | i = 1, \dots, r\} = \dim(X),$$

i.e., $\dim(Y) = d - 1$. By the induction hypothesis for d, there exist hyperplanes $H_1, \dots, H_{e-1} \subset \mathbb{P}^n_k$ such that $H_1 \cap \cdots \cap H_{e-1} \cap Y$ has dimension $(d-1) - (e-1) = d - e$, i.e., $H_1 \cap \cdots \cap H_e \cap X$ has dimension $d - e$. This proves the claim for e. By induction on e, the claim is true for d. By induction on d, the claim is true.

In particular, taking $e = d$, there exist hyperplanes $H_1, \dots, H_d \subset \mathbb{P}^n_k$ such that $H_1 \cap \cdots \cap H_d \cap X$ has dimension 0, i.e., every irreducible component is a point. Moreover, because it is a closed subset of $\mathbb{P}^n_k$, it is quasi-compact. Therefore it is a finite set. As used above, for every finite subset of $\mathbb{P}^n_k$ there exists a hyperplane $H_{d+1} \subset \mathbb{P}^n_k$ disjoint from this set. Therefore $H_1 \cap \cdots \cap H_{d+1} \cap X = \emptyset$.

Problem 11, An irreducible, separated variety that is not quasi-compact: Thanks to Genya for inspiring this problem. For every integer $n \geq 0$, define $U_n = \mathbb{A}^2_k - \{(0, 0)\}$ with coordinates (x_n, y_n), for every pair of integers $0 < m < n$, define $U_{m,n} = D(x_m) \subset U_m$, define $U_{n,m} = D(x_n) \subset U_n$, define $\phi_{m,n} : U_{m,n} \to U_{n,m}$ to be $(a_m, b_m) \mapsto (a_m, b_m/a_m^{n-m})$, and define $\phi_{n,m} : U_{n,m} \to U_{m,n}$ to be $(a_n, b_n) \mapsto (a_n, a_n^{n-m} b_n)$.

(a) For every pair of integers $0 \leq m < n$, prove $\phi_{m,n}$ and $\phi_{n,m}$ are inverse isomorphisms.

(b) Prove the datum $((U_m), (U_{m,n}), (\phi_{m,n}))$ satisfies the hypotheses of the gluing lemma for varieties. Denote the associated variety by $(U, (\phi_m : U_m \to U))$.

(c) To prove that U is separated, it suffices to prove for every pair of integers $0 \leq m \leq n$, that $\phi_m(D(*_m)) \cap \phi_n(D(-_n))$ is affine and $\mathcal{O}_U(\phi_m(D(*_m)) \cap \phi_n(D(-_n)))$ is generated as a k-algebra by $\mathcal{O}_{U_m}(D(*_m))$ and $\mathcal{O}_{U_n}(D-_n)$, for $* = x, y$ and $- = x, y$. Check this, in particular, for $* = y, - = y$.

(d) Prove that all the open sets $\phi_m(D(x_m y_m)) \subset U$ are equal. This is an irreducible open subset that is dense in each $\phi_m(U_m)$, thus it is dense in U. Conclude that U is irreducible of dimension 2.

(e) Prove that U is not quasi-compact. **Hint:** Consider $\phi_m((0,1)) \in U$.

(f) You don't have to do this part, it is just motivational. For every integer $n \geq 0$, define $F_n : U_n \to \mathbb{A}^2_k$ by $F_n(a_n, b_n) = (a_n, a_n^n b_n)$. Prove that the morphisms F_n satisfy the gluing lemma for morphisms. Denote the corresponding morphism by $G : U \to \mathbb{A}^2_k$. Prove that G is an isomorphism over $\mathbb{A}^2_k - (0,0)$. What is the fiber of G over $(0,0) \in \mathbb{A}^2_k$?